OPERATOR-VALUED DYADIC BMO SPACES

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Abstract. We investigate a scale of dyadic operator-valued BMO spaces, corresponding to the different yet equivalent characterizations of dyadic BMO in the scalar case. In the language of operator spaces, we investigate different operator space structures on the scalar dyadic BMO space which arise naturally from the different characterisations of scalar BMO. We relate some of these operator BMO classes to each other by forming certain norm averages over "transformed" versions of the original operator function.

Furthermore, we investigate a connection between John-Nirenberg type inequalities and Carleson-type inequalities via a product formula for paraproducts.

1. Introduction

Let $\mathcal{D}$ denote the collection of dyadic subintervals of the unit circle $\mathbb{T}$, and let $(h_I)_{I\in\mathcal{D}}$, where $h_I = \frac{1}{|I|^{1/2}}(\chi_{I^+} - \chi_{I^-})$, be the Haar basis of $L^2(\mathbb{T})$. For $I \in \mathcal{D}$ and $\phi \in L^2(\mathbb{T})$, let ϕ_I denote the formal Haar coefficients $\int_I \phi(t)h_I dt$, and $m_I\phi = \frac{1}{|I|}\int_I \phi(t)dt$ denote the average of ϕ over I. We write $P_I(\phi) = \sum_{J\subseteq I} \phi_J h_J$.

We will use the notation "$\approx$" to indicate equivalence of expression up to an absolute constant, and "$\lesssim$", "$\gtrsim$" for the corresponding one-sided estimates.

We say that $\phi \in L^2(\mathbb{T})$ belongs to dyadic BMO, written $\phi \in \mathrm{BMO^d}(\mathbb{T})$, if

$$\sup_{I\in\mathcal{D}} (\frac{1}{|I|}\int_I |\phi(t) - m_I\phi|^2 dt)^{1/2} < \infty. \tag{1}$$

It is well-known that this has the following equivalent formulations:

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} \|P_I(\phi)\|_{L^2} < \infty, \tag{2}$$

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}, J\subseteq I} |\phi_J|^2 < \infty, \tag{3}$$

$$\pi_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} \phi_I (m_I f) h_I \tag{4}$$

defines a bounded linear operator on $L^2(\mathbb{T})$.

Of course, due to John-Nirenberg's lemma, one can replace the $L^2(\mathbb{T})$ norm in (1) and (2) by any L^p-norm. That is, for $0 < p < \infty$, we have $\phi \in \mathrm{BMO^d}(\mathbb{T})$ if and

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only if

$$\sup_{I\in\mathcal{D}}(\frac{1}{|I|}\int_I |\phi(t)-m_I\phi|^p dt)^{1/p} = \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/p}}\|P_I(\phi)\|_{L^p} < \infty. \tag{5}$$

For real-valued functions, we can also replace the boundedness of π_ϕ by the boundedness of its adjoint operator

$$\Delta_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} \phi_I f_I \frac{\chi_I}{|I|}. \tag{6}$$

Another equivalent formulation comes from the duality

$$\mathrm{BMO}^{\mathrm{d}}(\mathbb{T}) = (H^1_d(\mathbb{T}))^*, \tag{7}$$

where $H^1_d(\mathbb{T})$ consists of those functions $\phi \in L^1(\mathbb{T})$ such that $\mathcal{S}\phi \in L^1(\mathbb{T})$, where $\mathcal{S}\phi = (\sum_{I\in\mathcal{D}} |\phi_I|^2 \frac{\chi_I}{|I|})^{1/2}$ stands for the dyadic square function. An equivalent characterization of $H^1_d(\mathbb{T})$ is the one through dyadic atoms. That is, $H^1_d(\mathbb{T})$ consists of functions $\phi = \sum_{k\in\mathbb{N}} \lambda_k a_k, \lambda_k \in \mathbb{C}$, where $\sum_{k\in\mathbb{N}} |\lambda_k| < \infty$ and for each k, a_k is a dyadic atom, i.e. $supp(a_k) \subset I_k$ for some $I_k \in \mathcal{D}$, $\int_{I_k} a_k(t)dt = 0$, and $\|a_k\|_\infty \le \frac{1}{|I_k|}$. The reader is referred to [M] or to [G] for the results concerning dyadic H^1 and BMO.

The aim of this paper is to investigate the spaces of operator-valued BMO functions corresponding to characterizations (1)-(7). In the operator-valued case, these characterizations are in general no longer equivalent. In the language of operator spaces, we investigate the different operator space structures on the scalar space $\mathrm{BMO}^{\mathrm{d}}$ which arise naturally from the different yet equivalent characterisations of $\mathrm{BMO}^{\mathrm{d}}$. The reader is referred to [BPo] and [PSm] for some recent results on dyadic BMO and Besov spaces connected to the ones in this paper.

First, we require some further notation for the operator-valued case. Let $\mathcal{H}$ be a separable, finite or infinite-dimensional Hilbert space. Let $\mathcal{F}_{00}$ denote the subspace of $\mathcal{L}(\mathcal{H})$-valued functions on $\mathbb{T}$ with finite formal Haar expansion. Given $e, f \in \mathcal{H}$ and $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ we denote by B_e the function in $L^2(\mathbb{T},\mathcal{H})$ defined by $B_e(t) = B(t)(e)$ and by $B_{e,f}$ the function in $L^2(\mathbb{T})$ defined by $B_{e,f}(t) = \langle B(t)(e), f\rangle$. As in the scalar case, let B_I denote the formal Haar coefficients $\int_I B(t)h_I dt$, and $m_I B = \frac{1}{|I|}\int_I B(t)dt$ denote the average of B over I for any $I \in \mathcal{D}$. Observe that for B_I and $m_I B$ to be well-defined operators, we shall be assuming that the $\mathcal{L}(\mathcal{H})$-valued function B is $weak^*$-integrable. That means, using the duality $\mathcal{L}(\mathcal{H}) = (\mathcal{H}\hat{\otimes}\mathcal{H})^*$, that $\langle B(\cdot)(e), f\rangle \in L^1(\mathbb{T})$ for $e, f \in \mathcal{H}$. In particular, for any measurable set A, there exist $B_A \in \mathcal{L}(\mathcal{H})$ such that $\langle B_A(e), f\rangle = \langle \int_A B(t)(e)dt, f\rangle$.

We can define the following notions corresponding to the previous formulations: We denote by $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of Bochner integrable $\mathcal{L}(\mathcal{H})$-valued functions B such that

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} = \sup_{I\in\mathcal{D}}(\frac{1}{|I|}\int_I \|B(t)-m_I B\|^2 dt)^{1/2} < \infty \tag{8}$$

Similarly, we denote by $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})$ the space of Bochner integrable $\mathcal{H}$-valued functions $b : \mathbb{T} \to \mathcal{H}$ such that

$$\|b\|_{\mathrm{BMO}^{\mathrm{d}}} = \sup_{I\in\mathcal{D}}(\frac{1}{|I|}\int_I \|b(t)-m_I b\|^2 dt)^{1/2} < \infty \tag{9}$$

and by $\mathrm{WBMO^d}(\mathbb{T},\mathcal{H})$ the space of Pettis integrable $\mathcal{H}$-valued functions $b:\mathbb{T}\to\mathcal{H}$ such that

$$\|b\|_{\mathrm{WBMO^d}} = \sup_{I\in\mathcal{D},e\in\mathcal{H},\|e\|=1}\Big(\frac{1}{|I|}\int_I |\langle b(t)-m_Ib,e\rangle|^2dt\Big)^{1/2}<\infty \tag{10}$$

This gives rise to the following definitions of operator-valued dyadic BMO spaces. We denote by $\mathrm{SBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of $\mathcal{L}(\mathcal{H})$-valued functions B such that $B(\cdot)e\in L^1(\mathbb{T},\mathcal{H})$ for all $e\in\mathcal{H}$ and such that

$$\|B\|_{\mathrm{SBMO^d}} = \sup_{I\in\mathcal{D},e\in\mathcal{H},\|e\|=1}\Big(\frac{1}{|I|}\int_I \|(B(t)-m_IB)e\|^2dt\Big)^{1/2}<\infty. \tag{11}$$

We shall also use the notation

$$\|B\|_{\mathrm{BMO^d_{so}}} = \|B\|_{\mathrm{SBMO^d}}+\|B^*\|_{\mathrm{SBMO^d}}, \tag{12}$$

and denote by $\mathrm{BMO^d_{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of functions for which this expression is finite. We denote by $\mathrm{WBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of *weak**-integrable $\mathcal{L}(\mathcal{H})$-valued functions B such that

$$\begin{aligned}\|B\|_{\mathrm{WBMO^d}} &= \sup_{I\in\mathcal{D},\|e\|=\|f\|=1}\Big(\frac{1}{|I|}\int_I |\langle (B(t)-m_IB)e,f\rangle|^2dt\Big)^{1/2}\\ &= \sup_{e\in\mathcal{H},\|e\|=1}\|B_e\|_{\mathrm{WBMO^d}(\mathbb{T},\mathcal{H})}<\infty,\end{aligned} \tag{13}$$

or, equivalently, such that

$$\|B\|_{\mathrm{WBMO^d}} = \sup_{A\in S_1,\|A\|_1\le 1}\|\langle B,A\rangle\|_{\mathrm{BMO^d}(\mathbb{T})}<\infty.$$

Here, S_1 denotes the ideal of trace class operators in $\mathcal{L}(\mathcal{H})$, and $\langle B,A\rangle$ stands for the scalar-valued function given by $\langle B,A\rangle(t)=\mathrm{trace}(B(t)A^*)$.

We now define another operator-valued BMO space, using the notion of Haar multipliers.

As in the scalar-valued case (see [Per]), a sequence $(\Phi_I)_{I\in\mathcal{D}}$, $\Phi_I\in L^2(I,\mathcal{L}(\mathcal{H}))$ for all $I\in\mathcal{D}$, is said to be an *operator-valued Haar multiplier*, if there exists $C>0$ such that

$$\|\sum_{I\in\mathcal{D}}\Phi_I(f_I)h_I\|_{L^2(\mathbb{T},\mathcal{H})}\le C\Big(\sum_{I\in\mathcal{D}}\|f_I\|^2\Big)^{1/2} \text{ for all } (f_I)_{I\in\mathcal{D}}\in l^2(\mathcal{D},\mathcal{H}).$$

We write $\|(\Phi_I)\|_{mult}$ for the norm of the corresponding operator on $L^2(\mathbb{T},\mathcal{H})$.

Letting, again as in the scalar valued case, $P_IB=\sum_{J\subseteq I}h_JB_J$, we denote the space of those *weak**-integrable $\mathcal{L}(\mathcal{H})$-valued functions for which $(P_IB)_{I\in\mathcal{D}}$ defines a bounded operator-valued Haar multiplier by $\mathrm{BMO_{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ and write

$$\|B\|_{\mathrm{BMO_{mult}}} = \|(P_IB)_{I\in\mathcal{D}}\|_{mult}. \tag{14}$$

We shall use the notation $\Lambda_B(f)=\sum_{I\in\mathcal{D}}(P_IB)(f_I)h_I$.

It is elementary to see that

$$\Lambda_B(f)=\sum_{I\in\mathcal{D}}B_I(m_If)h_I+\sum_{I\in\mathcal{D}}B_I(f_I)\frac{\chi_I}{|I|}. \tag{15}$$

Observe that $(\Lambda_B)^*=\Lambda_{B^*}$, hence $\|B\|_{\mathrm{BMO_{mult}}}=\|B^*\|_{\mathrm{BMO_{mult}}}$.

Let us now give the definition of a further BMO space, the space defined in terms of paraproducts.

Let $B \in \mathcal{F}_{00}$. We define

$$\pi_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \quad f = \sum_{I \in \mathcal{D}} f_I h_I \mapsto \sum_{I \in \mathcal{D}} B_I(m_I f) h_I,$$

and

$$\Delta_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \quad f = \sum_{I \in \mathcal{D}} f_I h_I \mapsto \sum_{I \in \mathcal{D}} B_I(f_I) \frac{\chi_I}{|I|}.$$

π_B is called the vector paraproduct with symbol B. One sees easily that $\Delta_B = \pi^*_{B^*}$ and that $\Lambda_B = \pi_B + \Delta_B$.

Writing $E_k B = \sum_{|I| > 2^{-k}} B_I h_I$, we denote the space of *weak**-integrable operator-valued functions for which $\sup_{k \in \mathbb{N}} \|\pi_{E_k B}\| < \infty$ by $\mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. For those functions, $\pi_B f = \lim_{k \to \infty} \pi_{E_k B} f$ defines a bounded linear operator on $L^2(\mathbb{T}, \mathcal{H})$, and we write

$$\|B\|_{\mathrm{BMO}_{\mathrm{para}}} = \|\pi_B\|. \tag{16}$$

The space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ is the space of *weak**-integrable operator-valued functions for which

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}} = \sup_{I \in \mathcal{D}} \Big(\frac{1}{|I|} \sum_{J \in \mathcal{D}, J \subseteq I} \|B_J\|^2\Big)^{1/2} < \infty. \tag{17}$$

Recall that for a given Banach space $(X, \|\cdot\|)$, a family of norms $(M_n(X), \|\cdot\|_n)$ on the spaces $M_n(X)$ of X-valued $n \times n$ matrices defines an *operator space structure* on X, if $\|\cdot\|_1 \approx \|\cdot\|$,

- M1 $\|A \oplus B\|_{n+m} \le \max\{\|A\|_n, \|B\|_m\}$ for $A \in M_n(X)$, $B \in M_m(X)$
- M2 $\|\alpha A \beta\|_m \le \|\alpha\|_{M_{n,m}(\mathbb{C})} \|A\|_n \|\beta\|_{M_{m,n}(\mathbb{C})}$ for all $A \in M_n(X)$ and all scalar matrices $\alpha \in M_{n,m}(\mathbb{C})$, $\beta \in M_{m,n}(\mathbb{C})$.

(see e. .g. [ER]). One verifies easily that all the norms above, apart from $\|\cdot\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}}$, taken for finite-dimensional $\mathcal{H}$, define operator space structures on $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T})$.

The paper is divided into four sections following this introduction. Section 2 is devoted to proving the following chain of strict inclusions for infinite-dimensional $\mathcal{H}$:

$$\begin{aligned} \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}} \\ \subsetneq \mathrm{SBMO}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{WBMO}(\mathbb{T}, \mathcal{L}(\mathcal{H})). \end{aligned} \tag{18}$$

This means that the corresponding embeddings of operator spaces over $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T})$ are completely bounded, but not completely isomorphic (for the notation, see again e. g. [ER]).

In the third section, we investigate the operator-valued paraproducts in terms of the so-called *sweep* of the symbol. Given $B \in \mathcal{F}_{00}$, we define the sweep of B as

$$S_B = \sum_{I \in \mathcal{D}} B_I^* B_I \frac{\chi_I}{|I|}. \tag{19}$$

Our main result of this section, Theorem 3.5, states that $\|B\|^2_{\mathrm{BMO}_{\mathrm{para}}} \approx \|S_B\|_{\mathrm{BMO}_{\mathrm{mult}}} + \|B\|^2_{\mathrm{SBMO}^{\mathrm{d}}}$.

Operator-valued paraproducts are of particular interest, because they can be seen as dyadic versions of vector Hankel operators or of vector Carleson embeddings, which are important in the real and complex analysis of matrix valued functions and also in the theory of infinite-dimensional linear systems with infinite-dimensional output space (see e.g. [JPP1]).

In Section 4, we investigate "average BMO conditions" in the following sense. For $\sigma \in \{-1,1\}^{\mathcal{D}}$, define the *dyadic martingale transform*

$$T_\sigma : L^2(\mathbb{T},\mathcal{H}) \to L^2(\mathbb{T},\mathcal{H}), \qquad f = \sum_{I\in\mathcal{D}} h_I f_I \mapsto \sum_{I\in\mathcal{D}} h_I \sigma_I f_I, \tag{20}$$

Let $\Sigma = \{-1,1\}^{\mathcal{D}}$, equipped with the natural product measure which assigns measure 2^{-n} to cylinder sets of length n.

While we do not know whether $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}} \subseteq \mathrm{BMO}_{\mathrm{para}}$, we show (see Theorem 4.1) that $\|B\|_{\mathrm{BMO}_{\mathrm{para}}} \le C(\int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma)^{1/2}$. More precisely, $\|B\|^2_{\mathrm{BMO}_{\mathrm{para}}} + \|B^*\|^2_{\mathrm{BMO}_{\mathrm{para}}} \approx \int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}_{\mathrm{mult}}} d\sigma$.

Moreover, the norms $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}}$, $\|B\|_{\mathrm{BMO}_{\mathrm{mult}}}$ and $\|B\|_{\mathrm{BMO}_{\mathrm{para}}}$ can be completely described in terms of average boundedness of certain operators involving either Λ_B or commutators $[T_\sigma, B]$. The results of this section complete those proved in [GPTV].

It was shown in [NTV] that $\mathrm{BMO}_{\mathrm{para}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{SBMO}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. The space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ is understood in [NTV] as the space of functions satisfying a natural operator Carleson condition, namely

$$\sup_{I\in\mathcal{D}} \Big\| \frac{1}{|I|} \sum_{J\subseteq I} B_J^* B_J \Big\| < \infty \tag{21}$$

Therefore, the result from [NTV] represents a breakdown of the Carleson embedding theorem in the operator case.

We investigate here a different version of the Carleson condition for the operator case, namely

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \Big\| \sum_{J\subseteq I} B_J^* B_J \frac{\chi_J}{|J|} \Big\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} < \infty \tag{22}$$

It is shown in Theorem 3.6 that (22) implies the boundedness of π_B, that is, the boundedness of a certain dyadic operator Carleson embedding.

In [K], [NTV] and [NPiTV], the correct rate of growth of the constant in the Carleson embedding theorem in the matrix case in terms of the dimension of Hilbert space $\mathcal{H}$ was determined, namely $\log(\dim \mathcal{H} + 1)$. Here, we want to show that this breakdown of the Carleson embedding theorem in the operator case is intimately connected to a breakdown of the John-Nirenberg Theorem, and that the dimensional growth for constants in the John-Nirenberg Theorem is the same. This answers a question left open in [GPTV].

The last section is devoted to the study of sweeps of functions in different BMO-spaces. The classical John-Nirenberg theorem on $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T})$ implies (and is essentially equivalent to) the fact that there exists a constant $C > 0$ such that

$$\|S_b\|_{\mathrm{BMO}^{\mathrm{d}}} \le C\|b\|^2_{\mathrm{BMO}^{\mathrm{d}}} \tag{23}$$

for any $b \in \mathrm{BMO}^{\mathrm{d}}$.

We will show that this formulation of John-Nirenberg does not hold for $\|B\|_{\mathrm{BMO}_{\mathrm{so}}}$. In fact, it is shown that if (23) holds for some space contained in $\mathrm{SBMO}^{\mathrm{d}}$ then this space is also contained in $\mathrm{BMO}_{\mathrm{para}}$.

2. Some operator-valued dyadic BMO spaces

Let us mention that by John-Nirenberg's lemma, we actually have that $f \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ if and only if

$$\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I \|B(t)-m_IB\|^p dt\Big)^{1/p}<\infty$$

for some (or equivalently, for all) $0<p<\infty$. Since $(B-m_IB)\chi_I = P_IB$ we can also say that $f\in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ if and only if

$$\sup_{I\in\mathcal{D}}\frac{1}{|I|^{1/p}}\|P_I(B)\|_{L^p(\mathcal{L}(\mathcal{H}))}<\infty.$$

Another elementary identity we shall use is

$$\|B\|_{\mathrm{WBMO}^{\mathrm{d}}} = \sup_{I\in\mathcal{D},\|e\|=\|f\|=1}\frac{1}{|I|^{1/2}}\|P_I(B_{e,f})\|_{L^2} = \sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\sum_{J\in\mathcal{D},J\subseteq I}|\langle B_Je,f\rangle|^2\Big)^{1/2}.$$

In particular,

$$\|B_J\|\le |J|^{1/2}\|B\|_{\mathrm{WBMO}^{\mathrm{d}}} \qquad (J\in\mathcal{D}). \tag{24}$$

The following characterizations of SBMO will be useful below. Most of it can be found in [GPTV], we give the proof for the convenience of the reader.

Proposition 2.1. *Let $B\in \mathrm{SBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. Then*

$$\begin{aligned}
\|B\|^2_{\mathrm{SBMO}^{\mathrm{d}}} &= \sup_{e\in\mathcal{H},\|e\|=1}\|B_e\|^2_{\mathrm{BMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})}\\
&= \sup_{I\in\mathcal{D},\|e\|=1}\frac{1}{|I|}\|P_I(B_e)\|^2_{L^2(\mathcal{H})}\\
&= \sup_{I\in\mathcal{D}}\frac{1}{|I|}\Big\|\sum_{J\subseteq I}B_J^*B_J\Big\|\\
&= \sup_{I\in\mathcal{D}}\left\|\frac{1}{|I|}\int_I (B(t)-m_IB)^*(B(t)-m_IB)dt\right\|\\
&= \sup_{I\in\mathcal{D}}\|m_I(B^*B)-m_I(B^*)m_I(B)\|\\
&\approx \sup_{I\in\mathcal{D}}\frac{1}{|I|}\Big\|\sum_{J\subsetneq I}B_J^*B_J\Big\|.
\end{aligned}$$

Proof. The two first equalities are obvious from the definition. Now observe

$$\Big\|\sum_{J\subseteq I}B_J^*B_J\Big\| = \sup_{\|e\|=1,\|f\|=1}\sum_{J\subseteq I}\langle B_J(e),B_J(f)\rangle = \sup_{\|e\|=1}\sum_{J\subseteq I}\|B_J(e)\|^2 = \|P_I(B_e)\|^2_{L^2(\mathcal{H})}.$$

The following equalities follows from

$$\begin{aligned}
\|m_I(B^*B)-m_I(B^*)m_I(B)\| &= \left\|\frac{1}{|I|}\int_I (B(t)-m_IB)^*(B(t)-m_IB)dt\right\|\\
&= \sup_{e\in\mathcal{H},\|e\|=1}\frac{1}{|I|}\int_I\langle (B(t)-m_IB)^*(B(t)-m_IB)e,e\rangle dt\\
&= \sup_{e\in\mathcal{H},\|e\|=1}\frac{1}{|I|}\int_I\|P_IBe\|^2 dt.
\end{aligned}$$

To show the equivalence up to constants in the last line, notice first that $\frac{1}{|I|}\|\sum_{J\subsetneq I} B_J^* B_J\| \le \frac{1}{|I|}\|\sum_{J\subseteq I} B_J^* B_J\|$ for each $I\in\mathcal{D}$.

On the other hand, let $C = \sup_{I\in\mathcal{D}} \frac{1}{|I|}\|\sum_{J\subsetneq I} B_J^* B_J\|$ and suppose that the supremum is finite. Then for any given $I\in\mathcal{D}$, $\|B_I^*B_I\| \le \|\sum_{J\subsetneq \tilde I} B_J^* B_J\| \le |\tilde I| C = 2|I|C$, where $\tilde I$ denotes the parent interval of I. It follows that

$$\frac{1}{|I|}\|\sum_{J\subseteq I} B_J^* B_J\| \le C + 2C$$

for each $I\in\mathcal{D}$. □

We would like to point out that while B belongs to one of the spaces $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$, $\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H})))$ or $B\in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ if and only if B^* does, this is not the case for the space $\mathrm{SBMO}^d(\mathbb{T},\mathcal{L}(\mathcal{H}))$. This leads to the following notion:

Definition 2.2. *(see* [GPTV], [Pet]*) We say that* $B\in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$*, if* B *and* B^* *belong to* $\mathrm{SBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H})$*. We define* $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} = \|B\|_{\mathrm{SBMO}^{\mathrm{d}}} + \|B^*\|_{\mathrm{SBMO}^{\mathrm{d}}}$.

Let r_k denote the Rademacher functions, that is

$$r_k = \sum_{|I|=2^{-k}} |I|^{1/2} h_I.$$

Lemma 2.3. *Let* $B=\sum_{k=1}^N B_k r_k$*. Then*

$$\|B\|_{\mathrm{SBMO}^{\mathrm{d}}} = \sup_{\|e\|=1} (\sum_{k=1}^N \|B_k e\|^2)^{1/2} \tag{25}$$

$$\|B\|_{\mathrm{BMO}_{\mathrm{so}}} = \sup_{\|e\|=1} (\sum_{k=1}^N \|B_k e\|^2)^{1/2} + \sup_{\|e\|=1} (\sum_{k=1}^N \|B_k^* e\|^2)^{1/2} \tag{26}$$

$$\|B\|_{\mathrm{WBMO}^d} = \sup_{\|f\|=\|e\|=1} (\sum_{k=1}^N |\langle B_k e, f\rangle|^2)^{1/2}. \tag{27}$$

Proof. This follows from standard Littlewood-Paley theory. □

Proposition 2.4. *Let* $\dim\mathcal{H}=\infty$*. Then*
$\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{SBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

Proof. The inclusions follow from the definitions.

Let us see that they are strict. For $x,y\in\mathcal{H}$ we denote by $x\otimes y$ the rank 1 operator in $\mathcal{L}(\mathcal{H})$ given by $(x\otimes y)(h) = \langle h,y\rangle x$. Hence it follows from (25) and (26) that if (e_k) is an orthonormal basis of $\mathcal{H}$ and $h\in\mathcal{H}$ with $\|h\|=1$, then $B=\sum_{k=1}^\infty h\otimes e_k\, r_k$ belongs to $\mathrm{SBMO}^{\mathrm{d}}$ but it does not belong to $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. It follows from (26) and (27) that $B=\sum_{k=1}^\infty e_k\otimes h\, r_k$ belongs to $\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H})$, but not to $\mathrm{SBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. □

Of course, if $(\Phi_I)_{I\in\mathcal{D}}$ is a Haar multiplier, then

$$\sup_{I\in\mathcal{D},\|e\|=1} |I|^{-1/2}\|\Phi_I(e)\|_{L^2(\mathbb{T},\mathcal{H})} \le \|(\Phi_I)\|_{mult}. \tag{28}$$

In case the Φ_I are constant operators T_I, one has

$$\|(\Phi_I)_{I\in\mathcal{D}}\|_{mult} = \sup_I \|T_I\|.$$

Proposition 2.5. $\mathrm{BMO}_{\mathrm{mult}} \subsetneq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}$.

Proof. The inclusion follows from (28), and the fact that $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \neq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ was shown in [GPTV]. □

Let us now describe the action of Λ_B in a different way.

Proposition 2.6. *Let $B \in \mathcal{F}_{00}$. Then*

$$\Lambda_B(f) = \sum_{I\in\mathcal{D}} \left(B_I(m_{I^+}f)\frac{\chi_{I^+}}{|I|^{1/2}} - B_I(m_{I^-}f)\frac{\chi_{I^-}}{|I|^{1/2}} \right).$$

Proof. Use the formulae

$$m_I f = \frac{1}{2}(m_{I^+}f + m_{I^-}f), \qquad f_I = \frac{|I|^{1/2}}{2}(m_{I^+}f - m_{I^-}f) \tag{29}$$

to obtain

$$\begin{aligned} m_I f h_I + f_I \frac{\chi_I}{|I|} &= \frac{1}{2}(m_{I^+}f + m_{I^-}f)(\chi_{I^+} - \chi_{I^-})|I|^{-1/2} \\ &+ \frac{1}{2}(m_{I^+}f - m_{I^-}f)(\chi_{I^+} + \chi_{I^-})|I|^{-1/2} \\ &= m_{I^+}f\frac{\chi_{I^+}}{|I|^{1/2}} - m_{I^-}f\frac{\chi_{I^-}}{|I|^{1/2}} \end{aligned}$$

□

Of course $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Using that

$$\Lambda_B f = Bf - \sum_{I\in\mathcal{D}} (m_I B)(f_I)h_I \tag{30}$$

one finds that

Proposition 2.7. $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Our next objective is to see that $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. For that, we need again some more notation.

Let S_1 denote the ideal of trace class operators on $\mathcal{H}$, and, as in the proof of Proposition 2.4, for $e, d \in \mathcal{H}$, let $e \otimes d$ denote the rank one operator given by $(e \otimes d)h = \langle h, d\rangle e$. One has that $S_1 = \mathcal{H}\hat{\otimes}\mathcal{H}$ and $(S_1)^* = \mathcal{L}(\mathcal{H})$ by the pairing $\langle U, (e \otimes d)\rangle = \langle U(e), d\rangle$.

It is easy to see that the space $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ can be embedded isometrically into the dual of a certain H^1 space of S_1 valued functions.

Definition 2.8. *Let $f, g \in L^2(\mathbb{T}, \mathcal{H})$. Define*

$$f \circledast g = \sum_{I\in\mathcal{D}} (m_{I^+}f \otimes m_{I^+}g - m_{I^-}f \otimes m_{I^-}g)\frac{1}{2}(\chi_{I^+} - \chi_{I^-}).$$

Let $H^1_\Lambda(\mathbb{T}, S_1)$ be the space of functions $f = \sum_{k=1}^\infty \lambda_k f_k \circledast g_k$ such that $f_k, g_k \in L^2(\mathbb{T}, \mathcal{H})$, $\|f_k\|_2 = \|g_k\|_2 = 1$ for all $k \in \mathbb{N}$, and $\sum_{k=1}^\infty |\lambda_k| < \infty$.

We endow the space with the norm given by the infimum of $\sum_{k=1}^\infty |\lambda_k|$ for all possible decompositions.

Proposition 2.9. *$H^1_\Lambda(\mathbb{T}, S_1)$ is continuously embedded into $L^1(\mathbb{T}, S_1)$.*

Proof. Writing $f_I = \frac{1}{2}|I|^{1/2}(m_{I^+}f - m_{I^-}f)$, $g_I = \frac{1}{2}|I|^{1/2}(m_{I^+}g - m_{I^-}g)$, one verifies the identity

$$f \circledast g = \sum_{I\in\mathcal{D}} h_I(f_I \otimes m_I g + m_I f \otimes g_I) = f \otimes g - \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} f_I \otimes g_I$$

from (29). Let $f, g \in L^2(\mathbb{T}, \mathcal{H})$,

$$\begin{aligned}\|f \circledast g\|_{L^1(\mathbb{T},S_1)} &= \|f \otimes g - \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} f_I \otimes g_I\|_{L^1(\mathbb{T},S_1)} \\ &\le \|f \otimes g\|_{L^1(\mathbb{T},S_1)} + \|\sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} f_I \otimes g_I\|_{L^1(\mathbb{T},S_1)} \\ &\le \|f\|_2\|g\|_2 + \sum_{I\in\mathcal{D}} \|f_I \otimes g_I\|_{S^1} \\ &\le \|f\|_2\|g\|_2 + \left(\sum_{I\in\mathcal{D}} \|f_I\|^2\right)^{1/2} \left(\sum_{I\in\mathcal{D}} \|g_I\|^2\right)^{1/2} = 2\|f\|_2\|g\|_2.\end{aligned}$$

□

With this notation, $B \in \mathrm{BMO}_{\mathrm{mult}}$ acts on $f \circledast g$ by

$$\langle B, f \circledast g\rangle = \int_{\mathbb{T}} \langle B(t), (f \circledast g)(t)\rangle dt = \langle \Lambda_B f, g\rangle.$$

By definition of $H^1_\Lambda(\mathbb{T}, S_1)$, $\|B\|_{(H^1_\Lambda(\mathbb{T},S_1))^*} = \|\Lambda_B\|$.

We will now define a further H^1 space of S_1-valued functions. For $F \in L^1(\mathbb{T}, S_1)$, define the dyadic Hardy-Littlewood maximal function F^* of F in the usual way,

$$F^*(t) = \sup_{I\in\mathbb{D}, t\in I} \frac{1}{|I|} \int_I \|F(s)\|_{S_1} ds.$$

Then let $H^1_{\mathrm{max,d}}(\mathbb{T}, S_1)$ be given by

$$\{F \in L^1(\mathbb{T}, S_1) : F^* \in L^1(\mathbb{T})\}.$$

By a result of Bourgain ([Bou], Th.12), $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ embeds continuously into $(H^1_{\mathrm{max,d}}(\mathbb{T}, S_1))^*$ (see also [B1, B2]).

Lemma 2.10. $H^1_\Lambda(\mathbb{T}, S_1) \subseteq H^1_{\mathrm{max,d}}(\mathbb{T}, S_1)$.

Proof. It is sufficient to show that there is a constant $C > 0$ such that for all $f, g \in L^2(\mathbb{T}, \mathcal{H})$, $f \circledast g \in H^1_{\mathrm{max,d}}(\mathbb{T}, S_1)$, and $\|f \circledast g\|_{H^1_{\mathrm{max,d}}(\mathbb{T},S_1)} \le C\|f\|_2\|g\|_2$. Observe as before that $f \circledast g = f \otimes g - \sum_{I\in\mathbb{D}} \frac{\chi_I}{|I|} f_I \otimes g_I$. For $k \in \mathbb{N}$, let E_k denote the expectation with respect to the σ-algebra generated by dyadic intervals of length 2^{-k}, $E_k F = \sum_{I\in\mathcal{D},|I|>2^{-k}} h_I F_I$. Then we have

$$E_k(f \circledast g) = (E_k f) \circledast (E_k g), \tag{31}$$

as

$$\sum_{I\in\mathcal{D},|I|>2^{-k}} h_I(f_I \otimes m_I g + m_I f \otimes g_I) = \sum_{I\in\mathcal{D}} h_I((E_k f)_I \otimes m_I(E_k g) + m_I(E_k f) \otimes (E_k g)_I).$$

Thus

$$(f\circledast g)^*(t) = \sup_{k\in\mathbb{N}} \|E_k(f\circledast g)(t)\|_{S^1} \le \sup_{k\in\mathbb{N}} \|(E_k f)(t)\|\|(E_k g)(t)\| + \sum_{I\in\mathcal{D}} \frac{\chi_I(t)}{|I|}\|f_I\|\|g_I\|$$
$$\le \|f^*(t)\|\|g^*(t)\| + \sum_{I\in\mathcal{D}} \frac{\chi_I(t)}{|I|}\|f_I\|\|g_I\|,$$

and

$$\|(f\circledast g)^*\|_1 \le \|f^*\|_2\|g^*\|_2 + \|f\|_2\|g\|_2 \le C\|f\|_2\|g\|_2$$

by the Cauchy-Schwarz inequality and boundedness of the dyadic Hardy-Littlewood maximal function on $L^2(\mathbb{T},\mathcal{H})$. □

Theorem 2.11. $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

Proof. The inclusion follows by Lemma 2.10, duality and Bourgain's result.

To see that they do not coincide, use the fact that $BMO(\ell_\infty) \subsetneq \ell_\infty(BMO) = (H^1(\ell_1))^*$ to find for each $N\in\mathbb{N}$ functions $b_k\in$ BMO, $k=1,...,N$, such that $\sup_{1\le k\le N}\|b_k\|_{\mathrm{BMO}} \le 1$, but $\|(b_k)_{k=1,\dots,N}\|_{\mathrm{BMO}^d(\mathbb{T},l^\infty_N)} \ge c_N$, where $c_N \overset{N\to\infty}{\to} \infty$.

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and consider the operator-valued function $B(t) = \sum_{k=1}^N b_k(t)e_k\otimes e_k \in L^2(\mathbb{T},\mathcal{L}(\ell_2))$. Clearly $B_I = \sum_{k=1}^N (b_k)_I e_k\otimes e_k$, and for each $\mathbb{C}^N$-valued function $f=\sum_{k=1}^N f_k e_k$, $f_1,\dots,f_N\in L^2(\mathbb{T})$, we have

$$\Lambda_B(f) = \sum_{k=1}^N \Lambda_{b_k}(f_k)e_k.$$

Choosing the f_k such that $\|f\|_2^2 = \sum_{k=1}^N \|f_k\|^2_{L^2(\mathbb{T})} = 1$, we find that

$$\|\Lambda_B(f)\|^2_{L^2(\mathbb{T},\ell_2)} = \sum_{k=1}^N \|\Lambda_{b_k}(f_k)\|^2_{L^2(\mathbb{T})} \le C\sum_{k=1}^N \|b_k\|^2_{BMO}\|f_k\|^2_{L^2(\mathbb{T})} \le C,$$

where C is a constant independent of N. Therefore, Λ_B is bounded.

But since $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} = \|(b_k)_{k=1,\dots,N}\|_{\mathrm{BMO}(\mathbb{T},l^\infty_N)} \ge c_N$, it follows that $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T})$ is not continuously embedded in $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. From the open mapping theorem, we obtain inequality of the spaces. □

3. Operator-valued paraproducts

We start by describing the action of a paraproduct π_B as a Haar multiplier.

Proposition 3.1. *Let $B\in\mathcal{F}_{00}$. Then*

$$\|\pi_B\| = \|(B_I^* h_I)_{I\in\mathcal{D}}\|_{mult}$$
$$= \|(P_{I^+}B + P_{I^-}B)_{I\in\mathcal{D}}\|_{mult}$$
$$= \|(\sum_{J\subsetneq I} B_J^* B_J \frac{\chi_J}{|J|})_{I\in\mathcal{D}}\|^{1/2}_{mult}.$$

In particular,

$$\|B_I\| \le \|\pi_B\||I|^{1/2},$$
$$\|P_{I^+}B(e) + P_{I^-}B(e)\|_{L^2(\mathbb{T},\mathcal{H})} \le \|\pi_B\||I|^{1/2}\|e\|$$

and

$$\|(\sum_{J\subsetneq I} B_J^* B_J \frac{\chi_J}{|J|})e\|_{L^2(\mathbb{T},\mathcal{H})} \leq \|\pi_B\|^2|I|^{1/2}\|e\|.$$

Proof. The first equality follows by writting $\Delta_{B^*}(f) = \sum_{I\in\mathcal{D}} B_I^* h_I f_I h_I$. Then use $\|\pi_B\| = \|\Delta_{B^*}\|$.

The second follows from the fact that $P_I B = (P_{I^+}B + P_{I^-}B) + B_I h_I$, which shows that

$$\pi_B(f) = \sum_{I\in\mathcal{D}} (P_{I^+}B + P_{I^-}B)(f_I)h_I.$$

For the third formulation, use $\|\pi_B\|^2 = \|\pi_B^*\pi_B\|$.

$$\begin{aligned}
\pi_B^*\pi_B(f)(t) &= \sum_{I\in\mathcal{D}} B_I^* B_I(m_I(f))\frac{\chi_I(t)}{|I|} \\
&= \sum_{I\in\mathcal{D}} B_I^* B_I(\sum_{I\subsetneq J} f_J m_I(h_J))\frac{\chi_I(t)}{|I|} \\
&= \sum_{I\in\mathcal{D}} B_I^* B_I(\sum_{I\subsetneq J} f_J)h_J(t)\frac{\chi_I(t)}{|I|} \\
&= \sum_{J\in\mathcal{D}} (\sum_{I\subsetneq J} B_I^* B_I \frac{\chi_I(t)}{|I|})f_J h_J(t).
\end{aligned}$$

□

It follows at once from Proposition 3.1 that

$$\mathrm{BMO}_{\mathrm{para}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subseteq \mathrm{SBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H})).$$

It is easily seen that, if B and B^* belong to $\mathrm{BMO}_{\mathrm{para}}$, then $B\in\mathrm{BMO}_{\mathrm{mult}}$. However, we want to remark that the boundedness of π_B alone does not imply boundedness of Λ_B.

To see this, choose some orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$, and choose a sequence of $\mathbb{C}^n$-valued function $(b_n)_{n\in\mathbb{N}}$ with finite Haar expansion such that $\|b_n\|_{\mathrm{BMO}^{\mathrm{d}}(\mathcal{L}(\mathcal{H}))} \geq Cn^{1/2}\|b_n\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathcal{L}(\mathcal{H}))}$ (for a choice of such a sequence, see [JPP1]). Let $B_n(t)$ be the column matrix with respect to the chosen orthonormal basis which has the vector $b_n(t)$ as its first column. Then it is easy to see that

$$\|\pi_{B_n}\| = \|\pi_{b_n}\| \sim \|b_n\|_{\text{ mod }(\mathbb{T},\mathcal{H})} \geq n^{1/2}C\|b_n\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})}.$$

As pointed out to us [PV] it follows from the first Theorem in the appendix in [PXu] that $\|\pi_{B_n^*}\| \leq C\|b_n\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})}$ for some absolute constant C and all $n\in\mathbb{N}$. Forming the direct sum

$$B = \bigoplus_{n=1}^{\infty} \frac{1}{\|\pi_{B_n^*}\|} B_n^*,$$

we find that $\|\pi_B\| = 1$, but $\Delta_B = (\pi_{B^*})^*$ is unbounded.

The next proposition shows that the space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}$ belongs to a different scale than the standard BMO-spaces.

Proposition 3.2. $L^\infty(\mathbb{T},\mathcal{L}(\mathcal{H})) \not\subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

Proof. Choose an orthonormal basis of $\mathcal{H}$ indexed by the elements of $\mathcal{D}$, say $(e_I)_{I\in\mathcal{D}}$, and let $\Phi_I = e_I \otimes e_I$, $\Phi_I h = \langle h, e_I\rangle e_I$. Let $\lambda_I = |I|^{1/2}$ for $I \in \mathcal{D}$, and define $B = \sum_{I\in\mathcal{D}} h_I \lambda_I \Phi_I$. Then $\sum_{I\in\mathcal{D}} \|B_I\|^2 = \sum_{I\in\mathcal{D}} |I| = \infty$, so in particular $B \notin \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. But the operator function B is diagonal with uniformly bounded diagonal entry functions $\phi_I(t) = \langle B(t)e_I, e_I\rangle = |I|^{1/2} h_I(t)$, so $B \in L^\infty(\mathcal{L}(\mathcal{H}))$. □

Proposition 3.3. $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Proof. The inclusion $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}} \subseteq \mathrm{BMO}_{\mathrm{para}}$ is easy, since (17) implies that for $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}$, the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}$ norm equals the norm of the scalar $\mathrm{BMO}^{\mathrm{d}}$ function given by $|B| := \sum_{I\in\mathcal{D}} h_I \|B_I\|$. For $f \in L^2(\mathcal{H})$, let $|f|$ denote the function given by $|f|(t) = \|f(t)\|$. Thus

$$\|\pi_B f\|_2^2 = \sum_{I\in\mathcal{D}} \|B_I m_I f\|^2 \leq \sum_{I\in\mathcal{D}} (\|B_I\| m_I |f|)^2 = \|\pi_{|B|} |f|\|.$$

The boundedness of π_{B^*} follows analogously.

To show that $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}} \neq \mathrm{BMO}_{\mathrm{para}}$, we can use the diagonal operator function B constructed in Proposition 3.2. There, it is shown that $B \notin \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}$, and that the diagonal entry functions $\phi_I = \langle Be_I, e_I\rangle$ are uniformly bounded. Since the paraproduct of each scalar-valued L^∞ function is bounded, we see that $\pi_B = \bigoplus_{I\in\mathcal{D}} \pi_{\phi_I}$ is bounded. Similarly, π_{B^*} is bounded. Thus $B \in \mathrm{BMO}_{\mathrm{para}}$. □

One of the main tools to investigate the connection between $\mathrm{BMO}_{\mathrm{mult}}$ and $\mathrm{BMO}_{\mathrm{para}}$ is the *dyadic sweep.* Given $B \in \mathcal{F}_{00}$, we define

$$S_B(t) = \sum_{I\in\mathcal{D}} B_I^* B_I \frac{\chi_I(t)}{|I|}.$$

Lemma 3.4. *Let* $B \in \mathcal{F}_{00}$. *Then*

$$\pi_B^* \pi_B = \pi_{S_B} + \pi_{S_B}^* + D_B = \Lambda_{S_B} + D_B, \tag{32}$$

where D_B *is defined by* $D_B h_I \otimes x = h_I \frac{1}{|I|} \sum_{J\subsetneq I} B_J^* B_J x$ *for* $x \in \mathcal{H}$, $I \in \mathcal{D}$ *and*

$$\|D_B\| \approx \|B\|^2_{\mathrm{SBMO}^{\mathrm{d}}}.$$

Proof. (32) is verified on elementary tensors $h_I \otimes x$, $h_J \otimes y$. We find that

(1) for $I \subsetneq J$,

$$\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y\rangle = \langle \pi_{S_B}^* h_I \otimes x, h_J \otimes y\rangle$$

(2) for $I \supsetneq J$,

$$\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y\rangle = \langle \pi_{S_B} h_I \otimes x, h_J \otimes y\rangle$$

(3) for $I = J$,

$$\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y\rangle = \langle D_B(h_I \otimes x), h_J \otimes y\rangle.$$

Since $\operatorname{supp} \pi_{S_B} h_I \subseteq I$ and $\operatorname{supp} \Delta_{S_B} h_I \subseteq I$, $\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y\rangle = 0$ in all other cases.

One sees easily that D_B is block diagonal with respect to the Hilbert space decomposition $L^2(\mathbb{T}, \mathcal{H}) = \bigoplus_{I\in\mathcal{D}} \mathcal{H}$ defined by the mapping $f \mapsto (f_I)_{I\in\mathcal{D}}$. The operator π_{S_B} is block-lower triangular with respect to this decomposition (using the natural partial order on $\mathcal{D}$), and Δ_{S_B} is block-upper triangular. Thus we obtain the required identity. Note that

$$\|D_B\| = \sup_{I\in\mathcal{D}, \|e\|=1} \frac{1}{|I|} \| \sum_{J\subsetneq I} B_J^* B_J e\| \approx \|B\|^2_{\mathrm{SBMO^d}}$$

by Proposition 2.1. □

Notice that $(S_B)^* = S_B$. Hence Lemma 3.4 gives

Theorem 3.5.

$$\|S_B\|_{\mathrm{BMO_{mult}}} + \|B\|^2_{\mathrm{SBMO^d}} \approx \|\pi_B\|^2.$$

Proof. It suffices to use that $\|D_B\| \approx \|B\|^2_{\mathrm{SBMO^d}}$ and that $\|B\|_{\mathrm{SBMO^d}} \lesssim \|\pi_B\|$ (using Proposition 3.1). □

We can now prove that a certain Carleson-type condition is sufficient for the boundedness of π_B.

Theorem 3.6. *There exists $C > 0$ such that for all $B \in \mathcal{F}_{00}$,*

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \| \sum_{J\subseteq I} B_J^* B_J \|_{\mathcal{L}(\mathcal{H})} \le \|\pi_B\|^2 \le C \sup_{I\in\mathcal{D}} \frac{1}{|I|} \| \sum_{J\subseteq I} B_J^* B_J \frac{\chi_J}{|J|} \|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))}.$$

Proof. The first inequality is the inclusion $\mathrm{BMO_{para}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{SBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

For the second one, use the fact that $P_I(S_B) = P_I \sum_{J\subseteq I} B_J^* B_J \frac{\chi_J}{|J|}$ for each $I \in \mathcal{D}$, together with Theorem 2.11, Theorem 3.5 and John-Nirenberg's lemma for $\mathrm{BMO^d_{norm}}$. □

4. Averages over martingale transforms and operator-valued BMO

As in the introduction, let $\Sigma = \{-1, 1\}^{\mathcal{D}}$, and let $d\sigma$ denote the natural product probability measure on Σ, which assigns measure 2^{-n} to cylinder sets of length n.

Given a Banach space X and $F \in L^1(\mathbb{T}, X)$, we write $\tilde{F}$ for the function defined a.e. on $\Sigma \times \mathbb{T}$ by

$$\tilde{F}(\sigma, t) = T_\sigma F(t) = \sum_I \sigma_I F_I h_I(t).$$

In case that X is a Hilbert space, $\|T_\sigma F\|_{L^2(\mathbb{T},X)} = \|F\|_{L^2(\mathbb{T},X)}$ for any $(\sigma_I)_{I\in\mathcal{D}}$, and therefore $\|\tilde{F}\|_{L^\infty(\Sigma,L^2(\mathbb{T},X))} = \|F\|_{L^2(\mathbb{T},X)}$.

More generally, we have for UMD spaces that $\|T_\sigma F\|_{L^2(\mathbb{T},X)} \approx \|F\|_{L^2(\mathbb{T},X)}$. However, $X = \mathcal{L}(\mathcal{H})$ is not a UMD space, unless $\mathcal{H}$ is finite dimensional. Nevertheless, we can use properties of $\tilde{B}$ to study the boundedness of operator valued paraproducts, using for example the identity

$$\Delta_B f = \int_\Sigma T_\sigma B T_\sigma f d\sigma. \tag{33}$$

This identity shows by an easy application of the Cauchy-Schwarz inequality that if $\int_\Sigma \|T_\sigma B\|^2_{L^2(\mathbb{T},\mathcal{L}(\mathcal{H}))} d\sigma < \infty$, then

$$\Delta_B : L^2(\mathbb{T},\mathcal{H}) \to L^1(\mathbb{T},\mathcal{H}) \text{ is a bounded operator.} \tag{34}$$

Whilst we do not know whether $\|B\|_{\mathrm{BMO_{para}}}$ can be estimated in terms of $\|B\|_{\mathrm{BMO_{mult}}}$, we will prove an estimate of $\|B\|_{\mathrm{BMO_{para}}}$ in terms of an average of $\|T_\sigma B\|_{\mathrm{BMO_{mult}}}$ over Σ.

Similarly, whilst we do not know whether $\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$ can be estimated in terms of $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$, we will prove an estimate of $\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$ in terms of an average of $\|T_\sigma B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$ over Σ.

For this, the following representation of the sweep will be useful:

$$S_B(t) = \int_\Sigma (T_\sigma B)^*(t)(T_\sigma B)(t)d\sigma. \tag{35}$$

Theorem 4.1. *Let $B \in \mathcal{F}_{00}$. Then*

$$\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} \lesssim (\int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma)^{1/2}.$$

Proof. This inequality follows from the estimate

$$\begin{aligned}
\|P_I S_B\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} &= \|P_I S_{P_I B}\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} \\
&\le 2\left\|\int_\Sigma (T_\sigma P_I B^*)(T_\sigma P_I B)d\sigma\right\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} \\
&\le 2\int_\Sigma \|(P_I T_\sigma B)^* P_I T_\sigma B\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} d\sigma \\
&= 2\int_\Sigma \|(P_I T_\sigma B)\|^2_{L^2(\mathbb{T},\mathcal{L}(\mathcal{H}))} d\sigma \\
&\le 2|I| \int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma.
\end{aligned}$$

Using John-Nirenberg's lemma for $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$, one concludes the result. □

We are going to describe the different operator-valued BMO spaces in terms of "average boundedness" of certain operators, namely Λ, π, or commutators with the martingale transforms.

First we see that the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}$-norm can be described by "average boundedness" of Λ_B.

Theorem 4.2. *Let $B \in \mathcal{F}_{00}$, and let Φ_B be the map*

$$\Phi_B : L^2(\mathbb{T},\mathcal{H}) \to L^2(\mathbb{T}\times\Sigma,\mathcal{H}), \quad f \mapsto \Lambda_B T_\sigma f.$$

Then

$$\|\Phi_B\| = \sup_{\|f\|_{L^2(\mathcal{H})}=1} (\int_\Sigma \|\Lambda_B(T_\sigma f)\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma)^{1/2} = \|B\|_{\mathrm{SBMO}^{\mathrm{d}}}.$$

In particular, $\|B\|_{\mathrm{BMO}_{\mathrm{so}}} = \|\Phi_B\| + \|\Phi_{B^*}\|$.

Proof. Since $\Lambda_B(T_\sigma f) = \sum_{I\in\mathcal{D}} P_I(B) f_I h_I \sigma_I$, we have

$$\begin{aligned}
\int_\Sigma\int_\mathbb{T} \|(\Phi_B f)(t,\sigma)\|^2 dt d\sigma &= \int_\Sigma\int_\mathbb{T} \|(\Lambda_B T_\sigma f)(t)\|^2 dt d\sigma \\
&= \sum_{I\in\mathcal{D}} \|P_I(B) f_I h_I\|^2_{L^2(\mathcal{H})} \\
&= \sum_{I\in\mathcal{D}} \frac{1}{|I|}\int_I \|(B(t) - m_I B)(\frac{f_I}{\|f_I\|})\|^2 \|f_I\|^2 dt \\
&\le \sup_{\|e\|=1} \|B_e\|^2_{BMO(\mathcal{H})} \sum_{J\in\mathcal{D}} \|f_J\|^2.
\end{aligned}$$

The reverse inequality follows by considering functions $f = h_I e$, where $e \in \mathcal{H}$, $I \in \mathcal{D}$. □

We require a further technical lemma, which shows that the L^2 norm of $\tilde{B}f$ can be decomposed in a certain way.

Lemma 4.3. *Let $B \in \mathcal{F}_{00}$ and $f \in L^2(\mathbb{T}, \mathcal{H})$. Write $Bf = \pi_B f + \Delta_B f + \gamma_B f$. Then*

$$\|\tilde{B}f\|^2_{L^2(\Sigma\times\mathbb{T},\mathcal{H})} = \int_\Sigma \|\pi_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma + \int_\Sigma \|\Delta_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma + \int_\Sigma \|\gamma_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma.$$

Proof. Observe that $m_I(T_\sigma B)h_I = (\sum_{I\subsetneq J} \sigma_J B_J h_J)h_I$. Hence

$$\gamma_{T_\sigma B}(f) = \sum_{I\in\mathcal{D}} m_I(T_\sigma B)(f_I)h_I = \sum_{J\in\mathcal{D}} \sigma_J B_J (\sum_{I\subsetneq J} f_I h_I)h_J.$$

This shows that

$$\int_\mathbb{T}\int_\Sigma \langle \pi_{T_\sigma B} f, \gamma_{T_\sigma B} g\rangle d\sigma dt = \sum_{I\in\mathcal{D}} \int_I \langle B_I m_I f, B_I(\sum_{J\subsetneq I} g_J h_J)\rangle \frac{\chi_I}{|I|} dt = 0$$

$$\int_\mathbb{T}\int_\Sigma \langle \gamma_{T_\sigma B} f, \Delta_{T_\sigma B} g\rangle d\sigma dt = \sum_{I\in\mathcal{D}} \int_I \langle B_I(\sum_{J\subsetneq I} f_J h_J), B_I g_I\rangle \frac{h_I}{|I|} dt = 0$$

$$\int_\mathbb{T}\int_\Sigma \langle \pi_{T_\sigma B} f, \Delta_{T_\sigma B} g\rangle d\sigma dt = \sum_{I\in\mathcal{D}} \int_I \langle B_I m_I f, B_I g_I\rangle \frac{h_I}{|I|} dt = 0.$$

To finish the proof, simply expand

$$\|\tilde{B}(f)\|^2_{L^2(\Sigma\times\mathbb{T},\mathcal{H})} = \int_\mathbb{T}\int_\Sigma \langle (T_\sigma B)f, (T_\sigma B)f\rangle d\sigma dt.$$

□

Theorem 4.4. *Let $B \in \mathcal{F}_{00}$. Let Ψ_B be the map*

$$\Psi_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}\times\Sigma, \mathcal{H}), \quad f \mapsto \Lambda_{T_\sigma B} f.$$

Then

$$\|\pi_B\| \le \|\Psi_B\| = \sup_{\|f\|=1} (\int_\Sigma \|\Lambda_{T_\sigma B}(f)\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma)^{1/2} \le (\|\pi_B\|^2 + \sup_{I\in\mathcal{D}} \frac{1}{|I|}\|B_I\|^2)^{1/2} \le \sqrt{2}\|\pi_B\|.$$

Proof. Using the orthogonality properties from Lemma 4.3, we obtain

$$\begin{aligned} \int_\Sigma \|\Lambda_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma &= \int_\Sigma \|\pi_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma + \int_\Sigma \|\Delta_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma \\ &= \|\pi_B(f)\|^2_{L^2(\mathcal{H})} + \int_\Sigma \|\Delta_B(T_\sigma f)\|^2_{L^2(\mathcal{H})} d\sigma \end{aligned}$$

Therefore

$$\|\pi_B\| \le \sup_{\|f\|=1} (\int_\Sigma \|\Lambda_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})})^{1/2}.$$

The second term is easily estimated by

$$\begin{aligned}
\int_\Sigma \|\Delta_B(T_\sigma f)\|^2_{L^2(\mathcal{H})} d\sigma &= \int_\Sigma \int_{\mathbb{T}} \| \sum_{I\in\mathcal{D}} \sigma_I B_I \frac{\chi_I(t)}{|I|} f_I\|^2 dt d\sigma \\
&= \sum_{I\in\mathcal{D}} \int_{\mathbb{T}} \|B_I \frac{\chi_I(t)}{|I|} f_I\|^2 dt \\
&= \sum_{I\in\mathcal{D}} \frac{1}{|I|}\|B_I f_I\|^2 \\
&\le (\sup_{I\in\mathcal{D}} \frac{1}{|I|}\|B_I\|^2)\|f\|^2_{L^2(\mathcal{H})} \le \|\pi_B\|^2\|f\|^2_{L^2(\mathcal{H})}.
\end{aligned}$$

by (24). □

Here is our desired estimate of $\|B\|_{\mathrm{BMO_{para}}} + \|B^*\|_{\mathrm{BMO_{para}}}$ in terms of an average over $\|\tilde{B}\|_{\mathrm{BMO_{mult}}}$.

Corollary 4.5. *Let $B \in \mathcal{F}_{00}$. Then*

$$\frac{1}{2}(\|\pi_B\| + \|\Delta_B\|) \le \|\tilde{B}\|_{L^2(\Sigma,\mathrm{BMO_{mult}})} \le \|\pi_B\| + \|\Delta_B\|.$$

Proof. To show the first estimate, it is sufficient to use Theorem 4.4, the identity $\|\Delta_B\| = \|\pi_{B^*}\|$ and the invariance of the right hand side under passing to the adjoint B^*.

For the reverse estimate, note that

$$\begin{aligned}
\int_\Sigma \|\tilde{B}\|^2_{\mathrm{BMO_{mult}}} d\sigma &\le \int_\Sigma (\|\Delta_{T_\sigma B}\| + \|\pi_{T_\sigma B}\|)^2 d\sigma \\
&= \int_\Sigma (\|\Delta_B T_\sigma\| + \|T_\sigma \pi_B\|)^2 d\sigma \\
&= \int_\Sigma (\|\Delta_B\| + \|\pi_B\|)^2 d\sigma = (\|\Delta_B\| + \|\pi_B\|)^2.
\end{aligned}$$

□

It was shown in [GPTV], Th. 3.5, that there exists $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}$ and $\sigma \in \{-1,1\}^{\mathcal{D}}$ such that $[T_\sigma, B]$ does not define a bounded operator on $L^2(\mathbb{T},\mathcal{H})$. We shall use averages of commutators to describe the spaces $\mathrm{BMO_{mult}}$ and $\mathrm{BMO_{para}}$.

Let us mention the following "commutator-type" characterization of $\mathrm{BMO_{mult}}$.

Proposition 4.6 ([GPTV], Cor 4.1)**.** *$B \in \mathrm{BMO_{mult}}$ if and only if the commutator $[T_\sigma, B]$ defines a bounded linear operator on $L^2(\mathbb{T},\mathcal{H})$ for each $\sigma \in \{-1,1\}^{\mathcal{D}}$, and $\|B\|_{\mathrm{BMO_{mult}}} \sim \sup_{\sigma\in\{-1,1\}^{\mathcal{D}}} \|[T_\sigma, B]\|$.*

We shall see that one can replace the "sup" condition by some average one. We formulate a general lemma from which this can be deduced.

Lemma 4.7. *Let $U \in \mathcal{L}(L^2(\mathbb{T},\mathcal{H}))$ such that*

$$\int (Ueh_I)(t)h_I(t)dt = 0 \quad (e \in \mathcal{H}, I \in \mathcal{D}). \tag{36}$$

If $f \in L^2(\mathcal{H})$ then

$$\int_\Sigma \|[T_\sigma, U](f)\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma = \|U(f)\|^2_{L^2(\mathbb{T},\mathcal{H})} + \sum_{I\in\mathcal{D}} \|U(f_I h_I)\|^2_{L^2(\mathbb{T},\mathcal{H})}. \tag{37}$$

In particular,

$$\|U\| \le \sup_{\|f\|=1} (\int_\Sigma \|[T_\sigma, U](f)\|^2 d\sigma)^{1/2} \le (\int_\Sigma \|[T_\sigma, U]\|^2 d\sigma)^{1/2} \le 2\|U\|. \tag{38}$$

Proof. Note that if we write $\Phi_I = h_I \otimes h_I \in \mathcal{L}(L^2(\mathbb{T}, \mathcal{H}))$, that is, $\Phi_I(f) = f_I h_I$, then

$$[T_\sigma, U](f) = \sum_{I\in\mathcal{D}} \sigma_I [\Phi_I, U](f).$$

Observe that (36) yields, for $I \in \mathcal{D}$,

$$\int_\mathbb{T} \langle \Phi_I(Uf)(t), U(\Phi_I f)(t)\rangle dt = \langle (Uf)_I, \int_\mathbb{T} U(f_I h_I)(t) h_I(t) dt\rangle = 0.$$

Hence

$$\begin{aligned}
\int_\Sigma \|[T_\sigma, U]f\|^2_{L^2(\mathcal{H})} d\sigma &= \sum_{I\in\mathcal{D}} \|[\Phi_I, U]f\|^2_{L^2(\mathcal{H})} \\
&= \sum_{I\in\mathcal{D}} \|\Phi_I(Uf) - U(\Phi_I f)\|^2_{L^2(\mathcal{H})} \\
&= \sum_{I\in\mathcal{D}} \|\Phi_I(Uf)\|^2_{L^2(\mathcal{H})} + \sum_{I\in\mathcal{D}} \|U(\Phi_I f)\|^2_{L^2(\mathcal{H})} \\
&= \|U(f)\|^2_{L^2(\mathcal{H})} + \sum_{I\in\mathcal{D}} \|U(f_I h_I)\|^2_{L^2(\mathcal{H})}.
\end{aligned}$$

Now (38) follows from the previous estimates and the fact

$$\|[T_\sigma, U]\| = \|T_\sigma U - U T_\sigma\| \le 2\|U\|.$$

□

We also obtain that average boundedness of the commutator $[\pi_B, T_\sigma]$ coincides with boundedness of π_B:

Corollary 4.8. *Let $B \in \mathcal{F}_{00}$.*

$$\|\pi_B\| \le \sup_{\|f\|=1} (\int_\Sigma \|[T_\sigma, \pi_B]f\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma)^{1/2} \le (\|\pi_B\|^2 + \|B\|^2_{\mathrm{SBMO^d}})^{1/2} \le C\|\pi_B\|. \tag{39}$$

Proof. Apply (37) in Lemma 4.7 with $U = \pi_B$ together with the fact

$$\sum_{I\in\mathcal{D}} \|\pi_B(f_I h_I)\|^2_{L^2(\mathcal{H})} = \sum_{I\in\mathcal{D}} \sum_{J\subsetneq I} \frac{\|B_J(f_I)\|^2}{|I|} \le \sup_{\|e\|=1} \|B_e\|^2_{BMO(\mathcal{H})} \sum_{I\in\mathcal{D}} \|f_I\|^2$$

□

We can also describe $\|B\|_{\mathrm{BMO_{mult}}}$ as an average condition of the commutator.

Corollary 4.9. *Let $B \in \mathcal{F}_{00}$ and $f \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ with $\int_\mathbb{T} f = 0$. Let us define*

$$[T_\sigma, B](f) = T_\sigma(Bf) - B(T_\sigma f).$$

Then

$$\int_\Sigma \|[T_\sigma, B]f\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma = \|\Lambda_B(f)\|^2_{L^2(\mathbb{T},\mathcal{H})} + \sum_{I\in\mathcal{D}} \|\Lambda_B(f_I h_I)\|^2_{L^2(\mathbb{T},\mathcal{H})}. \tag{40}$$

In particular,

$$\|B\|_{\mathrm{BMO_{mult}}} \approx \sup_{\|f\|=1} (\int_\Sigma \|[T_\sigma, B]f\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma)^{1/2} \approx (\int_\Sigma \|[T_\sigma, B]\|^2 d\sigma)^{1/2}. \tag{41}$$

Proof. It is elementary to see that

$$[T_\sigma, B] = [T_\sigma, \Lambda_B].$$

Now observe that $\langle \Delta_B(eh_I), h_I\rangle = \langle B_I(e)\frac{\chi_I}{|I|}, h_I\rangle = 0$ and that $\langle \pi_B(eh_I), h_I\rangle = 0$, and use Lemma 4.7 for $U = \Lambda_B$.

□

Theorem 4.10. *(see also* [GPTV]*) Let $B \in \mathcal{F}_{00}$. Then*

$$\sup_{f\in L^2(\mathbb{T},\mathcal{H}), \|f\|=1} (\int_\Sigma \int_\Sigma \|[T_\sigma, (T_\tau B)]f\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma d\tau)^{1/2} \approx \|\pi_B\| \tag{42}$$

and

$$(\int_\Sigma \int_\Sigma \|[T_\sigma, (T_\tau B)]\|^2_{\mathcal{L}(L^2(\mathbb{T},\mathcal{H}))} d\sigma d\tau)^{1/2} \approx \|\pi_B\| + \|\Delta_B\|. \tag{43}$$

Proof. To show (42), use Corollary 4.9 again to get for any $\tau \in \Sigma$,

$$\int_\Sigma \|[T_\sigma, (T_\tau B)]f\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma = \|\Lambda_{T_\tau B}(f)\|^2_{L^2(\mathbb{T},\mathcal{H})} + \sum_{I\in\mathcal{D}} \|\Lambda_{T_\tau B}(f_I h_I)\|^2_{L^2(\mathbb{T},\mathcal{H})}.$$

Now integrate over Σ and use Theorem 4.4.

For the estimate "$\gtrsim$" in (43), note (42) together with the invariance of the left hand side under passing to the adjoint function B^*. For the estimate "$\lesssim$", use that

$$\|[T_\sigma, (T_\tau B)]\| = \|[T_\sigma, \Lambda_{T_\tau B}]\| \le 2(\|\Delta_{T_\tau B}\| + \|\pi_{T_\tau B}\|) = 2(\|\Delta_B\| + \|\pi_B\|).$$

□

5. Sweeps of operator-valued functions.

In the final chapter, we investigate the action of the sweep on operator-valued BMO spaces. It turns out that the sweep can easily be extended to a sesquilinear map, which acts on cartesian products of BMO spaces. One way to express the John-Nirenberg inequality on scalar-valued $\mathrm{BMO^d}$ is to say that the mapping

$$\mathrm{BMO^d} \to \mathrm{BMO^d}, \quad b \mapsto S_b, \tag{44}$$

is bounded. In the operator-valued setting, this John-Nirenberg property breaks down. Our main result is that any space of operator-valued functions which is contained in $\mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ and on which the mapping (44) acts boundedly is already contained in $\mathrm{BMO_{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

However, we find that (44) acts boundedly between different operator-valued BMO spaces. We also obtain the precise rate of growth of the norm of the mapping (44) on $\mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ in terms of the dimension of $\mathcal{H}$.

Definition 5.1. *Let us denote by $\Delta : \mathcal{F}_{00} \times \mathcal{F}_{00} \to L^1(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ the bilinear map given by*

$$\Delta(B,F) = \sum_{I\in\mathcal{D}} B_I^* F_I \frac{\chi_I}{|I|}.$$

In particular $S_B = \Delta(B,B)$ and $\Delta(B,F)^* = \Delta(F,B)$.

Lemma 5.2. *Let $B \in \mathcal{F}_{00}$. Then*

$$P_I\Delta(B,F) = P_I\Delta(B,P_IF) = P_I \sum_{J\subseteq I} \frac{\chi_J}{|J|} B_J^* F_J = P_I \sum_{J\subsetneq I} \frac{\chi_J}{|J|} B_J^* F_J.$$

In particular, $P_I(S_B) = P_I(S_{P_I B}) = P_I(S_{(P_{I^+}+P_{I^-})B})$.

Proof. $P_I\Delta(B^*, (F_J h_J)) = P_I(B_J^* F_J \frac{\chi_J}{|J|}) = 0$ if $I \subseteq J$. Hence

$$P_I\Delta(B,F) = P_I\Delta(B,P_IF) = P_I\Delta(B,(P_{I^+}+P_{I^-})F).$$

□

A similar proof as in Lemma 3.4 shows that

Lemma 5.3. *Let $B, F \in \mathcal{F}_{00}$. Then*

$$\pi_B^*\pi_F = \pi_{\Delta(B,F)} + \pi_{\Delta(F,B)}^* + D_{B,F} = \Lambda_{\Delta(B,F)} + D_{B,F},$$

where $D_{B,F}$ is defined by $D_{B,F}(h_I \otimes x) = h_I \frac{1}{|I|} \sum_{J\subsetneq I} B_J^* F_J x$ for $x \in \mathcal{H}$, $I \in \mathcal{D}$. Moreover, $\|D_{B,F}\| \le \sup_{\|e\|=1} \|B_e\|_{BMO(\mathcal{H})} \sup_{\|e\|=1} \|F_e\|_{BMO(\mathcal{H})}$.

Let us now study the boundedness of the sesquilinear map Δ in various BMO norms.

Theorem 5.4. *There exists a constant $C > 0$ such that for $B, F \in \mathcal{F}_{00}$,*

1. $\|\Delta(B,F)\|_{\mathrm{BMO_{mult}}} \le C\|B\|_{\mathrm{BMO_{para}}}\|F\|_{\mathrm{BMO_{para}}}$
2. $\|\Delta(B,F)\|_{\mathrm{WBMO^d}} \le C\|B\|_{\mathrm{SBMO^d}}\|F\|_{\mathrm{SBMO^d}}$
3. $\|\Delta(B,F)\|_{\mathrm{SBMO^d}} \le C\|\pi_B\|\|F\|_{\mathrm{SBMO^d}}$
4. $\|\Delta(B,F)\|_{\mathrm{BMO^d_{norm}}} \le C\|B\|_{\mathrm{BMO^d_{Carl}}}\|F\|_{\mathrm{BMO^d_{Carl}}}$

Proof. (i) follows from Lemma 5.3.

(ii) Using Lemma 5.2, one obtains

$$\langle P_I\Delta(B,F)e, f\rangle = P_I \sum_{J\in\mathcal{D}} \langle (P_IF)_J e, (P_IB)_J f\rangle \frac{\chi_J}{|J|}$$

for $e, f \in \mathcal{H}$. Therefore,

$$\begin{aligned}
\|\langle P_I\Delta(B,F)e,f\rangle\|_{L^1} &= \|P_I \sum_{J\in\mathcal{D}} \langle (P_IF)_J e, (P_IB)_J f\rangle \frac{\chi_J}{|J|}\|_{L^1} \\
&\le 2\|\sum_{J\in\mathcal{D}} \langle (P_IF)_J e, (P_IB)_J f\rangle \frac{\chi_J}{|J|}\|_{L^1} \\
&\le 2\|(\sum_{J\in\mathcal{D}} \|(P_IB)_J f\|^2 \frac{\chi_J}{|J|})^{1/2}\|_{L^2}\|(\sum_{J\in\mathcal{D}} \|(P_IF)_J e\|^2 \frac{\chi_J}{|J|})^{1/2}\|_{L^2} \\
&\le 2(\sum_{J\in\mathcal{D}} \|(P_IB)_J f\|^2)^{1/2}(\sum_{J\in\mathcal{D}} \|(P_IF)_J e\|^2)^{1/2}.
\end{aligned}$$

Thus if $\|B\|_{\mathrm{BMO^d_{so}}} = \|F\|_{\mathrm{BMO^d_{so}}} = 1$, then

$$\|\langle P_I\Delta(B,F)e,f\rangle\|_{L^2} \le 2\|P_IB_f\|_{L^2(\mathcal{H})}\|P_IF_e\|_{L^2(\mathcal{H})} \le 2|I|.$$

This, again using John-Nirenberg's lemma, gives $\|\Delta(B,F)\|_{\mathrm{WBMO^d}(\mathcal{L}(\mathcal{H}))} \le C$.

(iii) From Lemma 5.2, we obtain

$$\|P_I\Delta(B,F)e\|_{L^2(\mathcal{H})} = \|\Delta_{B^*}(P_I F_e)\|_{L^2(\mathcal{H})} \le \|\pi_B\|\|P_I F_e\|_{L^2(\mathcal{H})}.$$

(iv) This follows from John-Nirenberg's lemma and

$$\begin{aligned}\|P_I\Delta(B,F)\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} &\le \|\sum_{J\subseteq I}\|B_J^*\|\|F_J\|\frac{\chi_J}{|J|}\|_{L^1}\\ &\le \sum_{J\subseteq I}\|B_J^*\|\|F_J\| \le C|I|\|B\|_{\mathrm{BMO^d_{Carl}}}\|F\|_{\mathrm{BMO^d_{Carl}}}.\end{aligned}$$

□

Here comes the main result of this section.

Theorem 5.5. *Let $\mathcal{H}$ be a separable, finite or infinite-dimensional Hilbert space. Let ρ be a positive homogeneous functional on the space $\mathcal{F}_{00}$ of $\mathcal{L}(\mathcal{H})$-valued functions on $\mathbb{T}$ with finite formal Haar expansion such that there exists constants c_1, c_2 with*

(1) $\|B\|_{\mathrm{BMO^d_{so}}} \le c_1\rho(B)$ and
(2) $\rho(S_B) \le c_2\rho(B)^2$ for all $B \in \mathcal{F}_{00}$.

Then there exists a constant C, depending only on c_1 and c_2, such that $\|B\|_{\mathrm{BMO_{para}}} \le C\rho(B)$ for all $B \in \mathcal{F}_{00}$.

Proof. For $n \in \mathbb{N}$, let E_n denote the subspace $\{f \in L^2(\mathbb{T},\mathcal{H}) : f_I = 0 \text{ for } |I| < 2^{-n}\}$ of $L^2(\mathbb{T},\mathcal{H})$. Let $c(n) = \sup\{\|\pi_B\|_{E_n} : \rho(B) \le 1\}$. An elementary estimate shows that $c(n)$ is well-defined and finite for each $n \in \mathbb{N}$. For $\varepsilon > 0$, $n \in \mathbb{N}$, we can find $f \in E_n$, $\|f\| = 1$, $B \in \mathcal{F}_{00}$, $\rho(B) \le 1$ such that

$$\begin{aligned}c(n)^2(1-\varepsilon)^2 \le \|\pi_B f\|^2 &= \langle \pi_{S_B}f, f\rangle + \langle f, \pi_{S_B}f\rangle + \langle D_B f, f\rangle\\ &\le 2c(n)\rho(S_B) + c_1\|B\|_{\mathrm{BMO^d_{so}}} \le 2c_2c(n) + c_1.\end{aligned}$$

It follows that the sequence $(c(n))_{n\in\mathbb{N}}$ is bounded by $C = c_2 + \sqrt{c_2^2 + c_1}$, and therefore $\|\pi_B\| \le C\rho(B)$ for all $B \in \mathcal{F}_{00}$. □

One immediate consequence is the following answer to Question 5.1 in [GPTV].

Theorem 5.6. *There exists an absolute constant $C > 0$ such that for each $n \in \mathbb{N}$ and each measurable function $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n\times n)$,*

$$\|S_B\|_{\mathrm{BMO^d_{so}}} \le C\log(n+1)\|B\|^2_{\mathrm{BMO^d_{so}}}, \tag{45}$$

and this is sharp.

Proof. From (iii) in Theorem 5.4 one obtains:

$$\|S_B\|_{\mathrm{BMO_{so}}} \le C\|B\|_{\mathrm{BMO_{para}}}\|B\|_{\mathrm{BMO^d_{so}}} \le C\log(n+1)\|B\|_{\mathrm{BMO^d_{so}}},$$

since there exists an absolute constant $C > 0$ with

$$\|B\|_{\mathrm{BMO_{para}}} \le C\log(n+1)\|B\|_{\mathrm{BMO^d_{so}}}$$

by [K] and [NTV]. On the other hand, denoting by C_n the smallest constant such that

$$\|S_B\|_{\mathrm{BMO^d_{so}}} \le C_n\|B\|^2_{\mathrm{BMO^d_{so}}},$$

for each integrable function $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$, we obtain from Theorem 5.5 that

$$\|B\|_{\mathrm{BMO_{para}}} \leq (C_n + \sqrt{C_n^2 + 1})\|B\|_{\mathrm{BMO^d_{so}}} \leq 3C_n\|B\|_{\mathrm{BMO^d_{so}}}$$

for each integrable B. It was shown in [NPiTV] that there exists an absolute constant $c > 0$ such that for each $n \in \mathbb{N}$, there exists $B^{(n)} : \mathbb{T} \to \mathrm{Mat}(n \times n, \mathbb{C})$ such that $\|B^{(n)}\|_{\mathrm{BMO_{para}}} \geq \log(n+1)c\|B^{(n)}\|_{\mathrm{BMO^d_{so}}}$. Therefore $C_n \geq \frac{c}{3}\log(n+1)$, and (45) is sharp. □

The following corollary gives an estimate of $\|\cdot\|_{\mathrm{BMO_{para}}}$ in terms of $\|\cdot\|_{\mathrm{SBMO^d}}$ with an "imposed" John-Nirenberg property. We need some notation: Let $S_B^{(0)} = B$ and let $S_B^{(n)} = S_{S^{(n-1)}B}$ for $n \in \mathbb{N}$, $B \in \mathcal{F}_{00}$.

Corollary 5.7. *There exists a constant $C > 0$ such that*

$$\|B\|_{\mathrm{BMO_{para}}} \leq C \sup_{n\geq 0} \|S_B^{(n)}\|_{\mathrm{SBMO^d}}^{1/2^n} \qquad (B \in \mathcal{F}_{00}).$$

Proof. Define $\rho(B) = \sup_{n\geq 0} \|S_B^{(n)}\|_{\mathrm{SBMO^d}}^{1/2^n}$. One sees easily that this expression is finite for $B \in \mathcal{F}_{00}$. Now apply 5.5. □

The space $\mathrm{SBMO^d}$ can be characterised by the test function condition $\sup_{e\in\mathcal{H},\|e\|=1,I\in\mathcal{D}} \|\pi_B e h_I\| < \infty$. Here is a test function characterization for $B, S_B \in \mathrm{SBMO^d}$.

Proposition 5.8.

$$\|S_B\|_{\mathrm{BMO^d_{so}}} + \|B\|^2_{\mathrm{SBMO^d}} \approx \sup_{e\in\mathcal{H},\|e\|=1,I\in\mathcal{D}} \|\pi_B^*\pi_B e h_I\| \qquad (B \in \mathcal{F}_{00}).$$

Proof. First notice that for $I \in \mathcal{D}$, $e \in \mathcal{H}$,

$$\frac{1}{|I|}\|P_I S_B e\|^2 = \frac{1}{|I|}\|P_I S_{(P_{I^+}+P_{I^-})B} e\|^2 \leq \frac{4}{|I|}\|S_{(P_{I^+}+P_{I^-})B} e\|^2 = 4\|\pi_B^*\pi_B h_I e\|^2$$

and

$$\frac{1}{|I|}\|P_I B e\|^2 = 2\|\pi_B h_{\tilde{I}} e\|^2 = 2\langle \pi_B^*\pi_B h_{\tilde{I}} e, h_{\tilde{I}} e\rangle,$$

where $\tilde{I}$ denotes the parent interval of I.

Conversely, note that

$$\begin{aligned}
\|\pi_B^*\pi_B e h_I\|^2 &= \frac{1}{|I|}\|S_{(P_{I^+}+P_{I^-})B} e\|^2 \\
&\leq \frac{1}{|I|}(\|P_I S_{(P_{I^+}+P_{I^-}B)} e\| + |I|^{1/2}\|m_I S_{(P_{I^+}+P_{I^-})B} e\|)^2 \\
&= \frac{1}{|I|}(\|P_I S_{(P_I B)} e\| + |I|^{-1/2}\|\sum_{J\subsetneq I} B_J^* B_J e\|)^2 \\
&\leq 2(\frac{1}{|I|}\|P_I S_B e\|^2 + \frac{1}{|I|^2}\|\sum_{J\subsetneq I} B_J^* B_J e\|^2) \\
&\leq 2(\|S_B\|^2_{\mathrm{BMO^d_{so}}} + \|B\|^4_{\mathrm{SBMO^d}})\|e\|^2.
\end{aligned}$$

□

This result can be used to characterise a type of L^4 average boundedness of π_B in terms of $\|B\|_{\mathrm{BMO^d_{so}}}$ and $\|S_B\|_{\mathrm{BMO^d_{so}}}$.

Theorem 5.9.

$$(\|S_B\|^2_{\mathrm{BMO_{so}}}+\|B\|^4_{\mathrm{BMO_{so}}})^{1/4} \approx (\sup_{f\in L^2(\mathbb{T},\mathcal{H}),\|f\|=1}\int_\Sigma \|\pi_B(T_\sigma f)\|^4 d\sigma)^{1/4} \qquad (B\in\mathcal{F}_{00}).$$

Proof. We obtain the estimate "$\lesssim$" from Proposition 5.8, setting $f = eh_I$, $e\in\mathcal{H}$, $\|e\| = 1$, $I\in\mathcal{D}$.

For the reverse estimate, use Lemma 3.4 to write

$$\|\pi_B(T_\sigma f)\|^2 = \langle \Lambda_{S_B}(T_\sigma f), T_\sigma f\rangle + \langle D_B(T_\sigma f), T_\sigma f\rangle$$

Hence

$$\|\pi_B(T_\sigma f)\|^2 \le \|\Lambda_{S_B}(T_\sigma f)\|\|f\| + \|D_B(T_\sigma f)\|\|f\|.$$

Now we can write

$$\begin{aligned} &\sup_{\|f\|=1}\int_\Sigma \|\pi_B(T_\sigma f)\|^4 d\sigma \\ \le\ & C\left(\sup_{\|f\|=1}\int_\Sigma \|\Lambda_{S_B}(T_\sigma f)\|^2 d\sigma + \sup_{\|f\|=1}\int_\Sigma \|D_B(T_\sigma f)\|^2 d\sigma\right) \\ \le\ & C(\|S_B\|^2_{\mathrm{BMO_{so}}} + \|B\|^4_{\mathrm{BMO_{so}}}). \end{aligned}$$

□

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