REMARKS ON OPERATOR BMO SPACES

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Abstract. Several spaces defined according to different formulations for operator-valued functions in BMO are defined and studied.

1. Introduction and notation

Recall that a function f is said to belong to $BMO(\mathbb{T})$ if

$$\sup_{I\subseteq\mathbb{T} \text{ interval}} \Big(\frac{1}{|I|}\int_I |f(t)-m_I f|^2 dt\Big)^{1/2} < \infty, \tag{1}$$

where I is an interval in $\mathbb{T}$ and $m_I(f)$ stands for the averarage $m_I f = \frac{1}{|I|}\int_I f(t)dt$.

It is well known that there are many other equivalent characterizations of BMO functions:

First we can replace averaging over intervals by averaging respect to the Poisson kernel (see [Ga]) , that is $f \in BMO(\mathbb{T})$ if and only if

$$\sup_{|z|<1} \Big(\int_{\mathbb{T}} |f(t)-P(f)(z)|^2 P_z(t)dt\Big)^{1/2} < \infty \tag{2}$$

where $P_z(t) = \frac{1-|z|}{1-\bar{z}t}$, $t \in \mathbb{T}$ and $P(f)$ stands for the Poisson integral of f.

Actually this is also equivalent to

$$\sup_{|z|<1} P(|f|^2)(z) - |P(f)(z)|^2 < \infty \tag{3}$$

Recall also that, due to John-Nirenberg's lemma, one can replace in (1) and (2) the L^2-norm by the L^p-norm for $0 < p < \infty$.

Another possibility is to describe functions in BMO by the Carleson condition: $f \in BMO(\mathbb{T})$ if and only if $|\nabla(f)(z)|^2(1-|z|^2)$ is a Carleson measure on $\mathbb{D}$, equivalently

$$\sup_{|z|<1} \int_{\mathbb{D}} (1-|w|^2)|\nabla f(w)|^2 P_z(w)dA(w) < \infty \tag{4}$$

where $\nabla(f)$ stands for the gradient of f and dA for the Lebesgue measure in the disc $\mathbb{D}$ (see [Ga, Z]).

Of course, one of the first and main descriptions of BMO is as the dual space of $ReH^1(\mathbb{T})$, where $ReH^1(\mathbb{T})$ stands for the space of functions f in $L^1(\mathbb{T})$ such that the Hilbert transform Hf belong to $L^1(\mathbb{T})$, endowed with the norm $\|f\|_{H^1} = \|f\|_1 + \|Hf\|_1$.

There are other descriptions of functions in $ReH^1(\mathbb{T})$ either in terms of maximal functions, as those functions $f \in L^1(\mathbb{T})$ such that $P^*f \in L^1(\mathbb{T})$, where $P^*f(t) =$

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$\sup_{0<r<1} P_r * f(t)$ is the radial Poisson maximal function, or in terms of atomic decompositions, as those functions f in $L^1(\mathbb{T})$ that can be decomposed as $f = \sum_{k\in\mathbb{N}} \lambda_k a_k, \lambda_k \in \mathbb{C}$, where a_k are atoms, and $\sum_{k\in\mathbb{N}} |\lambda_k| < \infty$.

Different proofs of the duality result (see [FS]) $BMO(\mathbb{T}) = (ReH^1(\mathbb{T}))^*$, can be done using those formulations of BMO and ReH^1. The reader is referred to [GR] for the general theory on Hardy spaces using real-variable techniques.

There is a counterpart of Hardy spaces defined in terms of martingales (see [G]) and a particular and simpler case concerning dyadic martingales (see [Per]) which we will discuss here.

Let $\mathcal{D}$ denote the collection of dyadic subintervals of the unit circle $\mathbb{T}$, and let $(h_I)_{I\in\mathcal{D}}$, where $h_I = \frac{1}{|I|^{1/2}}(\chi_{I^+} - \chi_{I^-})$, be the Haar basis of $L^2(\mathbb{T})$. If $f \in L^1(\mathbb{T})$ and $I \in \mathcal{D}$ then f_I denote the formal Haar coefficients $\int_I f(t)h_I dt$, and, as above, $m_I f = \frac{1}{|I|}\int_I f(t)dt$ denotes the average of f over I.

We say that $f \in \mathrm{BMO^d}(\mathbb{T})$, if

$$\|f\|_{\mathrm{BMO^d}} = \sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|}\int_I |f(t) - m_I f|^2 dt\Big)^{1/2} < \infty. \tag{5}$$

Denote $P_I(f) = \sum_{J\subseteq I} h_J f_J$ and, using that $(f - m_I f)\chi_I = P_I(f)$ one has $f \in \mathrm{BMO^d}(\mathbb{T})$ if and only if there exists a constant $C > 0$ such that, for all $I \in \mathcal{D}$,

$$\|P_I(f)\|_2 \leq C|I|^{1/2}. \tag{6}$$

On the other hand, since $\|P_I(f)\|_{L^2} = (\sum_{J\in\mathcal{D}, J\subseteq I} |f_J|^2)^{1/2}$ for $f \in L^2(\mathbb{T})$. Hence $f \in \mathrm{BMO^d}(\mathbb{T})$ if and only if

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}, J\subseteq I} |f_J|^2 < \infty. \tag{7}$$

We shall use later on the following characterization of $\mathrm{BMO^d}$ in terms the boundedness of the paraproducts. It is well known that $f \in \mathrm{BMO^d}(\mathbb{T})$ if and only if

$$\|\pi_g(f)\|_2 \leq C\|f\|_2 \tag{8}$$

and $\|\pi_g\| = \|g\|_{\mathrm{BMO^d_{norm}}(\mathbb{T})}$, where $\pi_g(f) = \sum_{I\in\mathcal{D}} g_I (m_I f) h_I$ (see [Per] for a survey on dyadic Harmonic Analysis and paraproducts).

Throughout the paper we shall review some of the results on the vector-valued versions of the previously defined characterizations of BMO and prove some new ones on operator dyadic BMO.

The paper is divided into three sections. The first one contains a survey of some results proved by the author about several vector valued versions of BMO.

The second section is devoted to operator-valued dyadic BMO spaces. We concentrate on the connections with operator-valued Carleson measures and paraproducts. Several spaces associated to different formulations of the previous notions are introduced and studied. Inclusions between them are analyzed.

The third section contains new material. Some natural generalizations for functions taking values in $\mathcal{L}(\mathcal{H})$ where one replaces the action $(T, h) \to Th$ in $\mathcal{L}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ for a more general one $\mathcal{L}(\mathcal{H}) \times \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is a subspace of linear operators in $\mathcal{L}(\mathcal{H})$ are introduced. This leads to definitons of new spaces whose properties and relations are studied.

2. Vector-valued BMO

Let X be a Banach space and let $f : \mathbb{T} \to X$ be a Bochner integrable function. We say that f belongs to $BMO(\mathbb{T}, X)$ (respect. $BMO_{weak}(\mathbb{T}, X)$) if

$$\sup_{I \subseteq \mathbb{T} \text{ interval}} \left(\frac{1}{|I|} \int_I \|f(t) - m_I f\|^2 dt\right)^{1/2} < \infty, \tag{9}$$

(respect.

$$\sup_{\|x^*\|=1, I \text{ interval}} \left(\frac{1}{|I|} \int_I |\langle f(t) - m_I f, x^*\rangle|^2 dt\right)^{1/2} < \infty,) \tag{10}$$

where $x^* \in X^*$ and $m_I(f)$ stands for the averarage $m_I f = \frac{1}{|I|} \int_I f(t) dt$.

Same proof as in the scalar-valued case allows to get $f \in BMO(\mathbb{T}, X)$ if and only if

$$\sup_{|z|<1} \left(\int_{\mathbb{T}} \|f(t) - P(f)(z)\|^2 P_z(t) dt\right)^{1/2} < \infty \tag{11}$$

where $P_z(t) = \frac{1-|z|}{1-\bar{z}t}$, $t \in \mathbb{T}$ and $P(f)$ stands for the Poisson integral of f. Making use of the John-Nirenberg's lemma, which holds true in the vector-valued case, one can also replace the L^2-norm by the L^p-norm in (9), (10) and (11).

We say that $f \in BMO_{\mathcal{P}}(\mathbb{T}, X)$ (see [BPa]) if

$$\sup_{|z|<1} P(\|f\|^2)(z) - \|P(f)(z)\|^2 < \infty.$$

We say that $f \in BMO_{\mathcal{C}}(\mathbb{T}, X)$ (see [B4]) if

$$\sup_{|z|<1} \int_{\mathbb{D}} (1 - |w|^2) \|\nabla f(w)\|^2 P_z(w) dA(w) < \infty$$

where $\nabla(f)$ stands for the gradient of f and dA for the Lebesgue measure in the disc $\mathbb{D}$.

It is known that embeddings between the just defined spaces depend upon some geometrical properties of the underlying Banach space. For instance, $BMO(\mathbb{T}, X) \subset BMO_{\mathcal{C}}(\mathbb{T}, X)$ implies X has cotype 2 and $BMO_{\mathcal{C}}(\mathbb{T}, X) \subset BMO(\mathbb{T}, X)$ implies X has type 2 (see [B5], Theorem 1.2).

On the other hand, if X is a 2-uniformly PL-convex space then $BMOA_{\mathcal{P}}(\mathbb{T}, X) \subset BMOA_{\mathcal{C}}(\mathbb{T}, X)$, where $BMOA_{\mathcal{P}}(\mathbb{T}, X)$ and $BMOA_{\mathcal{C}}(\mathbb{T}, X)$ stand for the analytic version of the spaces (see [BPa], Theorem 3.2).

The reader is referred to [W] for the notions on Geometry of Banach spaces and related questions to be used throughout the paper.

The duality in the vector-valued setting is also very well understood. One can define certain vector-valued Hardy spaces (see [B1, B2] and [Bou]) which will give the preduals of different versions of vector-valued BMO spaces.

Given a Banach space X we write $H^1_{\mathrm{at}}(\mathbb{T}, X)$ for the space of functions $F \in L^1(\mathbb{T}, X)$ such that $F = \sum_{k \in \mathbb{N}} \lambda_k a_k, \lambda_k \in \mathbb{C}$, where $\sum_{k \in \mathbb{N}} |\lambda_k| < \infty$ and a_k are X-valued atoms, that is to say $a_k \in L^\infty(\mathbb{T}, X)$, $supp(a_k) \subset I_k$ for some interval I_k, $\|a_k\|_\infty \leq \frac{1}{|I_k|}$ and $\int_{I_k} a_k(t) dt = 0$. We endow the space with the norm given by the infimum of $\sum_{k \in \mathbb{N}} |\lambda_k|$ over all possible decompositions.

We write $H^1_{con}(\mathbb{T}, X)$ for the space of functions $f \in L^1(\mathbb{T}, X)$ such that $Hf \in L^1(\mathbb{T}, X)$, with the norm given by $\|f\|_{con} = \|f\|_{L^1(\mathbb{T}, X)} + \|Hf\|_{L^1(\mathbb{T}, X)}$.

Proposition 2.1. *(see* [B3]*) If X is a real Banach space then $BMO_{weak}(\mathbb{T}, X)$ isometrically embedds into $\mathcal{L}(ReH^1(\mathbb{T}), X)$.*

Remark 2.2. *The reader is also referred to* [B3] *for the definition of the space of vector-valued measures of bounded mean oscillation which characterizes $\mathcal{L}(ReH^1(\mathbb{T}), X)$.*

Proposition 2.3. *(see* [RRT] *or* [B1]*, Th.3.1 and Prop. 3.3) If X is a real Banach space then* $\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, X^*)$ *isometrically embedds into $(H^1_{\mathrm{at}}(\mathbb{T}, X))^*$.*

Moreover $\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, X^*) = (H^1_{\mathrm{at}}(\mathbb{T}, X))^*$ *if and only if X^* has the RNP.*

Remark 2.4. *The reader is referred to* [B1, B4] *for the definition of the space of vector-valued measures of bounded mean oscillation which leads to the duality result without conditions on the Banach space X.*

Let $\Sigma = \{-1, 1\}^{\mathbb{D}}$, equipped with the natural product measure which assigns measure 2^{-n} to cylinder sets of length n. For each $\sigma \in \{-1, 1\}^{\mathbb{D}}$, define the dyadic martingale transform $T_\sigma : L^2(\mathbb{T}, X) \to L^2(\mathbb{T}, X)$, given by

$$f = \sum_{I\in\mathbb{D}} h_I f_I \mapsto \sum_{I\in\mathbb{D}} h_I \sigma_I f_I.$$

In the case that X is a Hilbert space, $\|T_\sigma F\|_{L^2(\mathbb{T},X)} = \|F\|_{L^2(\mathbb{T},X)}$ for any $(\sigma_I) \in \Sigma$ and then $\|\tilde{F}\|_{L^\infty(\Sigma, L^2(\mathbb{T},X))} = \|F\|_{L^2(\mathbb{T},X)}$.

Given $F \in L^1(\mathbb{T}, X)$ we write $\tilde{F}$ the function defined in $\Sigma \times \mathbb{T}$,

$$\tilde{F}(\sigma, t) = T_\sigma F(t) = \sum_I \sigma_I F_I h_I.$$

Recall that X is said to be UMD space if there exists $C > 0$ such that

$$\sup_{\sigma\in\Sigma} \|T_\sigma F\|_2 \le C\|F\|_2$$

for all $F \in L^2(\mathbb{T}, X)$.

In particular, for UMD spaces we have that $\|T_\sigma F\|_{L^2(\mathbb{T},X)} \approx \|F\|_{L^2(\mathbb{T},X)}$ and $\|\tilde{F}\|_{L^2(\Sigma, L^2(\mathbb{T},X))} \le \|F\|_{L^2(\mathbb{T},X)}$.

It is known that $L^p(\mu)$ spaces for $1 < p < \infty$ are UMD. Also the Schatten classes S_p are UMD spaces for $1 < p < \infty$ (see [BGM]) while $\mathcal{L}(\mathcal{H})$ or S_1 are never UMD spaces (unless $\mathcal{H}$ is finite dimesional). The reader is referred to [Bur] for a general survey on the UMD property.

We simply mention here that the UMD property is equivalent to the boundedness of the Hilbert transform on $L^2(\mathbb{T}, X)$ and the following connection with vector-valued BMO and duality.

Proposition 2.5. *(see* [B1]*)* $\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, X^*) = (H_{\mathrm{con}}(\mathbb{T}, X))^*$ *if and only if X is a UMD space.*

In the case $X = \mathcal{L}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space we shall use the notation $\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ for $BMO(\mathbb{T}, \mathcal{L}(\mathcal{H}))$, that is $B \in \mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if

$$\sup_{I\subseteq\mathbb{T} \text{ interval}} \Big(\frac{1}{|I|}\int_I \|B(t) - m_I B\|^2 dt\Big)^{1/2} < \infty. \tag{12}$$

In this situation we can still consider two related notions. One by considering the $weak^*$-topology, using $\mathcal{L}(\mathcal{H}) = (\mathcal{H}\hat{\otimes}\mathcal{H})^*$. Let $B :\ T \to \mathcal{L}(\mathcal{H})$ be such that

$\langle B(t)e, h\rangle \in L^2(\mathbb{T})$ for all $e, h \in \mathcal{H}$. We say that $B \in WBMO(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if

$$\sup_{\|e\|=\|h\|=1, I\subseteq \mathbb{T} \text{ interval}} \Big(\frac{1}{|I|}\int_I |\langle B(t)e - m_I Be, h\rangle|^2 dt\Big)^{1/2} < \infty. \tag{13}$$

Of course $BMO_{weak}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq WBMO(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Note that B belongs to $\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ or $WBMO(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only B^* does.

Another possiblity is the following (see [NTV]): Let $B : \mathbb{T} \to \mathcal{L}(\mathcal{H})$ be a function such that $B(t)e, B^*(t)e \in L^2(\mathbb{T}, \mathcal{H})$ for all $e \in \mathcal{H}$. We say that $B \in \mathrm{BMO}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if

$$\sup_{I\subseteq \mathbb{T}, I \text{ interval }, e\in\mathcal{H}, \|e\|=1} \Big(\frac{1}{|I|}\int_I \|(B(t) - m_I B)e\|^2 dt\Big)^{1/2} < \infty \tag{14}$$

and

$$\sup_{I\subseteq \mathbb{T}, I \text{ interval }, e\in\mathcal{H}, \|e\|=1} \Big(\frac{1}{|I|}\int_I \|(B^*(t) - m_I B^*)e\|^2 dt\Big)^{1/2} < \infty. \tag{15}$$

It is not difficult to show the following chain of strict inclusions.

$$\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq WBMO(\mathbb{T}, \mathcal{L}(\mathcal{H})).$$

Since the trace class operators can be described as $S_1 = \ell_2 \hat{\otimes} \ell_2$, where $X\hat{\otimes}Y$ stands for the completion of the projective tensor product of the spaces X and Y then $(S_1)^* = \mathcal{L}(\mathcal{H})$.

Hence it follows from Propositions 2.1 and 2.3 that

$$BMO_{weak}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \mathcal{L}(H^1_{at}(\mathbb{T}), S_1) \tag{16}$$

and

$$\mathrm{BMO}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq (H^1_{\mathrm{at}}(\mathbb{T}, S_1))^* \tag{17}$$

Proposition 2.6. $\mathrm{BMO}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \Big(\mathcal{H}\hat{\otimes}\big(H^1_{con}(\mathcal{H}) \oplus_1 H^1_{con}(\mathcal{H})\big)\Big)^*$.

Proof. Using that $(X\hat{\otimes}Y)^* = \mathcal{L}(X, Y^*)$ and Proposition 2.5, it suffices to see that $\mathrm{BMO}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \mathcal{L}(\mathcal{H}, BMO(\mathcal{H}) \oplus_\infty BMO(\mathcal{H}))$. Observe now that $B \in \mathrm{BMO}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ implies that $e \to (B(t)e, B^*(t)e)$ defines a bounded linear operator from $\mathcal{H}$ into $BMO(\mathcal{H}) \oplus_\infty BMO(\mathcal{H})$. □

3. Dyadic versions of operator-valued BMO.

Let $\mathcal{H}$ be a separable finite or infinite-dimensional Hilbert space, and let $B : \mathbb{T} \to \mathcal{L}(\mathcal{H})$ such that $\langle B(\cdot)e, h\rangle \in L^1(\mathbb{T})$ for any $e, h \in \mathcal{H}$. From the closed graph theorem $\mathcal{H}\times\mathcal{H} \to L^1(\mathbb{T})$ given by $(e, h) \to \langle B(\cdot)e, h\rangle$ defines a bounded bilinear map. Hence, for $I \in \mathcal{D}$, we can define the Haar coefficients $B_I = \int_I B(t)h_I(t)dt \in \mathcal{L}(\mathcal{H})$, and the average of B over I, $m_I B = \frac{1}{|I|}\int_I B(t)dt \in \mathcal{L}(\mathcal{H})$ as the operators given by $\langle B_I e, f\rangle = \int_I \langle B(t)e, f\rangle h_I(t)dt$ and $\langle m_I Be, f\rangle = \frac{1}{|I|}\int_I \langle B(t)e, f\rangle dt$.

We can now give similar notions to those introduced in Section 2, but only for dyadic intervals. Thus, we write $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} = \sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|}\int_I \|B(t) - m_I B\|^2 dt\Big)^{1/2} < \infty. \tag{18}$$

$B \in \mathrm{WBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if

$$(19) \quad \|B\|_{\mathrm{WBMO^d}} = \sup_{I\in\mathcal{D}, e,f\in\mathcal{H}, \|e\|=\|f\|=1} \left(\frac{1}{|I|}\int_I |\langle (B(t)-m_I B)e, f\rangle|^2 dt\right)^{1/2} < \infty.$$

$B \in \mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if

$$\gamma(B) = \sup_{I\in\mathcal{D}, \|e\|=} \frac{1}{|I|}\int_I \|(B(t)-m_I B)e\|^2 dt < \infty,$$

$$\gamma(B^*) = \sup_{I\in\mathcal{D}, \|h\|=1} \frac{1}{|I|}\int_I \|(B^*(t)-m_I B^*)h\|^2 dt < \infty.$$

We write

$$(20) \qquad \|B\|_{\mathrm{BMO^d_{so}}} = \gamma(B)^{1/2} + \gamma(B^*)^{1/2}$$

As in the introduction we have $P_I(B) = \sum_{J\subseteq I} h_J B_J = (B - m_I B)\chi_I$. Hence $B \in \mathrm{BMO^d_{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if there exists a constant $C > 0$ such that

$$(21) \qquad \|P_I(B)\|_{L^2(\mathcal{L}(\mathcal{H}))} \leq C|I|^{1/2}$$

for all $I \in \mathcal{D}$.

$B \in \mathrm{WBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if there exists a constant $C > 0$ such that

$$(22) \qquad \|\langle P_I(B)e, h\rangle\|_{L^2} \leq C|I|^{1/2}\|e\|\|h\|$$

for all $I \in \mathcal{D}$ and $e, h \in \mathcal{H}$.

$B \in \mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if there exists a constant $C > 0$ such that

$$(23) \qquad \max_{\|e\|=1, \|h\|=1} \{\|P_I B(e)\|_{L^2(\mathcal{H}))}, \|P_I B^*(h)\|_{L^2(\mathcal{H}))}\} \leq C|I|^{1/2}$$

for all $I \in \mathcal{D}$ and $e \in \mathcal{H}$.

As before one can replace in (18), (19), (20), (21), (22) and (23) the L^2-norm by any other L^p-norm for $0 < p < \infty$.

As in the scalar valued case we have

$$\|P_I(f)\|_{L^2} = \left(\sum_{J\in\mathcal{D}, J\subseteq I} \|f_J\|^2\right)^{1/2}$$

for $f \in L^2(\mathbb{T}, X)$ if X is a Hilbert space, but not in the case $X = \mathcal{L}(\mathcal{H})$. This leads us to consider the following space in terms of Carleson measures.

$B \in \mathrm{BMO^d_{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if

$$(24) \qquad \sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}, J\subseteq I} \|B_J\|^2 < \infty.$$

In the papers [GPTV, NTV, NPiTV] the study of the boundedness of the following version of the operator-valued paraproducts was iniciated and developped: The densely defined linear maps

$$\pi_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I(m_I f) h_I,$$

which is called the vector paraproduct with symbol B, and

$$\Lambda_B = \pi_B + \pi^*_{B^*} : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \qquad f \mapsto \sum_{I\in\mathcal{D}} B_I(m_I f) h_I + \sum_{I\in\mathcal{D}} B_I(f_I)\frac{\chi_I}{|I|}.$$

Recall that a sequence $\Phi_I \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ for all $I \in \mathcal{D}$ is said to be an operator-valued Haar multiplier (see [Per, BPo3]) if there exists $C > 0$ such that

$$\|\sum_{I\in\mathcal{D}} \Phi_I(f_I)h_I\|_{L^2(\mathbb{T},\mathcal{H})} \leq C(\sum_{I\in\mathcal{D}} \|f_I\|^2)^{1/2}$$

for any finite family of elements $(f_I) \subset \mathcal{H}$.

In the papers [NTV, BPo3] the spaces $\mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ and $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ were introduced.

A function B is said to belong to $\mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if π_B defines bounded linear operators on $L^2(\mathbb{T}, \mathcal{H})$.

A function B is said to belong to $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if the sequence $(P_I(B))_{I\in\mathcal{D}}$ defines a Haar multiplier.

Due to the equality

$$\Lambda_B(f) = \sum_{I\in\mathcal{D}} P_I(B)(f_I)h_I. \tag{25}$$

one has that $B \in \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if Λ_B is bounded on $L^2(\mathbb{T}, \mathcal{H})$.

It was shown that (see [NTV, BPo3])

$$\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{WBMO}^{\mathrm{d}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$$

and

$$\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H})).$$

The space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ can be understood as the space of functions satisfying a natural operator Carleson condition, namely

$$\max\{\sup_{I\in\mathcal{D}} \|\frac{1}{|I|}\sum_{J\subseteq I} B_J^* B_J\|, \sup_{I\in\mathcal{D}} \|\frac{1}{|I|}\sum_{J\subseteq I} B_J B_J^*\|\} < \infty \tag{26}$$

The result in [NTV] therefore represents a breakdown of the Carleson embedding theorem in the operator case.

It was shown in [BPo3] that the stronger condition

$$\sup_{I\in\mathcal{D}} \|\frac{1}{|I|}\sum_{J\subseteq I} B_J^* B_J \frac{\chi_J}{|J|}\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H})} < \infty \tag{27}$$

implies that $B \in \mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

The results on [BPo3] heavily depends upon the use of the notion of sweep of a function defined by $S_B = \sum_{J\subseteq\mathcal{D}} B_J^* B_J \frac{\chi_J}{|J|}$. In particular it was discovered that $B \in \mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if $S_B \in \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$

Another important difference for operator-valued functions is the no validity of John-Nirenberg theorems, meaning that $\|S_b\|_{\mathrm{BMO}} \leq C\|b\|^2_{\mathrm{BMO}}$, in the operator-valued case. Several replacements of the previous inequality were obtained in [BPo3].

Let us mention to finish this section two new lines of research related to operator-valued paraproducts which are now in progress. One is about connections between Hankel operators and Schatten classes that have been recently considered in [PSm] and another one about paraproducts and Haar multipliers on the bidisc. This last one is connected with operator-valued theory by taking $\mathcal{H} = L^2(\mathbb{T})$. In this case $B \in L^2(\mathbb{T}, L^2(\mathbb{T}))$ can be understood as a function in two variables, say $b(t, s)$, $(B_I)_J = b_{I\times J}$ where $b_{I\times J} = \int_{\mathbb{T}^2} b(t, s)h_I(t)h_J(s)dtds$ and $m_J(m_I B) = m_{I\times J} b$

where $m_{I\times J}b = \frac{1}{|I||J|}\int_{I\times J} b(t,s)(s)dtds$. The reader is referred to [BPo1, BPo2] for results about paraproducts and Haar multipliers on the bidisc.

4. Dyadic $\mathcal{A}$-valued BMO spaces

Throughout this section $\mathcal{A}$ denotes an *operator ideal*, that is $\mathcal{A}$ a Banach space such that there exist two continuous embeddings maps $B_1 : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $B_2 : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ in such a way that the composition maps $\mathcal{L}(\mathcal{H}) \times B_1(\mathcal{A}) \to B_1(\mathcal{A})$ and $B_2(\mathcal{A}) \times \mathcal{L}(\mathcal{H}) \to B_2(\mathcal{A})$ are bounded, i.e. $u \in \mathcal{A}, v \in \mathcal{L}(\mathcal{H}) \Longrightarrow vB_1(u) \in B_1(\mathcal{A}), B_2(u)v \in B_2(\mathcal{A})$ and $\max\{\|vB_1(u)\|_{\mathcal{A}}, \|B_2(u)v\|_{\mathcal{A}}\} \leq C\|v\|\|u\|_{\mathcal{A}}$, where we write $\|u_i\|_{\mathcal{A}}$ also the norm of the corresponding $u \in \mathcal{A}$ where $u_i = B_i(u) \in B_i(\mathcal{A})$ for $i = 1, 2$. We shall use simply vu and uv for $vB_1(u)$ and $B_2(u)v$ in the sequel.

We use the notation $e \otimes h$ for the operator $e \otimes h(x) = \langle h, x\rangle e$ for $x, y, e \in \mathcal{H}$. Clearly one has

$$T(e \otimes h) = Te \otimes h, \ (e \otimes h)T = e \otimes T^*h \tag{28}$$

$$(e \otimes h)(e' \otimes h') = \langle h, e'\rangle(e \otimes h') \tag{29}$$

$$(e \otimes h)^* = h \otimes e \tag{30}$$

Proposition 4.1. *Let $e_0 \in \mathcal{H}$ with $\|e_0\| = 1$. Then $\mathcal{H}$ is an operator ideal by selecting $B_1 : \mathcal{H} \to \mathcal{L}(\mathcal{H})$ given by $h \to h \otimes e_0$ and $B_2 : \mathcal{H} \to \mathcal{L}(\mathcal{H})$ given by $h \to e_0 \otimes h$.*

Proof. Observe that $\|B_i(h)\| = \|h\|$ for $i = 1, 2$ and $T(h \otimes e_0) = (Th) \otimes e_0$ and $(e_0 \otimes h)T = e_0 \otimes T^*(h)$. This corresponds to the actions $\mathcal{L}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ given by $(T, h) \to Th$ and $\mathcal{H} \times \mathcal{L}(\mathcal{H}) \to \mathcal{H}$ given by $(h, T) \to T^*h$. The properties are now straighforward. □

Remark 4.2. *$\mathcal{A} = \mathcal{L}(\mathcal{H})$ and the Schatten classes $\mathcal{A} = S_p$, $1 \leq p < \infty$ are operator ideals for $B_1 = B_2$ the inclusion maps.*

Recall that $S_1 = \mathcal{H}\hat{\otimes}\mathcal{H}$, $(S_p)^* = S_{p'}$, $1/p + 1/p' = 1$ and that $(S_1)^* = \mathcal{L}(\mathcal{H})$, with the duality given by

$$(u, e \otimes h) = \langle u(h), e\rangle,$$

where $u \in \mathcal{L}(\mathcal{H})$, and $e, h \in \mathcal{H}$.

As in the previous sections we write $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{A})$ if

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{A})} = \sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I \|B(t) - m_I B\|^2_{\mathcal{A}} dt\Big)^{1/2} < \infty. \tag{31}$$

and $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T}, \mathcal{A})$ if

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}, J\subseteq I} \|B_J\|^2_{\mathcal{A}} < \infty. \tag{32}$$

It was shown in [BPo3] that $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ was not contained into $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Let us see that if we replace $\mathcal{L}(\mathcal{H})$ by $\mathcal{A}$ the situation becomes different.

As usual r_k denote the Rademacher functions. Recall that a Banach space X is said to have type p, for some $1 < p \leq 2$, if there exists a constant $C > 0$ such that

$$\|\sum_{k=1}^N x_k r_k\|_{L^2(\mathbb{T},X)} \leq C\Big(\sum_{k=1}^N \|x_k\|^p\Big)^{1/p}$$

for all $x_1, ..., x_n \in X$. Similarly, a Banach space X is said to have cotype q, for some $2 \leq q < \infty$, if there exists a constant $C > 0$ such that

$$(\sum_{k=1}^{N} \|x_k\|^p)^{1/p} \leq C \| \sum_{k=1}^{N} x_k r_k \|_{L^2(\mathbb{T},X)}$$

for all $x_1, ..., x_n \in X$.

Proposition 4.3. *(i) If* $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},\mathcal{A}) \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{A})$ *then* $\mathcal{A}$ *has type 2.*
In particular, $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},S_p) \nsubseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},S_p)$ *for* $p > 2$.
(ii) If $\mathcal{A}$ *has cotype 2 then* $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{A}) \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},\mathcal{A})$.
In particular, $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},S_p) \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},S_p)$ *for* $1 \leq p \leq 2$.

Proof. It was shown (see [B4],Theorem 1.1) that for any Banach space X

$$\| \sum_{k=1}^{N} x_k r_k \|_{BMO(\mathbb{T},X)} \approx \| \sum_{k=1}^{N} x_k r_k \|_{L^2(\mathbb{T},X)} \tag{33}$$

for any x_k be a sequence of elements in X.

Take $B_I = B_k |I|^{-1/2}$ for $|I| = 2^{-k}$. Then $\sum_{I\in\mathcal{D}} B_I h_I = \sum_{k=1}^{\infty} B_k r_k$.

Note that

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}, J\subseteq I} \|B_J\|^2_{\mathcal{A}} = \sup_{I\in\mathcal{D}} \sum_{2^{-k}\leq |I|} \|B_k\|^2_{\mathcal{A}} (\sum_{J\subseteq I, |J|=2^{-k}} \frac{|J|}{|I|}) = \sum_{k=1}^{\infty} \|B_k\|^2_{\mathcal{A}}$$

Applying (33) one gets

$$\| \sum_{k=1}^{N} B_k r_k \|^2_{L^2(\mathbb{T},\mathcal{A})} \leq C \| \sum_{I\in\mathcal{D}} B_I h_I \|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}} = C \sum_{k=1}^{\infty} \|B_k\|^2_{\mathcal{A}}$$

Now the assumption in (i) gives type 2. Similarly the part (ii). □

We can use martingale transforms (see Section 2 for the notation) to analyze the validity of John-Nirenberg's lemma in our situation, that is to say to study whether $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ implies $S_B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$. Let us rewrite the sweep $S_B = \sum_{J\in\mathcal{D}} B^*_J B_J \frac{\chi_J}{|J|}$ by the formula

$$S_B = \int_{\Sigma} T_\sigma B^* T_\sigma B d\sigma, \tag{34}$$

Theorem 4.4. *Let* $B \in L^1(\mathbb{T},\mathcal{A})$. *Then*
(i) $\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathcal{A})} \leq \int_{\Sigma} \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathcal{A})} d\sigma$.
(ii) $\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(S_{p/2})} \leq C \|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(S_p)}$ *for* $2 \leq p < \infty$.

Proof. It is not difficult to show that $P_I(S_B) = P_I S_{P_I B}$ (see [BPo3]). Hence, using (34), one gets $P_I(S_B) = \int_{\Sigma} T_\sigma P_I B^* T_\sigma P_I B d\sigma$.

Therefore

$$\begin{aligned}
\|P_I(S_B)\|_{L^1(\mathbb{T},\mathcal{A})} &\leq \|\int_\Sigma (T_\sigma P_I B^*)(T_\sigma P_I B) d\sigma\|_{L^1(\mathbb{T},\mathcal{A})} \\
&\leq \int_\Sigma \|(P_I T_\sigma B^*)(P_I T_\sigma B)\|_{L^1(\mathbb{T},\mathcal{A})} d\sigma \\
&\leq \int_\Sigma \|P_I T_\sigma B\|_{L^2(\mathbb{T},\mathcal{L}(\mathcal{H}))} \|P_I T_\sigma B\|_{L^2(\mathbb{T},\mathcal{A})} d\sigma \\
&\leq \int_\Sigma \|P_I T_\sigma B\|^2_{L^2(\mathbb{T},\mathcal{A})} d\sigma \\
&\leq (\int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma)|I|.
\end{aligned}$$

Use now John-Nirenberg's lemma to obtain

$$\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathcal{A})} \leq C \sup_{I\in\mathcal{D}} \frac{1}{|I|} \|P_I(S_B)\|_{L^1(\mathbb{T},\mathcal{A})} \leq C(\int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma).$$

(ii) Use the argument above, together with the estimate $\|uv\|_{S_{p/2}} \leq \|u\|_{S_p}\|v\|_{S_p}$, to get that

$$\begin{aligned}
\|P_I(S_B)\|_{L^1(\mathbb{T},S_{p/2})} &\leq \int_\Sigma \|P_I T_\sigma B\|^2_{L^2(\mathbb{T},S_p)} d\sigma \\
&\leq (\int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(S_p)} d\sigma)|I| \\
&\leq |I| \sup_{\sigma\in\Sigma} \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(S_p)} \\
&\leq C|I|\|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(S_p)},
\end{aligned}$$

where the last inequality follows from the fact that S_p is a UMD space. Now finish the proof applying John-Nirenberg's lemma again. □

Definition 4.5. *Let $\mathcal{A}$ be an operator ideal and let $B : \mathbb{T} \to \mathcal{L}(\mathcal{H})$ such that $B(t)u, vB(t) \in L^2(\mathbb{T},\mathcal{A})$ for any $u, v \in \mathcal{A}$. We say that $B \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so},\mathcal{A}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$, if*

$$\gamma_{r,\mathcal{A}}(B) = \sup_{I\in\mathcal{D}, u\in\mathcal{A}, \|u\|_\mathcal{A}=1} (\frac{1}{|I|}\int_I \|(B(t)-m_I B)u\|^2_\mathcal{A} dt)^{1/2} < \infty \tag{35}$$

and

$$\gamma_{l,\mathcal{A}}(B) = \sup_{I\in\mathcal{D}, v\in\mathcal{A}, \|v\|_\mathcal{A}=1} (\frac{1}{|I|}\int_I \|v(B(t)-m_I B)\|^2_\mathcal{A} dt)^{1/2} < \infty. \tag{36}$$

The norm $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{A})} = \gamma_{r,\mathcal{A}}(B) + \gamma_{l,\mathcal{A}}(B)$.

Definition 4.6. *Let $\mathcal{A}$ be an operator ideal and let $B : \mathbb{T} \to \mathcal{L}(\mathcal{H})$ such that $vB(t)u \in L^2(\mathbb{T},\mathcal{A})$ for any $u, v \in \mathcal{A}$. We say that $B \in {\mathrm{WBMO}^{\mathrm{d}}}_\mathcal{A}(\mathbb{T},\mathcal{L}(\mathcal{H}))$, if*

$$\gamma_\mathcal{A}(B) = \sup_{I\in\mathcal{D}, u,v\in\mathcal{A}, \|v\|=\|u\|_\mathcal{A}=1} (\frac{1}{|I|}\int_I \|v(B(t)-m_I B)u\|^2_\mathcal{A} dt)^{1/2} < \infty. \tag{37}$$

Of course for $\mathcal{A} = \mathcal{H}$ (see Proposition 4.1), one has

$$\mathrm{BMO}_{\mathrm{so},\mathcal{H}}(\mathbb{T},\mathcal{L}(\mathcal{H})) = \mathrm{BMO}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H})),$$

$${\mathrm{WBMO}^{\mathrm{d}}}_\mathcal{H}(\mathbb{T},\mathcal{L}(\mathcal{H}) = \mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H}).$$

It is also elementary to show

$$\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{A})\subseteq \mathrm{BMO}_{\mathrm{so},\mathcal{A}}(\mathbb{T},\mathcal{L}(\mathcal{H}))\subseteq \mathrm{WBMO^d}_{\mathcal{A}}(\mathbb{T},\mathcal{L}(\mathcal{H})).$$

Although for $\mathcal{A}=\mathcal{H}$ it was shown (see [BPo3]) that the inclusions are strict, however for $\mathcal{A}=\mathcal{L}(\mathcal{H})$ one obviously has

$$\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H}))=\mathrm{BMO}_{\mathrm{so},\mathcal{L}(\mathcal{H})}(\mathbb{T},\mathcal{L}(\mathcal{H}))=\mathrm{WBMO^d}_{\mathcal{L}(\mathcal{H})}(\mathbb{T},\mathcal{L}(\mathcal{H})).$$

Let us study the situation for $\mathcal{A}=S_1$.

Proposition 4.7.
(i) $\mathrm{BMO}_{\mathrm{so},S_1}(\mathbb{T},\mathcal{L}(\mathcal{H}))=\mathrm{BMO}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.
(ii) $\mathrm{WBMO^d}_{S_1}(\mathbb{T},\mathcal{L}(\mathcal{H}))=\mathrm{WBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

Proof. (i) Let $B\in\mathrm{BMO}_{\mathrm{so},S_1}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. Take $u=e\otimes h$ with $\|e\|=\|h\|=1$ and observe that $(B(t)-m_IB)u=(B(t)-m_IB)e\otimes h$ and $u(B(t)-m_IB)=e\otimes(B^*(t)-m_IB^*)h$. Hence $\|(B(t)-m_IB)u\|_{S_1}=\|(B(t)-m_IB)e\|$ and $\|u(B(t)-m_IB)\|_{S_1}=\|(B^*(t)-m_IB^*)h\|$. This implies that $\mathrm{BMO}_{\mathrm{so},S_1}(\mathbb{T},\mathcal{L}(\mathcal{H}))\subseteq\mathrm{BMO}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

Conversely, let $B\in\mathrm{BMO}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H})$ and let $u=\sum_{k=1}^\infty\lambda_ke_k\otimes h_k$ where $\sum_{k=1}^\infty|\lambda_k|<\infty$ and $\|e_k\|=\|h_k\|=1$. Now

$$\begin{aligned}\|(B(t)-m_IB)u\|_{S_1} &= \|\sum_{k=1}^\infty\lambda_k(B(t)-m_IB)e_k\otimes h_k\|_{S_1}\\ &\leq \sum_{k=1}^\infty|\lambda_k|\|(B(t)-m_IB)e_k\|\end{aligned}$$

Then, for $I\in\mathcal{D}$ and $u\in S_1$, one gets

$$\begin{aligned}\frac{1}{|I|}\int_I\|(B(t)-m_IB)u\|_{S_1}dt &\leq \sum_{k=1}^\infty|\lambda_k|\frac{1}{|I|}\int_I\|(B(t)-m_IB)e_k\|dt\\ &\leq \sum_{k=1}^\infty|\lambda_k|\|B\|_{\mathrm{BMO}_{\mathrm{so}}}\end{aligned}$$

Hence using John-Nirenberg's lemma one gets $\gamma_{r,\mathcal{A}}(B)\leq\|u\|_{S_1}\|B\|_{\mathrm{BMO}_{\mathrm{so}}}$. Similarly one obtains $\gamma_{l,\mathcal{A}}(B)\leq\|u\|_{S_1}\|B\|_{\mathrm{BMO}_{\mathrm{so}}}$.

(ii) Note that for $\|e\|=\|h\|=\|e'\|=\|h'\|=1$, $(e\otimes h)(B(t)-m_IB)(e'\otimes h')=\langle h,(B(t)-m_IB)e'\rangle e\otimes h'$. Hence $\|(e\otimes h)(B(t)-m_IB)(e'\otimes h')\|_{S_1}=|\langle h,(B(t)-m_IB)e'\rangle$. Now similar arguments to the ones used in (i) allow to get the result. $\square$

Let $\mathcal{F}_{00}(\mathcal{A})$ denote the subspace of $\mathcal{A}$-valued functions on $\mathbb{T}$ with finite formal Haar expansion (we keep the notation $\mathcal{F}_{00}$ in the case $\mathcal{A}=\mathcal{L}(\mathcal{H})$) and write $L^2_0(\mathbb{T},\mathcal{A})$ the closure of $\mathcal{F}_{00}(\mathcal{A})$ in $L^2(\mathbb{T},\mathcal{A})$.

Definition 4.8. *Let $(\Phi_I)_{I\in\mathcal{D}}\subset L^2(\mathbb{T},\mathcal{L}(\mathcal{H}))$ be a sequence of operators. It is said to be an $\mathcal{A}$-Haar multiplier if there exists $C>0$ such that*

$$\|\sum_{I\in\mathcal{D}}\Phi_IF_Ih_I\|_{L^2(\mathbb{T},\mathcal{A})}\leq C\|\sum_{I\in\mathcal{D}}F_Ih_I\|_{L^2(\mathbb{T},\mathcal{A})}$$

for any $F\in\mathcal{F}_{00}(\mathcal{A})$.

We write $\|(\Phi_I)_{I\in\mathcal{D}}\|_{mult,\mathcal{A}}$ for the norm of the extension of the operator to $L^2_0(\mathbb{T},\mathcal{A})$.

Definition 4.9. *Let $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. We say that $B \in \mathrm{BMO}_{\mathrm{mult}\mathcal{A}}(\mathbb{T}, \mathcal{A})$ if $(P_I B)_{I\in\mathcal{D}}$ defines a $\mathcal{A}$-Haar multiplier and we write*

$$\|B\|_{\mathrm{BMO}_{\mathrm{mult}},\mathcal{A}} = \|(P_I B)_{I\in\mathcal{D}}\|_{mult,\mathcal{A}}.$$

It was shown in [BPo3] that $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Now one has the following

Proposition 4.10. $\mathrm{BMO}_{\mathrm{mult},\mathcal{L}(\mathcal{H})} \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Proof. Let $B \in \mathrm{BMO}_{\mathrm{mult},\mathcal{L}(\mathcal{H})}$. Take $F = \mathcal{I}h_J$ for fixed $J \in \mathcal{D}$. Observe that

$$\begin{aligned} \|\sum_{I\in\mathcal{D}} P_I(B)F_I h_I\|_{L^2_0(\mathcal{L}(\mathcal{H}))} &= \|P_J(B)h_J\|_{L^2_0(\mathcal{L}(\mathcal{H}))} \\ &= \frac{1}{|J|^{1/2}}\|P_J(B)\|_{L^2(\mathcal{L}(\mathcal{H}))} \\ &\leq \|B\|_{\mathrm{BMO}_{\mathrm{mult}\,\mathcal{L}(\mathcal{H})}}. \end{aligned}$$

□

Definition 4.11. *Let $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. We define the $\mathcal{A}$-paraproduct with symbol B by*

$$\pi_B^{\mathcal{A}} : \mathcal{F}_{00}(\mathcal{A}) \to \mathcal{F}_{00}(\mathcal{A})$$

given by

$$F = \sum_{I\in\mathcal{D}} F_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I m_I F h_I,$$

where $B_I m_I F$ stands for the composition of operators.

Definition 4.12. *Let $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. We say that $B \in \mathrm{BMO}_{\mathrm{para},\mathcal{A}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if $\pi_B^{\mathcal{A}}$ extends to a bounded operator from $L^2_0(\mathbb{T}, \mathcal{A})$ to $L^2_0(\mathbb{T}, \mathcal{A})$.*

We write

$$\|B\|_{\mathrm{BMO}_{\mathrm{para}},\mathcal{A}} = \|\pi_B^{\mathcal{A}}\|_{L^2_0(\mathbb{T},\mathcal{A})\to L^2_0(\mathbb{T},\mathcal{A})},$$

Theorem 4.13. $\mathrm{BMO}_{\mathrm{para},\mathcal{L}(\mathcal{H})} \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. *Moreover*

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathcal{L}(\mathcal{H}))} \leq \frac{\sqrt{2}}{\sqrt{2}-1}\|B\|_{\mathrm{BMO}_{\mathrm{para}},\mathcal{L}(\mathcal{H})}$$

Proof. Applying the assumption on functions $F = (e\otimes h)\phi$ for fixed $\|e\| = \|h\| = 1$ and $\|\phi\|_{L^2(\mathbb{T})} = 1$, one easily obtains that $\|B\|_{\mathrm{WBMO}^{\mathrm{d}}} \leq \|\pi_B^{\mathcal{L}(\mathcal{H})}\|$. In particular $\|B_J\| \leq \|\pi_B^{\mathcal{L}(\mathcal{H})}\||J|^{1/2}$ for all $J \in \mathcal{D}$.

Consider $F(t) = \mathcal{I}\chi_J(t)$ for some $J \in \mathcal{D}$ where $\mathcal{I}$ stands for the identity operator. Now

$$\pi_B^{\mathcal{L}(\mathcal{H})}(F) = \sum_{I\subseteq J} B_I h_I + |J|\sum_{J\subset I}\frac{B_I}{|I|}h_I = P_J(B) + \sum_{I=2^kJ,k\geq 1}\frac{B_I}{2^k}h_I.$$

Clearly

$$\|\sum_{I=2^kJ,k\geq 1}\frac{B_I}{2^k}h_I\|_{L^2(\mathcal{L}(\mathcal{H}))} \leq \sum_{I=2^kJ,k\geq 1}\frac{\|B_I\|}{2^k} \leq$$

$$\leq |J|^{1/2}\|\pi_B^{\mathcal{L}(\mathcal{H})}\|\sum_{k\geq 1}2^{-k/2} \leq \frac{1}{\sqrt{2}-1}\|\pi_B^{\mathcal{L}(\mathcal{H})}\||J|^{1/2}.$$

Therefore

$$\|P_J(B)\|_{L^2(\mathcal{L}(\mathcal{H}))} \leq \|\pi_B^{\mathcal{L}(\mathcal{H})}\| + \frac{1}{\sqrt{2}-1}\|B\|_{\mathrm{WBMO^d}}|J|^{1/2} \leq \frac{\sqrt{2}}{\sqrt{2}-1}\|\pi_B^{\mathcal{L}(\mathcal{H})}\||J|^{1/2}.$$

Thus the proof is complete. □

Definition 4.14. *Let $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. We can define*

$$\Delta_B^{\mathcal{A}} : \mathcal{F}_{00}(\mathcal{A}) \to L^2(\mathbb{T}, \mathcal{A})$$

given by

$$F = \sum_{I\in\mathcal{D}} F_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I F_I \frac{\chi_I}{|I|}.$$

Let us denote $\Lambda_B^{\mathcal{A}} = \pi_B^{\mathcal{A}} + \Delta_B^{\mathcal{A}}$.
We also define

$$\Gamma_B^{\mathcal{A}} : \mathcal{F}_{00}(\mathcal{A}) \to \mathcal{F}_{00}(\mathcal{A})$$

given by

$$F = \sum_{I\in\mathcal{D}} F_I h_I \mapsto \sum_{I\in\mathcal{D}} \frac{B_I}{|I|^{1/2}} F_I h_I.$$

Remark 4.15. *Clearly if $\Gamma_B^{\mathcal{A}}$ is bounded on $L^2(\mathbb{T}, \mathcal{A})$ then $sup_{\|u\|_{\mathcal{A}}=1}\|B_I u\| \leq C|I|^{1/2}$.*

For $\mathcal{A} = \mathcal{H}$ one has $\Gamma_B^{\mathcal{A}}$ is bounded if and only if $\|B_I\| \leq C|I|^{1/2}$.

Proposition 4.16. *$B \in \mathrm{BMO}_{\mathrm{mult}\,\mathcal{A}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ if and only if $\Lambda_B^{\mathcal{A}}$ extends to a bounded operator on $L_0^2(\mathbb{T}, \mathcal{A})$. Moreover*

$$\|B\|_{\mathrm{BMO_{mult}},\mathcal{A}} = \|\Lambda_B^{\mathcal{A}}\|_{L_0^2(\mathbb{T},\mathcal{A})\to L^2(\mathbb{T},\mathcal{A})}.$$

Proof. It follows from the formula

$$\Lambda_B^{\mathcal{A}} F = BF - \sum_{I\in\mathcal{D}} (m_I B) F_I h_I = \sum_{I\in\mathcal{D}} (P_I B) F_I h_I. \tag{38}$$

□

We observe that the boundedness of $\Delta_B^{\mathcal{A}}$ on $L_0^2(\mathbb{T}, \mathcal{A})$ can be pushed to $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{A})$.

Proposition 4.17.

$$\|\Delta_B^{\mathcal{A}}\|_{\mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{A})\to\mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{A})} \leq 2\|\Delta_B^{\mathcal{A}}\|_{L_0^2(\mathbb{T},\mathcal{A})\to L_0^2(\mathbb{T},\mathcal{A})}. \tag{39}$$

Proof. Assume $\Delta_B^{\mathcal{A}}$ is bounded on $L_0^2(\mathbb{T}, \mathcal{A})$. Let $F \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{A})$ of norm 1, that is $P_I F \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ with norm bounded by $|I|^{1/2}$.

It is not difficult to see that $\|P_I \Delta_B^{\mathcal{A}}(F)\|_{L^2(\mathbb{T},\mathcal{A})} = \|P_I \Delta_B^{\mathcal{A}}(P_I F)\|_{L^2(\mathbb{T},\mathcal{A})}$.

Since $P_I G = (G - m_I G)\chi_I$ then we have

$$\|P_I \Delta_B^{\mathcal{A}}(P_I F)\|_{L^2(\mathbb{T},\mathcal{A})} \leq 2\|\Delta_B^{\mathcal{A}}(P_I F)\|_{L^2(\mathbb{T},\mathcal{A})} \leq 2\|\Delta_B^{\mathcal{A}}\|_{L_0^2(\mathbb{T},\mathcal{A})\to L_0^2(\mathbb{T},\mathcal{A})}|I|^{1/2}.$$

Hence one gets the desired estimate. □

Definition 4.18. *We write $\Delta : \mathcal{F}_{00} \times \mathcal{F}_{00} \to L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ for the map*

$$\Delta(B, F) = \Delta_{B^*}^{\mathcal{L}(\mathcal{H})}(F) = \sum_{I\in\mathcal{D}} B_I^* F_I \frac{\chi_I}{|I|}.$$

And we denote $\Gamma : \mathcal{F}_{00} \times \mathcal{F}_{00} \to \mathcal{F}_{00}$ *given by*

$$\Gamma(B,F) = \sum_{I\in\mathcal{D}} \frac{B_I^* F_I}{|I|^{1/2}} h_I.$$

Of course $\Gamma(B,F) = \Gamma_{B^*}^{\mathcal{L}(\mathcal{H})}(F)$.

In particular, the "dyadic sweep" of $B \in \mathcal{F}_{00}$ *is given by*

$$S_B = \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} B_I^* B_I = \Delta_{B^*}^{\mathcal{L}(\mathcal{H})}(B) = \Delta(B,B). \tag{40}$$

Let us finish by giving the formulation of the main connection between $\mathrm{BMO}_{\mathrm{para}}$ and $\mathrm{BMO}_{\mathrm{mult}}$ (see [BPo3]) in the new situation.

Next result is the extension of the similar one shown in [BPo3], and the proof presented here is different from the one given there for $\mathcal{A} = \mathcal{H}$.

Theorem 4.19. *Let* $B, F \in \mathcal{F}_{00}$. *Then*

$$\Delta_F^{\mathcal{A}} \pi_B^{\mathcal{A}} = \Lambda_{\Delta(F^*,B)}^{\mathcal{A}} - \Gamma_{\Gamma(F,B)}^{\mathcal{A}}.$$

Proof.

$$\begin{aligned}
\Delta_F^{\mathcal{A}} \pi_B^{\mathcal{A}}(G) &= \Delta_F^{\mathcal{A}}\Big(\sum_{I\in\mathcal{D}} B_I m_I(G) h_I\Big) \\
&= \sum_{I\in\mathcal{D}} F_I B_I m_I(G) \frac{\chi_I}{|I|} \\
&= \sum_{I\in\mathcal{D}} F_I B_I \sum_{I\subsetneq J} G_J h_J \frac{\chi_I}{|I|} \\
&= \sum_{J\in\mathcal{D}} \Big(\sum_{I\subsetneq J} F_I B_I \frac{\chi_I}{|I|}\Big) G_J h_J \\
&= \sum_{J\in\mathcal{D}} \Big(\sum_{I\subseteq J} F_I B_I \frac{\chi_I}{|I|}\Big) G_J h_J - \sum_{J\in\mathcal{D}} \frac{F_J B_J}{|J|} G_J h_J \\
&= \Lambda_{\Delta(F^*,B)}^{\mathcal{A}}(G) - \Gamma_{\Gamma(F^*,B)}^{\mathcal{A}}(G).
\end{aligned}$$

□

Corollary 4.20. *Let* $B \in \mathcal{F}_{00}$. *Then*

$$\Delta_{B^*}^{\mathcal{L}(\mathcal{H})} \pi_B^{\mathcal{L}(\mathcal{H})} = \Lambda_{S_B}^{\mathcal{L}(\mathcal{H})} - \Gamma_{B'}^{\mathcal{L}(\mathcal{H})}$$

where $B' = \Gamma(B,B) = \sum_{I\in\mathcal{D}} \frac{B_I^* B_I}{|I|^{1/2}} h_I$.

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