# OPERATOR-VALUED DYADIC BMO SPACES

OSCAR BLASCO AND SANDRA POTT

Abstract. We consider BMO spaces of operator-valued functions, among them the space of operator-valued functions $B$ which define a bounded paraproduct on $L^2(\mathcal{H})$. We obtain several equivalent formulations of $\|\pi_B\|$ in terms of the norm of the "sweep" function of $B$ or of averages of the norms of martingales transforms of $B$ in related spaces. Furthermore, we investigate a connection between John-Nirenberg type inequalities and Carleson-type inequalities via a product formula for paraproducts and deduce sharp dimensional estimates for John-Nirenberg type inequalities.

## 1. Introduction

Spaces of BMO functions on the real numbers $\mathbb{R}$ or the circle $\mathbb{T}$, taking values in the bounded linear operators on a Hilbert space, have been investigated in a number of different contexts in recent years, for example non-commutative $L^p$ spaces [PXu], [Me1], matrix-weighted inequalities [GPTV1], [GPTV2], sharp estimates for vector Carleson Embedding Theorem [K], [NTV], [NPiTV], [Pet], observation operators in linear systems over contractive semigroups [JPa], [JPaP], and Hankel operators in several variables [PS].

The theory of operator valued BMO functions is much more complicated than the scalar theory and remains to be fully understood. Some of the different yet equivalent characterizations of scalar $\mathrm{BMO}(\mathbb{T})$ or $\mathrm{BMO}(\mathbb{R})$ lead to distinct spaces of operator valued BMO functions. In many cases, we can express this in the language of operator spaces by saying that different operator space structures on the scalar BMO space arise naturally from the different yet equivalent characterisations of scalar BMO. These difficulties reflect partly the subtle geometric properties of the dual Banach space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space.

It is often easier to consider dyadic versions of BMO and to work with dyadic versions of classical operators like the Hilbert transform $H$ or the Hankel operator with symbol $b$, $\Gamma_b$. Two such dyadic counterparts of a Hankel operator $\Gamma_b$ are the *dyadic paraproduct* $\pi_b$ and the operator $\Lambda_b = \pi_b + \pi^*_{\bar{b}}$. While the former has a natural interpretation as a Carleson Embedding operator, the latter connects more easily in the operator valued case to the theory of vector-valued BMO functions (in particular to the space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathcal{L}(\mathcal{H}))$. Estimates for Hankel operators can then be obtained by averaging techniques.

One important difference between the scalar-valued and the operator-valued settings is the failure of a certain version of the classical *John-Nirenberg Lemma*, or in

---

*Key words and phrases.* operator BMO, Carleson measures.

*2000 Mathematical Subjects Classifications.* Primary 42B30, 42B35, Secondary 47B35

The first author gratefully acknowledges support by the LMS and Proyectos MTM 2005-08350 and PR2006-0086. The second author gratefully acknowledges support by EPSRC and by the Nuffield Foundation.

other words, the lack of boundedness of the "*sweep*", which governs the behaviour of the dyadic paraproduct.

The purpose of the present paper is to study in particular the spaces arising from the operators $\pi_b$ and $\Lambda_b$, to investigate the relationship between dyadic paraproduct, its "real part" $\Lambda_b$ and the sweep, and to give sharp dimensional estimates for the sweep in the "strong" BMO norm $\|\cdot\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}}$ and other norms, answering a question in [GPTV1].

Let $\mathcal{D}$ denote the collection of dyadic subintervals of the unit circle $\mathbb{T}$, and let $(h_I)_{I\in\mathcal{D}}$, where $h_I = \frac{1}{|I|^{1/2}}(\chi_{I^+} - \chi_{I^-})$, be the Haar basis of $L^2(\mathbb{T})$. Let $\mathcal{H}$ be a separable, finite or infinite-dimensional Hilbert space and let $\mathcal{F}_{00}$ denote the subspace of $\mathcal{L}(\mathcal{H})$-valued functions on $\mathbb{T}$ with finite formal Haar expansion. Given $e, f \in \mathcal{H}$ and $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ we denote by $B_e$ the function in $L^2(\mathbb{T}, \mathcal{H})$ defined by $B_e(t) = B(t)(e)$ and by $B_{e,f}$ the function in $L^2(\mathbb{T})$ defined by $B_{e,f}(t) = \langle B(t)(e), f\rangle$. As in the scalar case, let $B_I$ denote the formal Haar coefficients $\int_I B(t)h_I dt$, and $m_I B = \frac{1}{|I|}\int_I B(t)dt$ denote the average of $B$ over $I$ for any $I \in \mathcal{D}$. Observe that for $B_I$ and $m_I B$ to be well-defined operators, we shall be assuming that the $\mathcal{L}(\mathcal{H})$-valued function $B$ is *weak*$^*$-integrable. That means, using the duality $\mathcal{L}(\mathcal{H}) = (\mathcal{H}\hat{\otimes}\mathcal{H})^*$, that $\langle B(\cdot)(e), f\rangle \in L^1(\mathbb{T})$ for $e, f \in \mathcal{H}$. In particular, for any measurable set $A$, there exist $B_A \in \mathcal{L}(\mathcal{H})$ such that $\langle B_A(e), f\rangle = \langle \int_A B(t)(e)dt, f\rangle$.

Let us denote by $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T}, \mathcal{H})$ the space of Bochner integrable $\mathcal{H}$-valued functions $b : \mathbb{T} \to \mathcal{H}$ such that

$$\|b\|_{\mathrm{BMO}^{\mathrm{d}}(\mathcal{H})} = \sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I \|b(t) - m_I b\|^2 dt\Big)^{1/2} < \infty \tag{1}$$

and by $\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T}, \mathcal{H})$ the space of Pettis integrable $\mathcal{H}$-valued functions $b : \mathbb{T} \to \mathcal{H}$ such that

$$\|b\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathcal{H})} = \sup_{I\in\mathcal{D}, e\in\mathcal{H}, \|e\|=1}\Big(\frac{1}{|I|}\int_I |\langle b(t) - m_I b, e\rangle|^2 dt\Big)^{1/2} < \infty \tag{2}$$

Let us define different version of dyadic operator-valued BMO to be considered throughout the paper.

We denote by $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ the space of Bochner integrable $\mathcal{L}(\mathcal{H})$-valued functions $B$ such that

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} = \sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I \|B(t) - m_I B\|^2 dt\Big)^{1/2} < \infty. \tag{3}$$

and denote by $\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ the space of *weak*$^*$-integrable $\mathcal{L}(\mathcal{H})$-valued functions $B$ such that

$$\begin{aligned}\|B\|_{\mathrm{WBMO}^{\mathrm{d}}} &= \sup_{I\in\mathcal{D}, \|e\|=\|f\|=1}\Big(\frac{1}{|I|}\int_I |\langle (B(t) - m_I B)e, f\rangle|^2 dt\Big)^{1/2} \\ &= \sup_{e\in\mathcal{H}, \|e\|=1} \|B_e\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})} < \infty,\end{aligned} \tag{4}$$

or, equivalently, such that

$$\|B\|_{\mathrm{WBMO}^{\mathrm{d}}} = \sup_{A\in S_1, \|A\|_1\le 1} \|\langle B, A\rangle\|_{\mathrm{BMO}^{\mathrm{d}}(\mathbb{T})} < \infty.$$

Here, $S_1$ denotes the ideal of trace class operators in $\mathcal{L}(\mathcal{H})$, and $\langle B, A\rangle$ stands for the scalar-valued function given by $\langle B, A\rangle(t) = \mathrm{trace}(B(t)A^*)$.

In the operator-valued setting one has another natural formulation. Denote by $\mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ the space of $\mathcal{L}(\mathcal{H})$-valued functions $B$ such that $B(\cdot)e \in \mathrm{BMO^d}(\mathbb{T}, \mathcal{H})$ for all $e \in \mathcal{H}$ and such that

$$\|B\|_{\mathrm{SBMO^d}} = \sup_{I\in\mathcal{D}, e\in\mathcal{H}, \|e\|=1} (\frac{1}{|I|}\int_I \|(B(t) - m_I B)e\|^2 dt)^{1/2} < \infty. \tag{5}$$

We would like to point out that while $B$ belongs to one of the spaces $\mathrm{BMO^d_{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ or $\mathrm{WBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H})))$ if and only if $B^*$ does, this is not the case for the space $\mathrm{SBMO}^d(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. This leads to the following notion:

**Definition 1.1.** *(see* [GPTV1], [Pet],[PXu] *) We say that* $B \in \mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$, *if* $B$ *and* $B^*$ *belong to* $\mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H})$. *We define* $\|B\|_{\mathrm{BMO^d_{so}}} = \|B\|_{\mathrm{SBMO^d}} + \|B^*\|_{\mathrm{SBMO^d}}$.

Continuous versions of this space in the more general setting of functions taking values in a von Neumann algebra with a semifinite normal faithful trace were studied by Pisier and Xu [PXu] and more recently by Mei [Me1], together with an $H^p$ theory and a rich duality and interpolation theory.

We now define another operator-valued BMO space, using the notion of Haar multipliers. As in the scalar-valued case (see [Per]), a sequence $(\Phi_I)_{I\in\mathcal{D}}$, $\Phi_I \in L^2(I, \mathcal{L}(\mathcal{H}))$ for all $I \in \mathcal{D}$, is said to be an *operator-valued Haar multiplier*, if there exists $C > 0$ such that

$$\|\sum_{I\in\mathcal{D}} \Phi_I(f_I)h_I\|_{L^2(\mathbb{T},\mathcal{H})} \le C(\sum_{I\in\mathcal{D}} \|f_I\|^2)^{1/2} \text{ for all } (f_I)_{I\in\mathcal{D}} \in l^2(\mathcal{D}, \mathcal{H}).$$

We write $\|(\Phi_I)\|_{mult}$ for the norm of the corresponding operator on $L^2(\mathbb{T}, \mathcal{H})$.

Let us observe that

$$\|\Phi_J\|_{L^2(\mathbb{T},\mathcal{H})} \le \|(\Phi_I)\|_{mult}|J|^{1/2}, \quad J \in \mathcal{D}. \tag{6}$$

**Definition 1.2.** *Let us define* $P_I B = \sum_{J\subseteq I} h_J B_J$, *and use the notation*

$$\Lambda_B(f) = \sum_{I\in\mathcal{D}} (P_I B)(f_I)h_I.$$

*We define* $\mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ *the space of those weak*$^*$*-integrable* $\mathcal{L}(\mathcal{H})$*-valued functions for which* $(P_I B)_{I\in\mathcal{D}}$ *defines a bounded operator-valued Haar multiplier and write*

$$\|B\|_{\mathrm{BMO_{mult}}} = \|\Lambda_B\| = \|(P_I B)_{I\in\mathcal{D}}\|_{mult}. \tag{7}$$

Let us now give the definition of a further BMO space, the space defined in terms of dyadic paraproducts.

Let $B \in \mathcal{F}_{00}$. We define

$$\pi_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I(m_I f)h_I,$$

and

$$\Delta_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I(f_I)\frac{\chi_I}{|I|}.$$

$\pi_B$ is called the vector paraproduct with symbol $B$.

It is elementary to see that

$$\Lambda_B(f) = \sum_{I\in\mathcal{D}} B_I(m_I f)h_I + \sum_{I\in\mathcal{D}} B_I(f_I)\frac{\chi_I}{|I|}. \tag{8}$$

This shows that $\Lambda_B = \pi_B + \Delta_B$. Observe that $\Delta_B = \pi^*_{B^*}$. Therefore $(\Lambda_B)^* = \Lambda_{B^*}$, and $\|B\|_{\mathrm{BMO_{mult}}} = \|B^*\|_{\mathrm{BMO_{mult}}}$.

**Definition 1.3.** *Let us denote* $E_k B = \sum_{|I|>2^{-k}} B_I h_I$. *The space* $\mathrm{BMO_{para}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ *consists of those weak*-integrable operator-valued functions for which* $\sup_{k\in\mathbb{N}} \|\pi_{E_k B}\| < \infty$ *For those functions,* $\pi_B f = \lim_{k\to\infty} \pi_{E_k B} f$ *defines a bounded linear operator on* $L^2(\mathbb{T},\mathcal{H})$*, and we write*

$$\|B\|_{\mathrm{BMO_{para}}} = \|\pi_B\|. \tag{9}$$

Let us notice that

$$\Lambda_B f = Bf - \sum_{I\in\mathcal{D}} (m_I B)(f_I)h_I. \tag{10}$$

From here one concludes immediately that

$$L^\infty(\mathbb{T},\mathcal{L}(\mathcal{H})) \subseteq \mathrm{BMO_{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H})). \tag{11}$$

However, Tao Mei [Me2] has shown recently that $L^\infty(\mathbb{T},\mathcal{L}(\mathcal{H})) \not\subseteq \mathrm{BMO_{para}}$ and therefore in particular $\mathrm{BMO_{mult}} \not\subseteq \mathrm{BMO_{para}}$. This is in contrast to the situation of scalar paraproducts in two variables, where $\mathrm{BMO_{mult}}(\mathbb{T}^2) = \mathrm{BMO_{para}}(\mathbb{T}^2)$ ([BPo], Thm 2.8).

The following chain of strict inclusions for infinite-dimensional $\mathcal{H}$ can be shown (see [BPo2]):

$$\begin{aligned} \mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})) &\subsetneq \mathrm{BMO_{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO^d_{so}} \\ &\subsetneq \mathrm{SBMO}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{WBMO}(\mathbb{T},\mathcal{L}(\mathcal{H})). \end{aligned} \tag{12}$$

The reader is referred to [B1], [BPo], [Me2], [PSm] for some recent results on dyadic BMO and Besov spaces connected to the ones in this paper.

Mei's result implies in particular that $\mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \not\subseteq \mathrm{BMO_{para}}$, and it is also easy to see that the reverse inclusion does not hold (see for example the proof of $\mathrm{BMO_{mult}} \not\subseteq \mathrm{BMO_{para}}$ at the beginning of Section 2).

To retrieve an estimate of the norm of the paraproduct in terms of the $\mathrm{BMO^d_{norm}}$ norm, we will consider the "*sweep*", which is of independent interest, in Section 2 and averages of martingale transforms in Section 4.

Given $B \in \mathcal{F}_{00}$, we define the sweep of $B$ as

$$S_B = \sum_{I\in\mathcal{D}} B_I^* B_I \frac{\chi_I}{|I|}. \tag{13}$$

Our main result of Section 2, Theorem 2.4, states that

$$\|B\|^2_{\mathrm{BMO_{para}}} \approx \|S_B\|_{\mathrm{BMO_{mult}}} + \|B\|^2_{\mathrm{SBMO^d}}.$$

In particular, using the result $\mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO_{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ (see [BPo2]), this shows that if $B \in \mathrm{SBMO^d}$ and $S_B \in \mathrm{BMO^d_{norm}}$ then $\pi_B$ is bounded.

Section 3 is devoted to the study of sweeps of functions in different $BMO$-spaces. The classical John-Nirenberg theorem on $\mathrm{BMO^d}(\mathbb{T})$ implies (and is essentially equivalent to) the fact that there exists a constant $C > 0$ such that

$$\|S_b\|_{\mathrm{BMO^d}} \leq C\|b\|^2_{\mathrm{BMO^d}} \tag{14}$$

for any $b \in \mathrm{BMO^d}$.

We will show that this formulation of John-Nirenberg does not hold for $\|B\|_{\mathrm{BMO_{so}}}$. In fact, it is shown that if (14) holds for some space contained in $\mathrm{SBMO^d}$ then this space is also contained in $\mathrm{BMO_{para}}$.

In [K], [NTV] and [NPiTV], the correct rate of growth of the constant in the Carleson embedding theorem in the matrix case in terms of the dimension of Hilbert space $\mathcal{H}$ was determined, namely $\log(\dim \mathcal{H} + 1)$. Here, we want to show that this breakdown of the Carleson embedding theorem in the operator case is intimately connected to a breakdown of the John-Nirenberg Theorem, and that the dimensional growth for constants in the John-Nirenberg Theorem is the same. This answers a question left open in [GPTV1].

In Section 4, we investigate "average BMO conditions" in the following sense. We show (see Theorem 4.1) that $\|B\|_{\mathrm{BMO_{para}}} \leq C(\int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO^d_{norm}}} d\sigma)^{1/2}$. More precisely, $\|B\|^2_{\mathrm{BMO_{para}}} + \|B^*\|^2_{\mathrm{BMO_{para}}} \approx \int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO_{mult}}} d\sigma$.

Moreover, the norms $\|B\|_{\mathrm{BMO^d_{so}}}$, $\|B\|_{\mathrm{BMO_{mult}}}$ and $\|B\|_{\mathrm{BMO_{para}}}$ can be completely described in terms of average boundedness of certain operators involving either $\Lambda_B$ or commutators $[T_\sigma, B]$. The results of this section complete those proved in [GPTV1].

## 2. Haar multipliers and paraproducts

We start by describing the action of a paraproduct $\pi_B$ as a Haar multiplier.

**Proposition 2.1.** *Let $B \in \mathcal{F}_{00}$. Then*

$$\begin{aligned}\|\pi_B\| &= \|(B_I^* h_I)_{I\in\mathcal{D}}\|_{mult} \\ &= \|(P_{I^+}B + P_{I^-}B)_{I\in\mathcal{D}}\|_{mult} \\ &= \|(\sum_{J\subsetneq I} B_J^* B_J \frac{\chi_J}{|J|})_{I\in\mathcal{D}}\|^{1/2}_{mult}.\end{aligned}$$

*In particular,*

$$\|B_I\| \leq \|\pi_B\||I|^{1/2},$$

$$\|P_{I^+}B(e) + P_{I^-}B(e)\|_{L^2(\mathbb{T},\mathcal{H})} \leq \|\pi_B\||I|^{1/2}\|e\|$$

*and*

$$\|(\sum_{J\subsetneq I} B_J^* B_J \frac{\chi_J}{|J|})e\|_{L^2(\mathbb{T},\mathcal{H})} \leq \|\pi_B\|^2|I|\|e\|.$$

*Proof.* The first and second equalities follow directly from the definitions and $\|\pi_B\| = \|\Delta_{B^*}\|$.

For the third equality, use $\|\pi_B\|^2 = \|\pi_B^*\pi_B\|$,

$$\begin{aligned}\pi_B^*\pi_B(f)(t) &= \sum_{I\in\mathcal{D}} B_I^* B_I(m_I(f))\frac{\chi_I(t)}{|I|} = \sum_{I\in\mathcal{D}} B_I^* B_I(\sum_{I\subsetneq J} f_J m_I(h_J))\frac{\chi_I(t)}{|I|} \\ &= \sum_{I\in\mathcal{D}} B_I^* B_I(\sum_{I\subsetneq J} f_J) h_J(t)\frac{\chi_I(t)}{|I|} = \sum_{J\in\mathcal{D}}(\sum_{I\subsetneq J} B_I^* B_I \frac{\chi_I(t)}{|I|}) f_J h_J(t).\end{aligned}$$

The estimates now follow from (6). □

The following characterizations of SBMO will be useful below.

**Proposition 2.2.** *(*[GPTV1]*) Let $B \in \mathrm{SBMO}^{\mathrm{d}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Then*

$$\|B\|^2_{\mathrm{SBMO}^{\mathrm{d}}} = \sup_{I \in \mathcal{D}, \|e\|=1} \frac{1}{|I|} \|P_I(B_e)\|^2_{L^2(\mathcal{H})} \approx \sup_{I \in \mathcal{D}} \frac{1}{|I|} \| \sum_{J \subseteq I} B_J^* B_J \|.$$

It follows at once from Propositions 2.1 and 2.2 that

$$\mathrm{BMO}_{\mathrm{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \mathrm{SBMO}^{\mathrm{d}}(\mathbb{T}, \mathcal{L}(\mathcal{H})).$$

It is easily seen that, if $B$ and $B^*$ belong to $\mathrm{BMO}_{\mathrm{para}}$, then $B \in \mathrm{BMO}_{\mathrm{mult}}$. However, we want to remark that the boundedness of $\pi_B$ alone does not imply boundedness of $\Lambda_B$.

To see this, choose some orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$, and choose a sequence of $\mathbb{C}^n$-valued function $(b_n)_{n\in\mathbb{N}}$ with finite Haar expansion such that $\|b_n\|_{\mathrm{BMO}^{\mathrm{d}}(\mathcal{L}(\mathcal{H}))} \geq Cn^{1/2}\|b_n\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathcal{L}(\mathcal{H}))}$ (for a choice of such a sequence, see [JPaP]). Let $B_n(t)$ be the column matrix with respect to the chosen orthonormal basis which has the vector $b_n(t)$ as its first column. Then it is easy to see that

$$\|\pi_{B_n}\| = \|\pi_{b_n}\| \sim \|b_n\|_{\mathrm{BMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})} \geq n^{1/2} C \|b_n\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})}.$$

As pointed out to us [PV], it follows from the first Theorem in the appendix in [PXu] that $\|\pi_{B_n^*}\| \leq C\|b_n\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{H})}$ for some absolute constant $C$ and all $n \in \mathbb{N}$. Forming the direct sum

$$B = \bigoplus_{n=1}^{\infty} \frac{1}{\|\pi_{B_n^*}\|} B_n^*,$$

we find that $\|\pi_B\| = 1$, but $\Delta_B = (\pi_{B^*})^*$ is unbounded.

One of the main tools to investigate the connection between $\mathrm{BMO}_{\mathrm{mult}}$ and $\mathrm{BMO}_{\mathrm{para}}$ is the *dyadic sweep.* Given $B \in \mathcal{F}_{00}$, we define

$$S_B(t) = \sum_{I \in \mathcal{D}} B_I^* B_I \frac{\chi_I(t)}{|I|}.$$

**Lemma 2.3.** *Let $B \in \mathcal{F}_{00}$. Then*

$$\pi_B^* \pi_B = \pi_{S_B} + \pi_{S_B}^* + D_B = \Lambda_{S_B} + D_B, \tag{15}$$

*where $D_B$ is defined by $D_B h_I \otimes x = h_I \frac{1}{|I|} \sum_{J \subsetneq I} B_J^* B_J x$ for $x \in \mathcal{H}$, $I \in \mathcal{D}$ and*

$$\|D_B\| \approx \|B\|^2_{\mathrm{SBMO}^{\mathrm{d}}}.$$

*Proof.* (15) is verified on elementary tensors $h_I \otimes x$, $h_J \otimes y$. We find that

(1) for $I \subsetneq J$,

$$\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y \rangle = \langle \pi_{S_B}^* h_I \otimes x, h_J \otimes y \rangle$$

(2) for $I \supsetneq J$,

$$\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y \rangle = \langle \pi_{S_B} h_I \otimes x, h_J \otimes y \rangle$$

(3) for $I = J$,

$$\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y \rangle = \langle D_B (h_I \otimes x), h_J \otimes y \rangle.$$

Since $\operatorname{supp} \pi_{S_B} h_I \subseteq I$ and $\operatorname{supp} \Delta_{S_B} h_I \subseteq I$, $\langle \pi_B^* \pi_B h_I \otimes x, h_J \otimes y\rangle = 0$ in all other cases.

One sees easily that $D_B$ is block diagonal with respect to the Hilbert space decomposition $L^2(\mathbb{T}, \mathcal{H}) = \bigoplus_{I\in\mathcal{D}} \mathcal{H}$ defined by the mapping $f \mapsto (f_I)_{I\in\mathcal{D}}$. The operator $\pi_{S_B}$ is block-lower triangular with respect to this decomposition (using the natural partial order on $\mathcal{D}$), and $\Delta_{S_B}$ is block-upper triangular. Thus we obtain the required identity. Note that

$$\|D_B\| = \sup_{I\in\mathcal{D}, \|e\|=1} \frac{1}{|I|} \| \sum_{J \subsetneq I} B_J^* B_J e\| \approx \|B\|^2_{\mathrm{SBMO^d}}$$

by Proposition 2.2. □

Notice that $(S_B)^* = S_B$. Hence Lemma 2.3 gives

**Theorem 2.4.**

$$\|S_B\|_{\mathrm{BMO_{mult}}} + \|B\|^2_{\mathrm{SBMO^d}} \approx \|\pi_B\|^2.$$

*Proof.* It suffices to use that $\|D_B\| \approx \|B\|^2_{\mathrm{SBMO^d}}$ and that $\|B\|_{\mathrm{SBMO^d}} \lesssim \|\pi_B\|$ (using Proposition 2.1). □

This provides, among other things, our first link between $\mathrm{BMO^d_{norm}}$ and $\mathrm{BMO_{para}}$:

**Corollary 2.5.**

$$\|\pi_B\|^2 \lesssim \|S_B\|_{\mathrm{BMO^d_{norm}}} + \|B\|^2_{\mathrm{BMO^d_{so}}}.$$

*Proof.* Theorem 2.4 and (12). □

## 3. Sweeps of operator-valued functions

Let us mention that by John-Nirenberg's lemma, we actually have that $f \in \mathrm{BMO^d_{norm}}$ if and only if

$$\sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|} \int_I \|B(t) - m_I B\|^p dt\Big)^{1/p} < \infty$$

for some (or equivalently, for all) $0 < p < \infty$. Since $(B - m_I B)\chi_I = P_I B$, we can also say that $f \in \mathrm{BMO^d_{norm}}$ if and only if

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/p}} \|P_I(B)\|_{L^p(\mathcal{L}(\mathcal{H}))} < \infty.$$

One way to express the John-Nirenberg inequality on scalar-valued $\mathrm{BMO^d}$ is to say that the mapping

$$\mathrm{BMO^d} \to \mathrm{BMO^d}, \quad b \mapsto S_b, \tag{16}$$

is bounded. In the operator-valued setting, this John-Nirenberg property breaks down. Our main result is that any space of operator-valued functions which is contained in $\mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ and on which the mapping (16) acts boundedly is already contained in $\mathrm{BMO_{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

However, we find that (16) acts boundedly between different operator-valued BMO spaces. We also obtain the sharp rate of growth of the norm of the mapping (16) on $\mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$, $\mathrm{BMO_{para}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$, $\mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ and $\mathrm{BMO^d_{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ in terms of the dimension of $\mathcal{H}$.

Before establishing this dimensional growth, we consider an extension of the sweep. In the scalar case, one can extend the sweep $\mathrm{BMO^d} \to \mathrm{BMO^d}$ to a sesquilinear map $\Delta : \mathrm{BMO^d} \times \mathrm{BMO^d} \to \mathrm{BMO^d}$. This map is motivated by the consideration of "products of paraproducts" $\pi_f^*\pi_g$, which in turn is motivated by the long-standing investigation of products of Hankel operators $\Gamma_f^*\Gamma_g$ in the literature (see [PSm] and the references therein).

**Definition 3.1.** *Let us denote by $\Delta : \mathcal{F}_{00} \times \mathcal{F}_{00} \to L^1(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ the bilinear map given by*

$$\Delta(B,F) = \sum_{I\in\mathcal{D}} B_I^* F_I \frac{\chi_I}{|I|}.$$

*In particular $S_B = \Delta(B,B)$ and $\Delta(B,F)^* = \Delta(F,B)$.*

**Lemma 3.2.** *Let $B \in \mathcal{F}_{00}$. Then*

$$P_I\Delta(B,F) = P_I\Delta(B,P_IF) = P_I \sum_{J\subseteq I} \frac{\chi_J}{|J|} B_J^* F_J = P_I \sum_{J\subsetneq I} \frac{\chi_J}{|J|} B_J^* F_J.$$

*In particular, $P_I(S_B) = P_I(S_{P_IB}) = P_I(S_{(P_{I^+}+P_{I^-})B})$.*

*Proof.* $P_I\Delta(B^*, (F_J h_J)) = P_I(B_J^* F_J \frac{\chi_J}{|J|}) = 0$ if $I \subseteq J$. Hence

$$P_I\Delta(B,F) = P_I\Delta(B,P_IF) = P_I\Delta(B,(P_{I^+}+P_{I^-})F).$$

□

A similar proof as in Lemma 2.3 shows that

**Lemma 3.3.** *Let $B, F \in \mathcal{F}_{00}$. Then*

$$\pi_B^*\pi_F = \pi_{\Delta(B,F)} + \pi_{\Delta(F,B)}^* + D_{B,F} = \Lambda_{\Delta(B,F)} + D_{B,F},$$

*where $D_{B,F}$ is defined by $D_{B,F}(h_I \otimes x) = h_I \frac{1}{|I|} \sum_{J\subsetneq I} B_J^* F_J x$ for $x \in \mathcal{H}$, $I \in \mathcal{D}$. Moreover, $\|D_{B,F}\| \leq \sup_{\|e\|=1} \|B_e\|_{BMO(\mathcal{H})} \sup_{\|e\|=1} \|F_e\|_{BMO(\mathcal{H})}$.*

Let us now study the boundedness of the sesquilinear map $\Delta$ in the various BMO norms. Again, the properties of the map $\Delta$ are more subtle in the operator-valued case than in the scalar case.

**Theorem 3.4.** *There exists a constant $C > 0$ such that for $B, F \in \mathcal{F}_{00}$,*
*(i)* $\|\Delta(B,F)\|_{\mathrm{BMO_{mult}}} \leq C\|B\|_{\mathrm{BMO_{para}}}\|F\|_{\mathrm{BMO_{para}}}$,
*(ii)* $\|\Delta(B,F)\|_{\mathrm{WBMO^d}} \leq C\|B\|_{\mathrm{SBMO^d}}\|F\|_{\mathrm{SBMO^d}}$
*(iii)* $\|\Delta(B,F)\|_{\mathrm{SBMO^d}} \leq C\|\pi_B\|\|F\|_{\mathrm{SBMO^d}}$.

*Proof.* (i) follows from Lemma 3.3.

(ii) Using Lemma 3.2, one obtains

$$\langle P_I\Delta(B,F)e, f\rangle = P_I \sum_{J\in\mathcal{D}} \langle (P_IF)_J e, (P_IB)_J f\rangle \frac{\chi_J}{|J|}$$

for $e, f \in \mathcal{H}$. Therefore,

$$\begin{aligned}
\|\langle P_I\Delta(B,F)e, f\rangle\|_{L^1} &= \|P_I \sum_{J\in\mathcal{D}} \langle (P_IF)_J e, (P_IB)_J f\rangle \frac{\chi_J}{|J|}\|_{L^1} \\
&\leq 2\|\sum_{J\in\mathcal{D}} \langle (P_IF)_J e, (P_IB)_J f\rangle \frac{\chi_J}{|J|}\|_{L^1} \\
&\leq 2\|(\sum_{J\in\mathcal{D}} \|(P_IB)_J f\|^2 \frac{\chi_J}{|J|})^{1/2}\|_{L^2}\|(\sum_{J\in\mathcal{D}} \|(P_IF)_J e\|^2 \frac{\chi_J}{|J|})^{1/2}\|_{L^2} \\
&\leq 2(\sum_{J\in\mathcal{D}} \|(P_IB)_J f\|^2)^{1/2}(\sum_{J\in\mathcal{D}} \|(P_IF)_J e\|^2)^{1/2}.
\end{aligned}$$

Thus if $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} = \|F\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} = 1$, then

$$\|\langle P_I \Delta(B,F)e, f\rangle\|_{L^1} \le 2\|P_I B_f\|_{L^2(\mathcal{H})}\|P_I F_e\|_{L^2(\mathcal{H})} \le 2|I|.$$

This, again using John-Nirenberg's lemma, gives $\|\Delta(B,F)\|_{\mathrm{WBMO}^{\mathrm{d}}(\mathcal{L}(\mathcal{H}))} \le C$.

(iii) From Lemma 3.2, we obtain

$$\|P_I \Delta(B,F)e\|_{L^2(\mathcal{H})} = \|\Delta_{B^*}(P_I F_e)\|_{L^2(\mathcal{H})} \le \|\pi_B\|\|P_I F_e\|_{L^2(\mathcal{H})}.$$

□

Here comes the main result of this section.

**Theorem 3.5.** *Let $\mathcal{H}$ be a separable, finite or infinite-dimensional Hilbert space. Let $\rho$ be a positive homogeneous functional on the space $\mathcal{F}_{00}$ of $\mathcal{L}(\mathcal{H})$-valued functions on $\mathbb{T}$ with finite formal Haar expansion such that there exists constants $c_1$, $c_2$ with*

(1) $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \le c_1\rho(B)$ *and*
(2) $\rho(S_B) \le c_2\rho(B)^2$ *for all* $B \in \mathcal{F}_{00}$.

*Then there exists a constant $C$, depending only on $c_1$ and $c_2$, such that $\|B\|_{\mathrm{BMO}_{\mathrm{para}}} \le C\rho(B)$ for all $B \in \mathcal{F}_{00}$.*

*Proof.* For $n \in \mathbb{N}$, let $E_n$ denote the subspace $\{f \in L^2(\mathbb{T},\mathcal{H}) : f_I = 0 \text{ for } |I| < 2^{-n}\}$ of $L^2(\mathbb{T},\mathcal{H})$. Let $c(n) = \sup\{\|\pi_B\|_{E_n} : \rho(B) \le 1\}$. An elementary estimate shows that $c(n)$ is well-defined and finite for each $n \in \mathbb{N}$. For $\varepsilon > 0$, $n \in \mathbb{N}$, we can find $f \in E_n$, $\|f\| = 1$, $B \in \mathcal{F}_{00}$, $\rho(B) \le 1$ such that

$$\begin{aligned} c(n)^2(1-\varepsilon)^2 \le \|\pi_B f\|^2 &= \langle \pi_{S_B} f, f\rangle + \langle f, \pi_{S_B} f\rangle + \langle D_B f, f\rangle \\ &\le 2c(n)\rho(S_B) + c_1\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \le 2c_2 c(n) + c_1. \end{aligned}$$

It follows that the sequence $(c(n))_{n\in\mathbb{N}}$ is bounded by $C = c_2 + \sqrt{c_2^2 + c_1}$, and therefore $\|\pi_B\| \le C\rho(B)$ for all $B \in \mathcal{F}_{00}$. □

One immediate consequence is the following answer to Question 5.1 in [GPTV1].

**Theorem 3.6.** *There exists an absolute constant $C > 0$ such that for each $n \in \mathbb{N}$ and each measurable function $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$,*

$$\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \le C\log(n+1)\|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}}, \tag{17}$$

*and this is sharp.*

*Proof.* From (iii) in Theorem 3.4 one obtains:

$$\|S_B\|_{\mathrm{BMO}_{\mathrm{so}}} \le C\|B\|_{\mathrm{BMO}_{\mathrm{para}}}\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \le C\log(n+1)\|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}},$$

since there exists an absolute constant $C > 0$ with

$$\|B\|_{\mathrm{BMO}_{\mathrm{para}}} \le C\log(n+1)\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}}$$

by [K] and [NTV]. On the other hand, denoting by $C_n$ the smallest constant such that

$$\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \le C_n\|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}},$$

for each integrable function $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$, we obtain from Theorem 3.5 that

$$\|B\|_{\mathrm{BMO}_{\mathrm{para}}} \le (C_n + \sqrt{C_n^2 + 1})\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \le 3C_n\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}}$$

for each integrable $B$. It was shown in [NPiTV] that there exists an absolute constant $c > 0$ such that for each $n \in \mathbb{N}$, there exists $B^{(n)} : \mathbb{T} \to \mathrm{Mat}(n \times n, \mathbb{C})$ such that $\|B^{(n)}\|_{\mathrm{BMO_{para}}} \geq \log(n+1) c \|B^{(n)}\|_{\mathrm{BMO^d_{so}}}$. Therefore $C_n \geq \frac{c}{3} \log(n+1)$, and (17) is sharp. □

Sharp rates of dimensional growth can also be determined for $S_B$ in $\mathrm{BMO^d_{norm}}$, $\mathrm{BMO_{para}}$ and $\mathrm{BMO_{mult}}$. Interestingly, the rate of growth for $\mathrm{BMO^d_{so}}$ and $\mathrm{BMO_{para}}$ is slower than the one for $\mathrm{BMO_{mult}}$ and $\mathrm{BMO^d_{norm}}$.

**Theorem 3.7.** *There exists an absolute constant $C > 0$ such that for each $n \in \mathbb{N}$ and each measurable function $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$,*

$$\|S_B\|_{\mathrm{BMO_{para}}} \leq C \log(n+1) \|B\|^2_{\mathrm{BMO_{para}}}, \tag{18}$$

$$\|S_B\|_{\mathrm{BMO_{mult}}} \leq C (\log(n+1))^2 \|B\|^2_{\mathrm{BMO_{mult}}}, \tag{19}$$

$$\|S_B\|_{\mathrm{BMO^d_{norm}}} \leq C (\log(n+1))^2 \|B\|^2_{\mathrm{BMO^d_{norm}}}, \tag{20}$$

*and this is sharp.*

*Corresponding estimates also hold for the sesquilinear map $\Delta$.*

*Proof.* This is contained in [BPo2]. □

Finally, the following corollary to Theorem 3.5 gives an estimate of $\|\cdot\|_{\mathrm{BMO_{para}}}$ in terms of $\|\cdot\|_{\mathrm{SBMO^d}}$ with an "imposed" John-Nirenberg property. We need some notation: Let $S_B^{(0)} = B$ and let $S_B^{(n)} = S_{S^{(n-1)}B}$ for $n \in \mathbb{N}$, $B \in \mathcal{F}_{00}$.

**Corollary 3.8.** *There exists a constant $C > 0$ such that*

$$\|B\|_{\mathrm{BMO_{para}}} \leq C \sup_{n \geq 0} \|S_B^{(n)}\|^{1/2^n}_{\mathrm{SBMO^d}} \qquad (B \in \mathcal{F}_{00}).$$

*Proof.* Define $\rho(B) = \sup_{n \geq 0} \|S_B^{(n)}\|^{1/2^n}_{\mathrm{SBMO^d}}$. One sees easily that this expression is finite for $B \in \mathcal{F}_{00}$. Now apply Theorem 3.5. □

## 4. Averages over martingale transforms and operator-valued BMO

Let $\Sigma = \{-1, 1\}^{\mathcal{D}}$, and let $d\sigma$ denote the natural product probability measure on $\Sigma$, which assigns measure $2^{-n}$ to cylinder sets of length $n$.

For $\sigma \in \{-1, 1\}^{\mathcal{D}}$, define the *dyadic martingale transform*

$$T_\sigma : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T}, \mathcal{H}), \qquad f = \sum_{I \in \mathcal{D}} h_I f_I \mapsto \sum_{I \in \mathcal{D}} h_I \sigma_I f_I, \tag{21}$$

Given a Banach space $X$ and $F \in L^1(\mathbb{T}, X)$, we write $\tilde{F}$ for the function defined a.e. on $\Sigma \times \mathbb{T}$ by

$$\tilde{F}(\sigma, t) = T_\sigma F(t) = \sum_I \sigma_I F_I h_I(t).$$

In case that $X$ is a Hilbert space, $\|T_\sigma F\|_{L^2(\mathbb{T},X)} = \|F\|_{L^2(\mathbb{T},X)}$ for any $(\sigma_I)_{I \in \mathcal{D}}$, and therefore $\|\tilde{F}\|_{L^\infty(\Sigma, L^2(\mathbb{T},X))} = \|F\|_{L^2(\mathbb{T},X)}$. More generally, we have for UMD spaces that $\|T_\sigma F\|_{L^2(\mathbb{T},X)} \approx \|F\|_{L^2(\mathbb{T},X)}$. However, $X = \mathcal{L}(\mathcal{H})$ is not a UMD space, unless $\mathcal{H}$ is finite dimensional.

Whilst $\|B\|_{\mathrm{BMO_{para}}}$ cannot be estimated in terms of $\|B\|_{\mathrm{BMO_{mult}}}$ [Me2], we will prove an estimate of $\|B\|_{\mathrm{BMO_{para}}}$ in terms of an average of $\|T_\sigma B\|_{\mathrm{BMO_{mult}}}$ over $\Sigma$. Similarly, whilst the result in [Me2] implies that $\|S_B\|_{\mathrm{BMO^d_{norm}}}$ cannot be estimated in terms of $\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$, we will prove an estimate of $\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$ in terms of an average of $\|T_\sigma B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}$ over $\Sigma$. For this, the following representation of the sweep will be useful:

$$S_B(t) = \int_\Sigma (T_\sigma B)^*(t)(T_\sigma B)(t)d\sigma. \tag{22}$$

**Theorem 4.1.** *Let $B \in \mathcal{F}_{00}$. Then*

$$\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} \lesssim \int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma.$$

*In particular $\|B\|^2_{\mathrm{BMO}_{\mathrm{para}}} \lesssim \int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma$.*

*Proof.* The first inequality follows from the estimate

$$\begin{aligned}
& \|P_I S_B\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} \\
= \; & \|P_I S_{P_I B}\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} \leq 2 \left\| \int_\Sigma (T_\sigma P_I B^*)(T_\sigma P_I B) d\sigma \right\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} \\
\leq \; & 2 \int_\Sigma \|(P_I T_\sigma B)^* P_I T_\sigma B\|_{L^1(\mathbb{T},\mathcal{L}(\mathcal{H}))} d\sigma = 2 \int_\Sigma \|(P_I T_\sigma B)\|^2_{L^2(\mathbb{T},\mathcal{L}(\mathcal{H}))} d\sigma \\
\leq \; & 2|I| \int_\Sigma \|T_\sigma B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} d\sigma.
\end{aligned}$$

Using John-Nirenberg's lemma for $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$, one concludes the result. The second inequality follows from the first, (12) and Theorem 2.4. □

We are going to describe the different operator-valued BMO spaces in terms of "average boundedness" of certain operators. First we see that the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}$-norm can be described by "average boundedness" of $\Lambda_B$.

**Theorem 4.2.** *Let $B \in \mathcal{F}_{00}$, and let $\Phi_B$ be the map*

$$\Phi_B : L^2(\mathbb{T}, \mathcal{H}) \to L^2(\mathbb{T} \times \Sigma, \mathcal{H}), \quad f \mapsto \Lambda_B T_\sigma f.$$

*Then*

$$\|\Phi_B\| = \sup_{\|f\|_{L^2(\mathcal{H})}=1} \left(\int_\Sigma \|\Lambda_B(T_\sigma f)\|^2_{L^2(\mathbb{T},\mathcal{H})} d\sigma\right)^{1/2} = \|B\|_{\mathrm{SBMO}^{\mathrm{d}}}.$$

*In particular, $\|B\|_{\mathrm{BMO}_{\mathrm{so}}} = \|\Phi_B\| + \|\Phi_{B^*}\|$.*

*Proof.* Since $\Lambda_B(T_\sigma f) = \sum_{I\in\mathcal{D}} P_I(B) f_I h_I \sigma_I$, we have

$$\begin{aligned}
& \int_\Sigma \int_\mathbb{T} \|(\Phi_B f)(t,\sigma)\|^2 dt d\sigma \\
= \; & \int_\Sigma \int_\mathbb{T} \|(\Lambda_B T_\sigma f)(t)\|^2 dt d\sigma = \sum_{I\in\mathcal{D}} \|P_I(B) f_I h_I\|^2_{L^2(\mathcal{H})} \\
= \; & \sum_{I\in\mathcal{D}} \frac{1}{|I|} \int_I \|(B(t) - m_I B)(\frac{f_I}{\|f_I\|})\|^2 \|f_I\|^2 dt \leq \sup_{\|e\|=1} \|B_e\|^2_{BMO(\mathcal{H})} \sum_{J\in\mathcal{D}} \|f_J\|^2.
\end{aligned}$$

The reverse inequality follows by considering functions $f = h_I e$, where $e \in \mathcal{H}$, $I \in \mathcal{D}$. □

We require a further technical lemma, which shows that the $L^2$ norm of $\tilde{B}f$ can be decomposed in a certain way.

**Lemma 4.3.** *Let $B \in \mathcal{F}_{00}$ and $f \in L^2(\mathbb{T}, \mathcal{H})$. Write $Bf = \pi_B f + \Delta_B f + \gamma_B f$. Then*

$$\|\tilde{B}f\|^2_{L^2(\Sigma\times\mathbb{T},\mathcal{H})} = \int_\Sigma \|\pi_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma + \int_\Sigma \|\Delta_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma + \int_\Sigma \|\gamma_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma$$

*and*

$$\|\Lambda_{\tilde{B}} f\|^2_{L^2(\Sigma\times\mathbb{T},\mathcal{H})} = \int_\Sigma \|\pi_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma + \int_\Sigma \|\Delta_{T_\sigma B}(f)\|^2_{L^2(\mathcal{H})} d\sigma. \tag{23}$$

*Proof.* Observe that $m_I(T_\sigma B)h_I = (\sum_{I\subsetneq J} \sigma_J B_J h_J)h_I$. Hence

$$\gamma_{T_\sigma B}(f) = \sum_{I\in\mathcal{D}} m_I(T_\sigma B)(f_I)h_I = \sum_{J\in\mathcal{D}} \sigma_J B_J (\sum_{I\subsetneq J} f_I h_I) h_J.$$

This shows that

$$\int_\mathbb{T}\int_\Sigma \langle \pi_{T_\sigma B} f, \gamma_{T_\sigma B} g\rangle d\sigma dt = \sum_{I\in\mathcal{D}} \int_I \langle B_I m_I f, B_I(\sum_{J\subsetneq I} g_J h_J)\rangle \frac{\chi_I}{|I|} dt = 0$$

$$\int_\mathbb{T}\int_\Sigma \langle \gamma_{T_\sigma B} f, \Delta_{T_\sigma B} g\rangle d\sigma dt = \sum_{I\in\mathcal{D}} \int_I \langle B_I(\sum_{J\subsetneq I} f_J h_J), B_I g_I\rangle \frac{h_I}{|I|} dt = 0$$

$$\int_\mathbb{T}\int_\Sigma \langle \pi_{T_\sigma B} f, \Delta_{T_\sigma B} g\rangle d\sigma dt = \sum_{I\in\mathcal{D}} \int_I \langle B_I m_I f, B_I g_I\rangle \frac{h_I}{|I|} dt = 0.$$

To finish the proof, simply expand $\|\tilde{B}(f)\|^2_{L^2(\Sigma\times\mathbb{T},\mathcal{H})}$ and $\|\Lambda_{\tilde{B}}(f)\|^2_{L^2(\Sigma\times\mathbb{T},\mathcal{H})}$. □

Here is our desired estimate of $\|B\|_{\mathrm{BMO}_{\mathrm{para}}} + \|B^*\|_{\mathrm{BMO}_{\mathrm{para}}}$ in terms of an average over $\|\tilde{B}\|_{\mathrm{BMO}_{\mathrm{mult}}}$.

**Corollary 4.4.** *Let $B \in \mathcal{F}_{00}$. Then*

$$\frac{1}{2}(\|\pi_B\| + \|\Delta_B\|) \le \|\tilde{B}\|_{L^2(\Sigma,\mathrm{BMO}_{\mathrm{mult}})} \le \|\pi_B\| + \|\Delta_B\|.$$

*Proof.* To show the first estimate, it is sufficient to use (23) in Lemma 4.3, the identity $\|\Delta_B\| = \|\pi_{B^*}\|$ and the invariance of the right hand side under passing to the adjoint $B^*$.

For the reverse estimate, note that

$$\int_\Sigma \|\tilde{B}\|^2_{\mathrm{BMO}_{\mathrm{mult}}} d\sigma \le \int_\Sigma (\|\Delta_{T_\sigma B}\| + \|\pi_{T_\sigma B}\|)^2 d\sigma = (\|\Delta_B\| + \|\pi_B\|)^2.$$

□

## 5. Acknowledgement

We thank V. Paulsen for a helpful discussion on operator space structures. We also thank Tao Mei for his personal communication of a preliminary version of [Me2].

## References


[B1] O. Blasco, *Remarks on operator-valued BMO spaces*, Rev. Uni. Mat. Argentina. **345** (2004), 63-78.

[BPo] O. Blasco, S. Pott *Dyadic BMO on the bidisk*, Rev. Mat. Iberoamericana **21** 2,(2005), 483-510.

[BPo2] O. Blasco, S. Pott *Embeddings between operator-valued dyadic BMO spaces*, Preprint

[G] A. M. Garsia, *Martingale inequalities: Seminar Notes on recent progress*, Benjamin, Reading, 1973.

[GPTV1] T.A. Gillespie, S. Pott, S. Treil, A. Volberg, *Logarithmic growth for matrix martingale transforms*, J. London Math. Soc. (2) 64 (2001), no. 3, 624-636.

[GPTV2] T.A. Gillespie, S. Pott, S. Treil, A. Volberg, *Logarithmic growth for matrix Hilbert transform*,

[JPa] B. Jacob, J. R. Partington, *The Weiss conjecture on admissibility of observation operators for contraction semigroups*, Integral Equations and Operator Theory 40 (2001), 231–243

[JPaP] B. Jacob, J. R. Partington, S. Pott, *Admissible and weakly admissible observation operators for the right shift semigroup*, Proc. Edinb. Math. Soc. (2) **45**(2002), no. 2, 353-362

[K] N. H. Katz, *Matrix valued paraproducts*, J. Fourier Anal. Appl. **300** (1997), 913-921

[Me1] T. Mei, *Operator valued Hardy spaces*, to appear in MAMS 2006

[Me2] T. Mei, *Notes on Matrix Valued Paraproducts* Indiana Univ. Math. J. **55** 2,(2006), 747-760.

[M] Y. Meyer, *Wavelets and operators* Cambridge Univ. Press, Cambridge, 1992.

[NTV] F. Nazarov, S. Treil, A. Volberg, *Counterexample to the infinite dimensional Carleson embedding theorem*, C. R. Acad. Sci. Paris **325** (1997), 383-389.

[NPiTV] F. Nazarov, G. Pisier, S. Treil, A. Volberg, *Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts*, J. Reine Angew. Math. **542** (2002), 147-171

[Per] M.C. Pereyra, Lecture notes on dyadic harmonic analysis. Second Summer School in Analysis and Mathematical Physics (Cuernavaca, 2000), 1–60, Contemp. Math. **289**, Amer. Math. Soc., Providence, RI, 2001.

[Pet] S. Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C. R. Acad. Sci. Paris Sr. I Math. **330** (2000), no. 6, 455-460

[PV] G. Pisier and A. Volberg, personal communication

[PXu] G. Pisier and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Physics, 189 (1997) 667-698

[PS] S. Pott, C. Sadosky, *Bounded mean oscillation on the bidisk and Operator BMO*, J. Funct. Anal. **189**(2002), 475-495

[PSm] S. Pott, M. Smith, *Vector paraproducts and Hankel operators of Schatten class via p-John-Nirenberg theorem*, J. Funct. Anal. **217**(2004), no. 1, 38–78.



Department of Mathematics, Universitat de Valencia, Burjassot 46100 (Valencia) Spain

*E-mail address*: oscar.blasco@uv.es

Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW

*E-mail address*: sp@maths.gla.ac.uk