ON OPERATOR VALUED SEQUENCES OF MULTIPLIERS AND R-BOUNDEDNESS.

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ABSTRACT. In recent papers (cf [3], [4],[5], [20]) the concept of (p, q)-summing multiplier was considered in both general and special context. It has been shown that some geometric properties of Banach spaces and some classical theorems can be described using spaces of (p, q)-summing multipliers. The present paper is a continuation of this study, whereby multiplier spaces for some classical Banach spaces are considered. The scope of this research is also broaden, by studying other classes of summing multipliers. Generally spoken, a sequence of bounded linear operators $(u_n) \subset \mathcal{L}(X, Y)$ is called a *multiplier sequence* from $E(X)$ to $F(Y)$ if $(u_n x_n) \in F(Y)$ for all $(x_i) \in E(X)$, whereby $E(X)$ and $F(Y)$ are two Banach spaces whose elements are sequences of vectors in X and Y, respectively. Several cases where $E(X)$ and $F(Y)$ are different (classical) spaces of sequences, including for instance the spaces $Rad(X)$ of almost unconditionally summable sequences in X, are considered. Several examples, properties and relations among spaces of summing multipliers are discussed. Important concepts like R-bounded, semi-R-bounded and weak-R-bounded from recent papers are also considered in this context.

1. INTRODUCTION.

Let X and Y be two real or complex Banach spaces and let $E(X)$ and $F(Y)$ be two Banach spaces whose elements are sequences of vectors in X and Y (containing all eventually null sequence in X or Y), respectively. A sequence of operators $(u_n) \in \mathcal{L}(X, Y)$ is called a *multiplier sequence* from $E(X)$ to $F(Y)$ if there exists a constant $C > 0$ such that

$$\left\|(u_j x_j)_{j=1}^n\right\|_{F(Y)} \leq C\left\|(x_j)_{j=1}^n\right\|_{E(X)}$$

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for all finite families $x_1, \dots, x_n$ in X.

The set of all multiplier sequences from $E(X)$ to $F(Y)$ is denoted by $(E(X), F(Y))$. The reader is referred to [1] where $(E(X), F(Y))$ is considered in the setting of spaces of distributions. We refer to [7, 8, 10, 9, 13] for the case of vector-valued Hardy and BMO spaces $E(X) = H^1(\mathbb{T}, X)$ and $F(Y) = \ell_p(Y)$ or $F(Y) = BMOA(\mathbb{T}, Y)$, to [2] for the case $E(X) = B_p(X)$ and $F(Y) = B_q(Y)$ or $F(Y) = \ell_q(Y)$ where $B_p(X)$ stands for vector-valued Bergman spaces and to [11] for the case $E(X) = Bloch(X)$ and $F(Y) = \ell_q(Y)$. Also, the cases $E(X) = Rad(X)$ and $F(Y) = Rad(Y)$, were introduced by E. Berkson and T.A. Gillespie [6] and used for different purposes.

In the papers [4, 12] the cases $E(X) = \ell_p^w(X)$ and $F(Y) = \ell_p(Y)$ where considered (see also [3]). These spaces are defined as follows. Given a real or complex Banach space X and $1 \leq p \leq \infty$, we denote by $\ell_p(X)$, $\ell_p^w(X)$ and $\ell_p\langle X\rangle$ the Banach spaces of sequences in X, which are endowed with the norms $\|(x_n)\|_{\ell_p(X)} = \|(\|x_n\|)\|_{\ell_p}$,

$$\varepsilon_p((x_j)) = \sup\{\|(x^*x_j)\|_{\ell_p} : x^* \in X^*,\ \|x^*\| \leq 1\} \text{ and}$$

$$\|(x_j)\|_{\langle p\rangle} = \sup\{\|(x_j^*x_j)\|_{\ell_1} : \varepsilon_{p'}((x_j^*)) = 1\}, \text{ respectively.}$$

The space $\ell_p\langle X\rangle$ was first introduced in [16] and recently it has been described in different ways (see [3] for a description as the space of integral operators from $\ell_{p'}$ into X or [15] and [20] for the identification with the projective tensor product $\ell_p\hat{\otimes}X$).

We recall some basic notions in Banach space theory. Following standard notation, $\mathcal{L}(X, Y)$ will denote the space of bounded linear operators between Banach spaces X and Y, B_X denotes the unit ball in X and by (e_j) we denote the canonical basis of the classical sequence spaces ℓ_p $(1 \leq p < \infty)$ and c_0. For $1 \leq p < \infty$, p' will be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and (e_j^*) will sometimes be used to denote the canonical basis of $(\ell_p)^* = \ell_{p'}$ for $1 < p < \infty$ and $c_0^* = \ell_1$ to distinct between the standard bases of the classical sequence space and its dual space. $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$ if no difference is relevant. Sequences in Banach spaces are denoted by (x_i), (y_i), etc. and

$$(x_i)(\leq n) := (x_1, x_2, \dots, x_n, 0, 0 \dots).$$

For $1 \leq q \leq p < \infty$, the space $\Pi_{p,q}(X, Y)$ of *(p,q)-summing operators* is the vector space of those operators which map sequences in $\ell_q^w(X)$ onto sequences in $\ell_p(Y)$; more precisely, $u \in \mathcal{L}(X, Y)$ is in $\Pi_{p,q}(X, Y)$ if there exists $C > 0$ such that

$$\|(ux_j)\|_{\ell_p(Y)} \leq C\epsilon_q((x_j))$$

for all finite family of vectors x_j in X; the least (meaning, infimum) of such $C > 0$ is called the (p,q)-summing norm of u and is denoted by $\pi_{p,q}(u)$. Thus, $u \in \Pi_{p,q}(X,Y) \iff \hat{u} : \ell_q^w(X) \to \ell_p(Y) :: (x_i) \mapsto (ux_i)$ is a bounded linear operator. Usually, (p,p)-summing is called p-summing and 1-summing operators are also called *absolutely summing*, because for a 1-summing operator $u \in \mathcal{L}(X,Y)$ we have that $\sum ux_j$ is absolutely convergent in Y for every unconditionally convergent series $\sum x_j$ in X.

Grothendieck's theorem, in this setting, says that, for any measure space (Ω,μ) and any Hilbert space H, $\mathcal{L}(L^1(\mu),H) = \Pi_1(L^1(\mu),H)$. Because of this, a Banach space X is called a GT- space, i.e. X satisfies the Grothendieck theorem, if $\mathcal{L}(X,\ell_2) = \Pi_1(X,\ell_2)$ (see [25], page 71).

For each $1 \le p \le \infty$, we denote by $\mathrm{Rad}_p(X)$ the space of sequences (x_n) in X such that

$$\|(x_n)\|_{R_p} = sup_{n\in\mathbb{N}} \|\sum_{j=1}^{n} r_j x_j\|_{L^p([0,1],X)} < \infty,$$

where $(r_j)_{j\in\mathbb{N}}$ are the Rademacher functions on $[0,1]$ defined by $r_j(t) = sign(\sin 2^j \pi t)$.

The reader is referred to [26, 19, 27] for the difference between this space and the space of sequences (x_n) for which the series $\sum_{n=1}^{\infty} x_n r_n$ is convergent in $L^p([0,1],X)$. It is easy to see that $\mathrm{Rad}_\infty(X)$ coincides with $\ell_1^w(X)$.

Making use of the Kahane's inequalities (see [19], page 211) it follows that the spaces $\mathrm{Rad}_p(X)$ coincide up to equivalent norms for all $1 \le p < \infty$. The unique vector space so obtained, will therefore be denoted by $\mathrm{Rad}(X)$, and we agree to (mostly) use the norm $\|\cdot\|_{R_2}$ on $Rad(X)$.

We recall the fundamentals on type and cotype. For $1 \le p \le 2$ (respectively, $q \ge 2$), a Banach space X is said to have *(Rademacher) type* p (respectively, *(Rademacher) cotype* q) if there exists a constant $C > 0$ such that

$$\int_0^1 ||\sum_{j=1}^{n} x_j r_j(t)||dt \le C\big(\sum_{j=1}^{n} \|x_j\|^p\big)^{1/p}$$

(respectively,

$$\big(\sum_{j=1}^{n} \|x_j\|^q\big)^{1/q} \le C \int_0^1 ||\sum_{j=1}^{n} x_j r_j(t)||dt)$$

for any finite family $x_1, x_2, \dots x_n$ of vectors in X. Furthermore, a Banach space X is said to have the *Orlicz property* if there exists a constant $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \|x_j\|^2\Big)^{1/2} \leq C \sup_{t\in[0,1]} \Big|\Big|\sum_{j=1}^{n} x_j r_j(t)\Big|\Big|$$

for any finite family $x_1, x_2, \dots x_n$ of vectors in X.

The basic theory of p-summing and (p,q)-summing operators, type and cotype can be found, for example, in the books [18, 19, 23, 25, 26, 28].

In this paper we shall consider some connections between different notions of sequences of operators.

Definition 1.1. *(see* [4], [12]*) Let X and Y be Banach spaces, and let $1 \leq p, q \leq \infty$. A sequence $(u_j)_{j\in\mathbb{N}}$ of operators in $\mathcal{L}(X,Y)$ is called a (p,q)-summing multiplier, if there exists a constant $C > 0$ such that, for any finite collection of vectors $x_1, x_2, \dots x_n$ in X, it holds that*

$$\Big(\sum_{j=1}^{n} \|u_j x_j\|^p\Big)^{1/p} \leq C \sup\Big\{\Big(\sum_{j=1}^{n} |x^* x_j|^q\Big)^{1/q} \ :\ x^* \in B_{X^*}\Big\}.$$

The vector space of all (p,q)-summing multipliers from X into Y is denoted by $(\ell_q^w(X), \ell_p(Y))$. Note that the constant sequence $u_j = u$ for all $j \in \mathbb{N}$ belonging to $(\ell_q^w(X), \ell_p(Y))$, corresponds to u being an operator in $\Pi_{p,q}(X,Y)$. Also the case $(u_j) = (\lambda_j.u) \in (\ell_q^w(X), \ell_1(Y))$ for all $(\lambda_j) \in \ell_{p'}$, where $(1/p) + (1/p') = 1$, corresponds to $u \in \Pi_{p,q}(X,Y)$. These facts suggest the use of the notation $\ell_{\pi_{p,q}}(X,Y)$ instead of $(\ell_q^w(X), \ell_p(Y))$ and $\ell_{\pi_p}(X,Y)$ for the case $q = p$.

In the recent paper [3], J.L. Arregui and O. Blasco have considered the previous notion for $Y = \mathbb{K}$ and have shown that some geometric properties on X can be described using $\ell_{\pi_{p,q}}(X,\mathbb{K})$ and also that classical theorems, like Grothendieck theorem and others, can be rephrased into this setting. Some results on the spaces $\ell_{\pi_{p,q}}(X,Y)$ can be found in [12] and [4]. The reader is also referred to [5, 20] for the particular case $p = q$, $X = Y$ and $u_j = \alpha_j Id_X$. In these papers a scalar sequence (α_j) is defined to be a p-summing multiplier if $(u_j) = (\alpha_j Id_X)$ belongs to $\ell_{\pi_{p,q}}(X,Y)$.

In Section 2 we summarize some (recent) results on (p,q)-summing multipliers and discuss some examples of (p,q)-summing multipliers on classical Banach spaces. We extend the idea of (p,q)-summing multiplier to other families of multiplier sequences from $E(X)$ to $F(Y)$, considering some well known and important Banach spaces of vector valued sequences in place of $E(X)$ and $F(Y)$. Some duality results with application to spaces of operators are also considered.

In Section 3, we study R-bounded sequences and other variants thereof, like for instance, semi-R-bounded and weak-R-bounded sequences in Banach spaces. Relations of several types of sequences of bounded linear operators (like R-bounded, weak-R-bounded, semi-R-bounded, uniformly bounded, unconditionally bounded and almost summing) are studied. These relations build on well known results on type and cotype and characterizations of different families of operators.

2. (p,q)-SUMMING MULTIPLIERS.

We refer to Definition 1.1 for the definition of (p,q)-summing multiplier. Some easy examples can be constructed by taking tensor products of some elements in classical spaces.

Proposition 2.1. *(see [4]) Let X and Y be Banach spaces, and $1 \leq p,q \leq \infty$.*

(1) $\ell_{\pi_{r,q}}(X,\mathbb{K})\hat{\otimes}\ell_s(Y) \subset \ell_{\pi_{p,q}}(X,Y)$ *for* $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.
(2) $\ell_s\hat{\otimes}\Pi_{r,q}(X,Y) \subset \ell_{\pi_{p,q}}(X,Y)$ *for* $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. *In particular* $\ell_p\hat{\otimes}X \subset \ell_{\pi_{1,p'}}(X) = \ell_p\langle X\rangle$. *Moreover,* $\ell_p\hat{\otimes}X = \ell_p\langle X\rangle$ *isometrically (different proofs of this fact are discussed in [20] and [15]).*
(3) $\ell_s(Y)\hat{\otimes}X^* \subset \ell_{\pi_{p,q}}(X,Y)$ *for* $p<q$ *and* $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$.

In particular, notice that

Remark 2.1. *Let $p,q,s \geq 1$ be real numbers such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$.*

(i) *If $p<q$, $x^* \in X^*$ and $(y_n) \in \ell_s(Y)$ then $(u_n) = (x^* \otimes y_n) \in \ell_{\pi_{p,q}}(X,Y)$.*
(ii) *If $(\lambda_n) \in \ell_s$ and $u \in \Pi_{r,q}(X,Y)$, then $(u_n) = (\lambda_n u) \in \ell_{\pi_{p,q}}(X,Y)$.*

We consider some (elementary) examples:

Example 2.1. *Let K be a compact set and μ a probability measure on the Borel sets of K. Let $1 \leq p < q < \infty$, $1/r = 1/p - 1/q$ and (ϕ_j) a sequence of continuous functions on K. Consider $u_j : C(K) \to L^p(\mu)$ given by $u_j(\psi) = \phi_j\psi$. Then $(u_j) \in \ell_{\pi_{p,q}}(C(K), L^p(\mu))$ if and only if*

$$(\sum_j |\phi_j|^r)^{1/r} \in L^p(\mu).$$

Example 2.2. *Let (Ω,Σ,μ) and (Ω',Σ',μ') be finite measure spaces. Let $1 \leq p \leq q < \infty$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. For each $n \in \mathbb{N}$, let $f_n \in L^p(\mu, L^1(\mu'))$ and consider the operator $u_n : L^\infty(\mu') \to L^p(\mu)$, defined by*

$$u_n(\phi)(\cdot) = \int_{\Omega'} \phi(\omega')f_n(\cdot)(\omega')\ d\mu'(\omega').$$

Put $f_n(\cdot,\omega') = f_n(\cdot)(\omega')$ *and* $(\sum_{k=1}^n |f_k|^r)^{\frac{1}{r}}(\omega)(\cdot) = (\sum_{k=1}^n |f_k(\omega,\cdot)|^r)^{\frac{1}{r}}$. *Then,* $(\sum_{k=1}^n |f_k|^r)^{\frac{1}{r}} \in L^p(\mu, L^1(\mu')) \implies (u_n) \in \ell_{\pi_{p,q}}(L^\infty(\mu'), L^p(\mu))$.

Proof. Given $n \in \mathbb{N}$ and $\phi_1, \phi_2, \cdots, \phi_n \in L^\infty(\mu')$, then

$$\sum_{k=1}^n \|u_k(\phi_k)\|^p_{L^p(\mu)} = \int_\Omega \|(\int_{\Omega'} \phi_k(\omega')f_k(\omega,\omega')d\mu'(\omega'))_{k\le n}\|^p_{\ell_p}\ d\mu(\omega)$$

$$\le \int_\Omega(\int_{\Omega'} \|(\phi_k(\omega')f_k(\omega,\omega'))_{k\le n}\|_{\ell_p}\ d\mu'(\omega'))^p\ d\mu(\omega)$$

$$\le \|(\sum_{k=1}^n |\phi_k(\cdot)|^q)^{\frac{1}{q}}\|^p_{L^\infty(\mu')} \int_\Omega(\int_{\Omega'}(\sum_{k=1}^n |f_k(\omega,\omega')|^r)^{\frac{1}{r}} d\mu'(\omega'))^p\ d\mu(\omega).$$

Hence, since $\|(\phi_n)\|_{\ell_q^w(L^\infty(\mu'))} = \|(\sum_{k=1}^n |\phi_k(\cdot)|^q)^{\frac{1}{q}}\|_{L^\infty(\mu')}$, it follows that

$$\pi_{p,q}((u_k)) \le \|(\sum_{k=1}^n |f_k(\omega,\omega')|^r)^{\frac{1}{r}}\|_{L^p(\mu,L^1(\mu'))}.$$

□

Example 2.3. *Let* $1 \le p \le q < \infty$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ *and* (A_n) *be a sequence of infinite matrices. Consider* $T_n \in L(c_0, \ell_p)$ *given by* $T_n((\lambda_k)) = (\sum_{k=1}^\infty A_n(k,j)\lambda_k)_j$. *If*

$$\sum_{k=1}^\infty \{\sum_{n=1}^\infty (\sum_{j=1}^\infty |A_n(k,j)|^p)^{\frac{r}{p}}\}^{\frac{1}{r}} < \infty \text{ then, } (T_n) \in \ell_{\pi_{p,q}}(c_0, \ell_p).$$

Proof. (T_n) is of the form $T_n = \sum_{k=1}^\infty e_k^* \otimes y_{n,k}$, where $y_{n,k} \in \ell_p$ is given by $y_{n,k} = (A_n(k,j))_j$. Using the usual Hölder type inequalities, one verifies easily for $(x_n) \subset c_0$ that

$$\sum_{n=1}^\infty \|T_n(x_n)\|^p \le \|(x_n)\|^p_{\ell_q^w(c_0)}[\sum_{k=1}^\infty(\sum_{n=1}^\infty \|y_{n,k}\|^r)^{\frac{1}{r}}]^p\,.$$

Therefore, we conclude that

$$(\sum_{n=1}^\infty \|T_n(x_n)\|^p)^{\frac{1}{p}} \le \|(x_n)\|_{\ell_q^w(c_0)} \sum_{k=1}^\infty\{\sum_{n=1}^\infty(\sum_{j=1}^\infty |A_n(k,j)|^p)^{\frac{r}{p}}\}^{\frac{1}{r}}.$$

□

Definition 2.2. *Let* X *and* Y *be Banach spaces, and let* $1 \le p, q \le \infty$. *A sequence* $(u_j)_{j\in\mathbb{N}}$ *of operators in* $\mathcal{L}(X,Y)$ *belongs to* $(\ell_q(X), \ell_p\langle Y\rangle)$, *if there exists a constant* $C > 0$ *such that*

$$\sum_{j=1}^n |< u_j x_j, y_j^* >| \le C\Big(\sum_{j=1}^n \|x_j\|^q\Big)^{1/q} \sup_{\|y\|=1}\Big(\sum_{j=1}^n |y_j^* y|^{p'}\Big)^{1/p'}$$

for all finite collections of vectors $x_1, x_2, \dots x_n$ *in* X *and* $y_1^*, y_2^*, \dots y_n^*$ *in* Y^*. *The infimum of the numbers* $C > 0$ *for which the inequality holds, is denoted by* $\|(u_i)\|_{(\ell_q(X),\ell_p\langle Y\rangle)}$.

Proposition 2.3. *Let* X *and* Y *be Banach spaces,* $1 \leq p, q \leq \infty$ *and let* $(u_j)_{j\in\mathbb{N}}$ *be a sequence of operators in* $\mathcal{L}(X, Y)$. *Then* $(u_j) \in (\ell_q(X), \ell_p\langle Y\rangle)$ *if and only if* $(u_j^*) \in \ell_{\pi_{q',p'}}(Y^*, X^*)$. *In this case*

$$\|(u_i)\|_{(\ell_q(X),\ell_p\langle Y\rangle)} = \pi_{q',p'}((u_i^*)).$$

Proof. Let $(u_j^*) \in \ell_{\pi_{q',p'}}(Y^*, X^*)$. If $x_1, \cdots, x_n$ is a finite set in X and if $(y_i^*) \in \ell_{p'}^w(Y^*)$, we have

$$\begin{aligned}\sum_{i=1}^n |\langle u_i x_i, y_i^*\rangle| &\leq (\sum_{i=1}^n \|u_i^*(y_i^*)\|^{q'})^{\frac{1}{q'}} (\sum_{i=1}^n \|x_i\|^q)^{\frac{1}{q}} \\ &\leq \pi_{q',p'}((u_i^*))\epsilon_{p'}((y_i^*))\|(x_i)\|_{\ell_q(X).}\end{aligned}$$

Taking the supremum over the unit ball in $\ell_{p'}^w(Y^*)$, we conclude that $(u_j) \in (\ell_q(X), \ell_p\langle Y\rangle)$ and $\|(u_i)\|_{(\ell_q(X),\ell_p\langle Y\rangle)} \leq \pi_{q',p'}((u_i^*))$.

Conversely, assume $(u_j) \in (\ell_q(X), \ell_p\langle Y\rangle)$. Let $y_1^*, \cdots, y_n^*$ be a finite set in Y^* and let $(x_i) \in \ell_q(X)$. It follows that

$$\begin{aligned}\sum_{i=1}^n |\langle u_i^* y_i^*, x_i\rangle| &\leq \|(u_i x_i)\|_{\langle p\rangle}\epsilon_{p'}((y_i^*)) \\ &\leq \|(u_i)\|_{(\ell_q(X),\ell_p\langle Y\rangle)}\|(x_i)\|_{\ell_q(X)}\epsilon_{p'}((y_i^*)).\end{aligned}$$

If we take the supremum over the unit ball in $\ell_q(X)$, we obtain $(u_i^*) \in \ell_{\pi_{q',p'}}(Y^*, X^*)$ and $\pi_{q',p'}((u_i^*)) \leq \|(u_i)\|_{(\ell_q(X),\ell_p\langle Y\rangle)}$. □

Example 2.4. *Let* μ *be a probability measure on* Ω. *Let* $1 \leq p < q < \infty$, $1/r = 1/p - 1/q$ *and* (ϕ_j) *a sequence of functions in* $L^{q'}(\mu)$. *Consider* $u_j : L^q(\mu) \to L^1(\mu)$ *given by* $u_j(\psi) = \phi_j\psi$. *Then*

$$(\sum_j |\phi_j|^r)^{1/r} \in L^{q'}(\mu) \implies (u_j) \in (\ell_q(L^q(\mu)), \ell_p\langle L^1(\mu)\rangle).$$

Proof. Let $\psi_1, \psi_2, ..., \psi_n \in L^q(\mu)$. Taking into account that $\ell_p\langle L^1(\mu)\rangle = \ell_p\hat{\otimes}L^1(\mu) = L^1(\mu, \ell_p)$, we have

$$\begin{aligned}\|(u_j\psi_j)\|_{\ell_p\langle L^1(\mu)\rangle} &= \|(\sum_{j=1}^n |\phi_j\psi_j|^p)^{1/p}\|_{L^1(\mu)} \\ &\le \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}(\sum_{j=1}^n |\psi_j|^q)^{1/q}\|_{L^1(\mu)} \\ &\le \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^{q'}(\mu)}(\sum_{j=1}^n \|\psi_j\|^q_{L^q(\mu)})^{1/q}.\end{aligned}$$

□

Remarks 2.1. (1) *Under the conditions of Example 2.4, we let $\nu_j : L^\infty(\mu) \to L^{q'}(\mu)$, be defined by $\nu_j(\chi) = \phi_j\chi$. Then $\nu_j = u_j^*$, $\forall j$ and Example 2.4 and Proposition 2.3 yield that $(\nu_j) \in \ell_{\pi_{q',p'}}(L^\infty(\mu), L^{q'}(\mu))$.*

(2) *Let $1 \le p, q < \infty$. If X is a Banach lattice and Y a Banach space, then we call an operator $u \in \mathcal{L}(X, Y)$ strongly $(p, q)-$concave (and write $u \in \mathcal{SC}_{p,q}(X, Y)$) if there exists a $c > 0$ such that for all $x_1, \cdots, x_n$ in X we have*

$$\|(ux_i)(i \le n)\|_{\langle p\rangle} \le c\ \|(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}\|_X.$$

The infimum of the numbers $c > 0$ such that the inequality holds for all choices of finite sets in X, is denoted by $\|u\|_{\mathcal{SC}_{p,q}}$.

$u \in \mathcal{L}(L^q(\mu), Y)$ is strongly (p, q)-concave iff there exists a $c > 0$ such that for all finite sets $\chi_1, \chi_2, \cdots, \chi_n$ in $L^q(\mu)$, we have

$$\begin{aligned}\|(u(\chi_i))(i \le n)\|_{\langle p\rangle} &\le c\ \|(\sum_{i=1}^n |\chi_i|^q)^{\frac{1}{q}}\|_{L^q(\mu)} \\ &= c\ (\sum_{i=1}^n \|\chi_i\|^q_{L^q(\mu)})^{\frac{1}{q}}.\end{aligned}$$

Thus it follows that $u \in \mathcal{L}(L^q(\mu), Y)$ is strongly $(p, q)-$concave iff the constant sequence $(u, u, \cdots)$ belongs to $(\ell_q(L^q(\mu)), \ell_p\langle Y\rangle)$ and moreover, $\|u\|_{\mathcal{SC}_{p,q}} = \|(u, u, \cdots)\|_{(\ell_q(L^q(\mu)),\ \ell_p\langle Y\rangle)}$. Proposition 2.3 tells us that this is the case iff

$$(u^*, u^*, \cdots) \in \ell_{\pi_{q',p'}}(Y^*, L^{q'}(\mu)) = (\ell^w_{p'}(Y^*), \ell_{q'}(L^{q'}(\mu))),$$

which corresponds to $u^* \in \Pi_{q',p'}(Y^*, L^{q'}(\mu))$.

We have thus proved that $u : L^q(\mu) \to Y$ is strongly $(p, q)-$concave iff $u^* : Y^* \to L^{q'}(\mu)$ is (q', p')-summing, with $\|u\|_{\mathcal{SC}_{p,q}} = \|u^*\|_{\pi_{q',p'}}$.

The following two examples are conclusions of Proposition 2.3 and ([12], Example 2.2, 2.3).

Example 2.5. *Let* (Ω, Σ, μ) *and* (Ω', Σ', μ') *be finite measure spaces and* $1 \le p < \infty$. *Let* $(f_n) \subset L^p(\mu, L^1(\mu'))$ *and consider the operator* $S_n : L^{p'}(\mu) \to L^1(\mu')$ *defined by*

$$S_n(g)(\cdot) = \int_\Omega g(\omega) f_n(\omega, \cdot)\, d\mu(\omega),$$

where, as before, we let $f_n(\omega, \cdot) := f_n(\omega)(\cdot)$. *If* $\sup_n |f_n| \in L^p(\mu, L^1(\mu'))$ *(where,* $\sup_n |f_n|(\omega)(\cdot) = \sup_n |f_n(\omega, \cdot)|$*), then* $(S_n) \in (\ell_{p'}(L^{p'}(\mu)), \ell_{p'}\langle L^1(\mu')\rangle)$.

Example 2.6. *Let* $1 \le p < \infty$ *and* (A_n) *be a sequence of matrices. Consider the bounded operator* $S_n : \ell_{p'} \to \ell_1$ *given by*

$$S_n((\xi_j)) = (\sum_{j=1}^{\infty} A_n(k, j)\xi_j)_k.$$

Then $(S_n) \in (\ell_\infty(\ell_{p'}), \ell_\infty\langle \ell_1\rangle)$ *if* $\sum_{k=1}^{\infty} \sup_n (\sum_{j=1}^{\infty} |A_n(k, j)|^p)^{\frac{1}{p}} < \infty$.

Definition 2.4. *Let* X *and* Y *be Banach spaces, and let* $1 \le p, q \le \infty$. *A sequence* $(u_j)_{j\in\mathbb{N}}$ *of operators in* $\mathcal{L}(X, Y)$ *belongs to* $(\ell_q^w(X), \ell_p\langle Y\rangle)$, *if there exists a constant* $C > 0$ *such that, for any finite collections of vectors* $x_1, x_2, \dots x_n$ *in* X *and* $y_1^*, y_2^*, \dots y_n^*$ *in* Y^*, *it holds that*

$$\sum_{j=1}^{n} | < u_j x_j, y_j^* > | \le C \sup_{\|x^*\|=1} \Big(\sum_{j=1}^{n} |x^* x_j|^q\Big)^{1/q} \sup_{\|y\|=1} \Big(\sum_{j=1}^{n} |y_j^* y|^{p'}\Big)^{1/p'}.$$

The infimum of all $C > 0$ *such that the inequality holds for all finite sets in* X *and* Y^*, *is denoted by* $\|(u_i)\|_{(\ell_q^w(X), \ell_p\langle Y\rangle)}$.

Proposition 2.5. *Let* X *and* Y *be Banach spaces,* $1 \le p, q \le \infty$ *and let* $(u_j)_{j\in\mathbb{N}}$ *be a sequence of operators in* $\mathcal{L}(X, Y)$. *Then* $(u_j) \in (\ell_q^w(X), \ell_p\langle Y\rangle)$ *if and only if* $(u_j^*) \in (\ell_{p'}^w(Y^*), \ell_{q'}\langle X^*\rangle)$ *and*

$$\|(u_i)\|_{(\ell_q^w(X), \ell_p\langle Y\rangle)} = \|(u_i^*)\|_{(\ell_{p'}^w(Y^*), \ell_{q'}\langle X^*\rangle)}.$$

Proof. Consider $(u_j^*) \in (\ell_{p'}^w(Y^*), \ell_{q'}\langle X^*\rangle)$ and let $x_1, x_2, \cdots, x_n \in X$. Verifying the inequalities

$$\begin{aligned}\sum_{i=1}^{n} |\langle u_i x_i, z_i^*\rangle| &\le \|(u_i^* z_i^*)(i \le n)\|_{\langle q'\rangle} \epsilon_q((x_i)(i \le n)) \\ &\le \|(u_i^*)\|_{(\ell_{p'}^w(Y^*), \ell_{q'}\langle X^*\rangle)} \epsilon_{p'}((z_i^*)(i \le n)) \epsilon_q((x_i)(i \le n)),\end{aligned}$$

for all $(z_i^*) \in \ell_{p'}^w(Y^*)$, one obtains that

$$\|(u_i x_i)(i \le n)\|_{\langle p\rangle} \le \|(u_i^*)\|_{(\ell_{p'}^w(Y^*), \ell_{q'}\langle X^*\rangle)} \epsilon_q((x_i)(i \le n))$$

and hence that $\|(u_i)\|_{(\ell_q^w(X),\ell_p\langle Y\rangle)} \leq \|(u_i^*)\|_{(\ell_{p'}^w(Y^*),\ell_{q'}\langle X^*\rangle)}$.

Conversely, take $(u_i) \in (\ell_q^w(X), \ell_p\langle Y\rangle)$. Let $y_1^*, \cdots, y_n^*$ be a finite set in Y^* and let $(x_i) \in B_{\ell_q^w(X)}$. Then

$$\sum_{i=1}^n |\langle x_i, u_i^* y_i^*\rangle| = \sum_{i=1}^n |\langle u_i x_i, y_i^*\rangle| \leq \|(u_i)\|_{(\ell_q^w(X),\ell_p\langle Y\rangle)} \epsilon_q((x_i)) \epsilon_{p'}((y_i^*)).$$

Taking the supremum over all sequences $(x_i) \in B_{\ell_q^w(X)}$, we conclude that $(u_i^*) \in (\ell_{p'}^w(Y^*), \ell_{q'}\langle X^*\rangle)$, $\|(u_i^*)\|_{(\ell_{p'}^w(Y^*),\ell_{q'}\langle X^*\rangle)} \leq \|(u_i)\|_{(\ell_q^w(X),\ell_p\langle Y\rangle)}$.

□

Example 2.7. *Let K be a compact set and μ a probability measure on the Borel sets of K. Let $1 \leq p < q < \infty$, $1/r = 1/p - 1/q$ and (ϕ_j) a sequence of continuous functions on K. Consider $u_j : C(K) \to L^1(\mu)$ given by $u_j(\psi) = \phi_j \psi$. Then*

$$(\sum_j |\phi_j|^r)^{1/r} \in L^{q'}(\mu) \implies (u_j) \in (\ell_q^w(C(K)), \ell_p\langle L^1(\mu)\rangle).$$

Proof. As in Example 2.4, if $\psi_1, \psi_2, ..., \psi_n \in C(K)$ we have

$$\|(u_j(\psi_j))_j\|_{\ell_p\langle L^1(\mu)\rangle} \leq \|(\sum_{j=1}^n |\phi_j|^r)^{1/r} (\sum_{j=1}^n |\psi_j|^q)^{1/q}\|_{L^1(\mu)}$$

$$\leq \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^1(\mu)} \sup_{t\in K} (\sum_{j=1}^n |\psi_j(t)|^q)^{1/q}$$

$$\leq \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^1(\mu)} \sup_{\|\nu\|_{M(K)}=1} (\sum_{j=1}^n | < \psi_j, \nu > |^q)^{1/q}$$

□

In the discussion above we restricted ourselves to the Banach spaces $(\ell_q^w(X), \ell_p(Y))$, $(\ell_q^w(X), \ell_p\langle Y\rangle)$ and $(\ell_q(X), \ell_p\langle Y\rangle)$; thus we considered special cases of the vector space $(E(X), F(X))$ of multiplier sequences – introduced in Section 1 – and defined suitable norms on them. Continuing in this fashion, we shall in the following section discuss the important concept of R-boundedness of sequences of operators and some related concepts in the setting of multiplier sequences.

3. R-bounded sequences

In this section we consider notions that have been shown to be relevant in some recent problems.

Definition 3.1. *(cf. [17] and [21]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \in \mathcal{L}(X,Y)$ is said to be* **Rademacher bounded i.e. R-bounded** *if there exists $C > 0$ such that*

$$(\int_0^1 \|\sum_{j=1}^n u_j(x_j)r_j(t)\|^2 dt)^{\frac{1}{2}} \leq C\ (\int_0^1 \|\sum_{j=1}^n x_j r_j(t)\|^2 dt)^{\frac{1}{2}}$$

for all finite collections $x_1, x_2, ..., x_n \in X$.

The space of R-bounded sequences of operators from X into Y is denoted by $R(X,Y)$ and $\|(u_j)\|_R$ denotes the infimum of the constants satisfying the previous inequality for all finite subsets of X. It is easy to see that $(\mathrm{Rad}(X,Y), \|(u_j)\|_R)$ is a Banach space which coincides with the multiplier space $(\mathrm{Rad}(X), \mathrm{Rad}(Y))$.

Definition 3.2. *(cf. [24]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \subset \mathcal{L}(X,Y)$ is called* **weakly Rademacher bounded, shortly WR-bounded** *if there exists a constant $C > 0$ such that for all finite collections $x_1, \cdots, x_n \in X$ and $y_1^*, \cdots, y_n^* \in Y^*$ we have*

$$\sum_{k=1}^n |\langle u_k x_k, y_k^* \rangle| \leq C(\int_0^1 \|\sum_{j=1}^n x_j r_j(t)\|^2\ dt)^{\frac{1}{2}}(\int_0^1 \|\sum_{j=1}^n y_j^* r_j(t)\|^2\ dt)^{\frac{1}{2}}.$$

The space of WR-bounded sequences in $\mathcal{L}(X,Y)$, is denoted by $WR(X,Y)$ and $\|(u_n)\|_{WR}$ is the infimum of the constants in the previous inequality, taken over all finite subsets of X and Y^*. Then $\|(u_n)\|_{WR}$ is a norm on $WR(X,Y)$, which is exactly the norm of the bilinear map $\mathrm{Rad}(X) \times \mathrm{Rad}(Y^*) \to \ell_1$ defined by $((x_k), (y_k^*)) \to (\langle u_k x_k, y_k^* \rangle)$.

Definition 3.3. *(cf. [12]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \in \mathcal{L}(X,Y)$ is said to be* **almost summing** *if there exists $C > 0$ such that for any finite set of vectors $\{x_1, \cdots, x_n\} \subset X$ we have*

$$(3.1)\quad (\int_0^1 \|\sum_{j=1}^n u_j(x_j)r_j(t)\|^2)^{1/2}\ dt \leq C \sup_{\|x^*\|=1} (\sum_{j=1}^n |\langle x^*, x_j \rangle|^2)^{\frac{1}{2}}.$$

(or, equivalently, $(u_j) \in \mathcal{L}(X,Y)$ is almost summing if there exists $C' > 0$ such that for any finite set of vectors $\{x_1, \cdots, x_n\} \subset X$ we have

$$\int_0^1 \|\sum_{j=1}^n u_j(x_j)r_j(t)\|\ dt \leq C' \sup_{\|x^*\|=1} (\sum_{j=1}^n |\langle x^*, x_j \rangle|^2)^{\frac{1}{2}}.)$$

We write $\ell_{\pi_{as}}(X,Y)$ for the space of almost summing sequences, which is endowed with the norm

$$\|(u_i)\|_{as} := \inf\{C > 0 \mid \text{such that (3.1) holds}\}.$$

Notice that $\ell_{\pi_{as}}(X,Y) = (\ell_2^w(X), Rad(Y))$. If the constant sequence $(u,u,u,\dots)$ is in $\ell_{\pi_{as}}(X,Y)$, then the operator u is called almost summing (see [19], page 234). The space of almost summing operators is denoted by $\Pi_{as}(X,Y)$ and the norm on this space is given by

$$\pi_{as}(u) = \|(u,u,u\dots)\|_{as} = \|\hat{u}\|,$$

where in this case $\hat{u} : \ell_2^w(X) \to Rad(Y)$ is given by $\hat{u}((x_j)) = (ux_j)$.

Definition 3.4. *(cf.* [24]*) Let X and Y be Banach spaces. A sequence of operators $(u_j) \in \mathcal{L}(X,Y)$ is called* **unconditionally bounded or U-bounded** *if there exists a constant $C > 0$ such that for all finite collections $x_1, \cdots, x_n \in X$ and $y_1^*, \cdots, y_n^* \in Y^*$ we have*

$$\sum_{k=1}^n |\langle u_k x_k, y_k^*\rangle| \le C \max_{\epsilon_k=\pm1} \|\sum_{k=1}^n \epsilon_k x_k\| \max_{\epsilon_k=\pm1} \|\sum_{k=1}^n \epsilon_k y_k^*\|.$$

We write $UR(X,Y)$ for the space of U-bounded sequences in $\mathcal{L}(X,Y)$. The space $UR(X,Y)$ is endowed with the norm $\|(u_n)\|_{UR}$, which is given by the infimum (taken over all finite subsets of X and Y^*) of the constants in the previous inequality.

Proposition 3.5. *Let X and Y be Banach spaces. The following inclusions hold.*

$$\ell_{\pi_{as}}(X,Y) \subseteq R(X,Y) \subseteq WR(X,Y) \subseteq UR(X,Y) \subseteq \ell_\infty(\mathcal{L}(X,Y)).$$

Proof. The inclusion $\ell_{\pi_{as}}(X,Y) \subseteq R(X,Y)$ is a trivial consequence of the embedding $Rad(X) \subseteq \ell_2^w(X)$.

Suppose $(u_i) \in R(X,Y)$. Orthogonality of the Rademacher variables, duality and the contraction principle, allow us to write

$$\begin{aligned}
&\sum_{k=1}^n |\langle u_k x_k, y_k^*\rangle| = \sup_{\epsilon_k=\pm1} \sum_{k=1}^n \langle u_k x_k, \epsilon_k y_k^*\rangle \\
= \quad & \sup_{\epsilon_k=\pm1} \int_0^1 \langle \sum_{k\le n} r_k(t) u_k x_k, \sum_{k\le n} r_k(t)\epsilon_k y_k\rangle \, dt \\
\le \quad & \sup_{\epsilon_k=\pm1} (\int_0^1 \|\sum_{k=1}^n u_k x_k r_k(t)\|^2 dt)^{1/2} (\int_0^1 \|\sum_{k=1}^n \epsilon_k y_k^* r_k(t)\|^2 dt)^{1/2} \\
\le \quad & \|(u_j)\|_R (\int_0^1 \|\sum_{k=1}^n x_k r_k(t)\|^2 dt)^{1/2} (\int_0^1 \|\sum_{k=1}^n y_k^* r_k(t)\|^2 dt)^{1/2}.
\end{aligned}$$

This proves the inclusion $R(X,Y) \subseteq WR(X,Y)$. The inclusion $WR(X,Y) \subseteq UR(X,Y)$ is clear from the definitions.

If $(u_n) \in UR(X,Y)$, then from the definition of unconditional boundedness there exists $C > 0$ such that for $x \in X, y^* \in Y^*$, we have

$$|\langle u_k x, y^* \rangle| \leq C\|x\|\|y^*\|$$

for all $k \in \mathbb{N}$. Thus the inclusion $UR(X,Y) \subseteq \ell_\infty(\mathcal{L}(X,Y))$ also follows. □

Remark 3.1. *If $u \in \mathcal{L}(X,Y)$ then $(u,u,\dots) \in R(X,Y)$ and $\|(u,u,\dots)\|_R = \|u\|$. However, $(u,u,\dots) \in \ell_{\pi_{as}}(X,Y)$ if and only if $u \in \Pi_{as}(X,Y)$. This shows that $\ell_{\pi_{as}}(X,Y) \subset R(X,Y)$ is strict.*

Recall that for $1 \leq p < \infty$, the p-convexity and p-concavity of $L^p(\mu)$ imply the following equivalence of norms:

$$\|(\phi_j)\|_{Rad(L^p(\mu))} \approx \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^p(\mu)}$$

for any collection $\phi_1, \phi_2, ..., \phi_n$ in $L^p(\mu)$ (cf [19], 16.11).

Also, if $X = C(K)$ for any compact set K or if $X = \ell_\infty$, then

$$\epsilon_p((\phi_j)) \approx \|(\sum_{j=1}^{n} |\phi_j|^p)^{1/p}\|_X$$

for all finite subsets $\phi_1, \phi_2, ..., \phi_n$ of X.

Therefore we have the following versions of Definitions 3.1, 3.2, 3.3 and 3.4 in some special cases:

Proposition 3.6. *(i) Let $X = C(K)$ and $Y = L^q(\nu)$ for $1 \leq q < \infty$. Then $(u_j) \in \ell_{\pi_{as}}(X,Y)$ if and only if there exists $C > 0$ such that*

$$\|(\sum_{j=1}^{n} |u_j(\phi_j)|^2)^{1/2}\|_{L^q(\nu)} \leq C\|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{C(K)}$$

for any finite collection $\phi_1, \phi_2, ..., \phi_n$ in $C(K)$.

(ii) Let $X = L^p(\mu)$ and $Y = L^q(\nu)$ for $1 \leq p, q < \infty$. Then $(u_j) \in R(X,Y)$ if and only if there exists $C > 0$ such that

$$\|(\sum_{j=1}^{n} |u_j(\phi_j)|^2)^{1/2}\|_{L^q(\nu)} \leq C\|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^p(\mu)}$$

for all finite collections $\phi_1, \phi_2, ..., \phi_n$ in $L^p(\mu)$.

(iii) Let $X = \ell_p$ and $Y = c_0$ for $1 \leq p < \infty$. Then $(u_j) \in WR(X,Y)$ if and only if there exists $C > 0$ such that

$$\sum_{j=1}^{n} |\langle u_j(\phi_j), \varphi_j \rangle| \leq C\|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{\ell_p}\|(\sum_{j=1}^{n} |\varphi_j|^2)^{1/2}\|_{\ell_1}$$

for all collections $\phi_1, \phi_2, ..., \phi_n$ *in* ℓ_p *and* $\varphi_1, \varphi_2, ..., \varphi_n$ *in* ℓ_1.

(iv) Let $X = \ell_\infty$ *and* $Y = \ell_1$. *Then* $(u_j) \in UR(X, Y)$ *if and only if there exists* $C > 0$ *such that*

$$\sum_{j=1}^{n} |\langle u_j(\phi_j), \varphi_j \rangle| \leq C \| (\sum_{j=1}^{n} |\phi_j|^2)^{1/2} \|_{\ell_\infty} \| (\sum_{j=1}^{n} |\varphi_j|^2)^{1/2} \|_{\ell_\infty}$$

for all finite collections $\phi_1, \phi_2, ..., \phi_n$ *and* $\varphi_1, \varphi_2, ..., \varphi_n$ *in* ℓ_∞.

Proposition 3.7. *Let* $2 \leq r \leq \infty$. *If* $u_j = \lambda_j u$ *for* $u \in \Pi_{as}(X, Y)$ *and* $(\lambda_j) \in \ell_r$ *then* $(u_j) \in (\ell_q^w(X), Rad(Y))$ *for* $1/q = 1/2 - 1/r$.

In particular, if $u \in \Pi_{as}(X, Y)$ *and* $(\lambda_j) \in \ell_\infty$ *then* $(u_j) = (\lambda_j u) \in \ell_{\pi_{as}}(X, Y)$.

Proof. From $u \in \Pi_{as}(X, Y)$, we have

$$\begin{aligned} & (\int_0^1 \| \sum_{j=1}^{n} u_j(x_j) r_j(t) \|^2 \, dt)^{1/2} \\ \leq \quad & \pi_{as}(u) \| (\lambda_j) \|_{\ell_r} \sup_{\|x^*\|=1} \Big(\sum_{j=1}^{n} |x^* x_j|^q \Big)^{1/q}. \end{aligned}$$

□

Remark 3.2. *We would like to point out that* $\cup_p \Pi_{p,p}(X, Y) \subset \Pi_{as}(X, Y)$ *(see* [19], *12.5). Nevertheless this is not the case for sequences of operators. Indeed, it suffices to take* $u_n = x^* \otimes y_n$ *for fixed* $x^* \in X^*$ *and* $(y_n) \in \ell_\infty(Y)$. *In this case,* (u_n) *belongs to* $\ell_{\pi_{2,2}}(X, Y)$, *but not to* $\ell_{\pi_{as}}(X, Y)$ *(consider for example* $Y = c_0$ *and* $y_n = e_n$ *the canonical basis).*

Proposition 3.8. *Let* Y *be a Banach space of type* $1 \leq p = p(Y)$ *and cotype* $q = q(Y) \leq \infty$. *Then* $\ell_{\pi_{p,2}}(X, Y) \subset \ell_{\pi_{as}}(X, Y) \subset \ell_{\pi_{q,2}}(X, Y)$.

In particular if Y *is a Hilbert space then* $\ell_{\pi_{2,2}}(X, Y) = \ell_{\pi_{as}}(X, Y)$.

Proof. It follows from the fact $\ell_p(Y) \subset Rad(Y) \subset \ell_q(Y)$. □

Let us mention that it was pointed out in ([24]) that if X has nontrivial type then $WR(X, X) = R(X, X)$. Actually the assumption only needs to be taken in the second space.

Recall that the notion of nontrivial type is equivalent to K-convexity (see [19], page 260). X is said to be K-convex if $f \to (\int_0^1 f(t) r_n(t) dt)_n$ defines a bounded operator from $L^p([0, 1])$ onto $Rad_p(X)$ for some (equivalently for all) $1 < p < \infty$.

For K-convex spaces one has $Rad(X^*) = Rad(X)^*$ (see [26], or [14] for more general systems).

Let us point out that this shows that there are no infinite dimensional K-convex GT-spaces of cotype 2. Indeed, assume X is K-convex and a GT-space of cotype 2. On the one hand $Rad(X) = \ell_2\langle X\rangle$ and on the other hand $Rad(X)^* = Rad(X^*)$ with equivalent norms. Therefore $Rad(X^*) = (\ell_2\langle X\rangle)^* = \ell_2^w(X^*)$. Hence the identity on X^* is almost summing and then X^* is finite dimensional.

It is well known that, in general, one can only expect $Rad(X^*)$ to be continuously embedded in $Rad(X)^*$, but that the embedding needs not even be isomorphically. Take, for instance, $X = \ell_1$. Then $Rad(\ell_1) = \ell_2\langle\ell_1\rangle = \ell_2\hat{\otimes}\ell_1$, that is to say $(x_n)_n \subset \ell_1$ (with $x_n = (x_n(k))_k$) belongs to $Rad(\ell_1)$ if and only if

$$\sum_{k\in\mathbb{N}}(\sum_{n\in\mathbb{N}}|x_n(k)|^2)^{1/2} < \infty.$$

As a matter of fact, it follows from earlier discussions that

$$Rad(\ell_1) = \ell_2\langle\ell_1\rangle = \ell_2\hat{\otimes}\ell_1 = \ell_1\hat{\otimes}\ell_2 = \ell_1\langle\ell_2\rangle = \ell_1(\ell_2).$$

Therefore $Rad(X)^*$ can be identified with $L(\ell_2, \ell_\infty)$ or with $\ell_\infty(\ell_2)$, and

$$\|(x_n^*)\|_{Rad(X)^*} = \sup_{k\in\mathbb{N}}(\sum_{n\in\mathbb{N}}|x_n^*(k)|^2)^{1/2}.$$

However

$$\|(x_n^*)\|_{Rad(X^*)} = \int_0^1 \sup_{k\in\mathbb{N}}|\sum_{n\in\mathbb{N}} x_n^*(k)r_n(t)|dt.$$

Proposition 3.9. *If Y is a K-convex space then $WR(X,Y) = R(X,Y)$.*

Proof. Let $(u_n) \in WR(X,Y)$ and let $x_i \in X$ for $i = 1, ..., n$. Using that $Rad(Y)^* = Rad(Y^*)$, we have

$$\begin{aligned} &(\int_0^1 \|\sum_{j=1}^n u_j(x_j)r_j(t)\|^2 dt)^{1/2} \\ \approx\ & \sup\{|\sum_{j=1}^n \langle u_j(x_j), y_j^*\rangle| : \|\sum_{j=1}^n y_j^* r_j\|_{L^2(Y^*)} \leq 1\} \\ \leq\ & \|(u_n)\|_{WR}\|\sum_{j=1}^n x_j r_j\|_{L^2(X)}. \end{aligned}$$

□

It is clear from the proof of Proposition 3.9 that $WR(X,Y) = R(X,Y)$ for all Banach spaces Y such that $Rad(Y)^* = Rad(Y^*)$.

For later use, we point out that

Lemma 3.10. *Let $1 \le p, q \le \infty$. For a sequence (u_j) in $\mathcal{L}(X,Y)$ we have $(u_j) \in \ell_{\pi_{p,q}}(X,Y)$ if and only if $F : \ell_q^w(X) \times \ell_{p'}(Y^*) \to \ell_1$ defined by $F((x_n),(y_n^*)) = (\langle u_n x_n, y_n^* \rangle)$ is a bounded bilinear operator. In this case $\|F\| = \pi_{p,q}((u_j))$.*

Theorem 3.11. *Let $1 \le p \le 2$.*
(i) If Y has type p then $\ell_{\pi_{p,2}}(X,Y) \subset \ell_{\pi_{as}}(X,Y)$.
(ii) If Y^* has cotype p' then $\ell_{\pi_{p,2}}(X,Y) \subset WR(X,Y)$.
(iii) If Y^* has cotype p' then $\ell_{\pi_{p,1}}(X,Y) \subset UR(X,Y)$.
(iv) If Y^* has the Orlicz property then $\ell_{\pi_{2,1}}(X,Y) \subset UR(X,Y)$.

Proof. (i) This follows from $\ell_p(Y) \subset Rad(Y)$.

(ii) Assume Y^* has cotype p'. Then $Rad(Y^*) \subset \ell_{p'}(Y^*)$ continuously, whereby $\|(y_i^*)\|_{\ell_{p'}(Y^*)} \le C_{p'}(Y^*)\|(y_i^*)\|_{Rad(Y^*)}$ and $C_{p'}(Y^*)$ is the cotype p' constant of Y^* (cf. [19]). Also, $Rad(X) \subset \ell_2^w(X)$, with $\epsilon_2((x_i)) \le \|(x_i)\|_{Rad(X)}$ (cf. [19], p. 234). Suppose $(u_j) \in \ell_{\pi_{p,2}}(X,Y)$. Then $F : \ell_2^w(X) \times \ell_{p'}(Y^*) \Rightarrow \ell_1 : ((x_n),(y_n^*)) \mapsto (\langle u_n x_n, y_n^* \rangle)$ is bounded with $\|F\| = \pi_{p,2}((u_i))$. Thus for all finite sets of elements $x_1, x_2, \cdots, x_n$ in X and $y_i^*, \cdots, y_n^*$ in Y^*, we have

$$\begin{aligned}\sum_{k=1}^{n} |\langle u_k x_k, y_k^* \rangle| &= \|F((x_i),\ (y_i^*))\| \\ &\le \pi_{p,2}((u_i)) C_{p'}(Y^*) \|(x_i)\|_{Rad(X)} \|(y_i^*)\|_{Rad(Y^*)}.\end{aligned}$$

(iii) Use Lemma 3.10 and the fact that Y^* of cotype p' gives $\ell_1^w(Y^*) \subset \ell_{p'}(Y^*)$.

(iv) Same argument as in the proof of (iii), now using that by the Orlicz property of Y^*, we have $\ell_1^w(Y^*) \subset \ell_2(Y^*)$. □

Theorem 3.12. *Let $1 \le p \le 2$.*
(i) If Y has cotype p' then $\ell_{\pi_{as}}(X,Y) \subset \ell_{\pi_{p',2}}(X,Y)$.
(ii) If Y has cotype p' then $R(X,Y) \subset \ell_{\pi_{p',1}}(X,Y)$.
(iii) If Y^* has type p then $WR(X,Y) \subset \ell_{\pi_{p',1}}(X,Y)$.

Remark 3.3. *Let $1 \le p \le 2 \le q \le \infty$ and denote by $C_q(X,Y)$ and $T_p(X,Y)$ the spaces of operators of cotype q and type p, that is*

$$C_q(X,Y) = \{u : X \to Y : (u_j)_j \in (Rad(X), \ell_q(Y)), u_j = u, j \in \mathbb{N}\}$$

and

$$T_p(X,Y) = \{u : X \to Y : (u_j)_j \in (\ell_p(X), Rad(Y)), u_j = u, j \in \mathbb{N}\}.$$

Let X and Y be Banach spaces.

(1) *If $(u_j) \in Rad(X,Y)$ and $u \in C_q(Y,Z)$ then $(uu_j) \in \ell_{\pi_{q,1}}(X,Z)$.*
(2) *If $(u_j) \in Rad(X,Y)$ and $u \in T_p(Z,X)$ then $(u_j u) \in (\ell_p(Z), \ell_2(Y))$.*

(3) *If* $(u_j) \in Rad(X,Y)$, $v \in C_q(Y,U)$ *and* $u \in \Pi_{as}(Z,X)$ *then* $(vu_ju) \in \ell_{\pi_{q,2}}(Z,U)$.

Theorem 3.13. *Let* $1 \leq p \leq 2$ *and* X *be a Banach space such that* X *has cotype* p', *let* Y *be a* GT*-space of cotype 2 and let* $u_j : X \to Y$ *be bounded linear operators for all* $j \in \mathbb{N}$. *Then*

$$(u_j^*) \in \ell_{\pi_{p,2}}(Y^*, X^*) \implies (u_j) \in R(X,Y).$$

Proof. Recall from Proposition 2.3 that $(u_j) \in (\ell_{p'}(X), \ell_2\langle Y\rangle)$. Since we can identify $Rad(Y)$ with $\ell_2\langle Y\rangle$ (see [3] and [20]), it follows that there exists a $C > 0$ such that

$$\begin{aligned}
&\int_0^1 \|\sum_{j=1}^n u_j(x_j)r_j(t)\|dt \leq \|(u_j(x_j))\|_{\ell_2\langle Y\rangle} \\
\leq\ & C\|(u_i)\|_{(\ell_{p'}(X),\ell_2\langle Y\rangle)}\|(x_j)\|_{p'} \\
\leq\ & K\int_0^1 \|\sum_{j=1}^n x_jr_j(t)\|dt, \text{ where, } K = C\|(u_i)\|_{(\ell_{p'}(X),\ell_2\langle Y\rangle)}C_{p'}(X).
\end{aligned}$$

□

Corollary 3.14. *Let* $1 \leq r \leq \infty$ *and* $u_j : L^r(\mu) \to L^1(\nu)$ *be bounded operators. If* $(u_j^*) \in \ell_{\pi_{p,2}}(L^\infty(\nu), L^{r'}(\mu))$ *for* $p = \min\{r', 2\}$, *then there exists* $C > 0$ *such that*

$$\|(\sum_{j=1}^n |u_j(\phi_j)|^2)^{1/2}\|_{L^1(\nu)} \leq C\|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{L^r(\mu)}$$

for any collection $\phi_1, \phi_2, ..., \phi_n$ *in* $L^r(\mu)$.

Another related notion is the following:

Definition 3.15. *(cf.* [22]*) Let* X *and* Y *be Banach spaces. A sequence of operators* $(u_j) \in \mathcal{L}(X,Y)$ *is said to be* **semi-R-bounded** *(i.e.* $(u_n) \in SR(X,Y)$*) if there exists* $C > 0$ *such that for every* $x \in X$ *and* $a_1, \cdots, a_n \in \mathbb{C}$ *we have*

$$(\int_0^1 \|\sum_{j=1}^n u_j(x)r_j(t)a_j\|^2\, dt)^{1/2} \leq C\ (\sum_{j=1}^n |a_j|^2)^{\frac{1}{2}}\|x\|. \tag{3.2}$$

$\|(u_i)\|_{SR} := \inf\{C > 0 \mid$ *such that (3.2) holds*$\}$ *is the norm on* $SR(X,Y)$.

It was observed (see [22], Prop 2.1) that $SR(X,X) = \ell_\infty(\mathcal{L}(X,X))$ if and only if X is of type 2. Note that R-boundedness of sequences in $\mathcal{L}(X,Y)$ implies semi-R-boundedness of the same. It is known that if X is a Hilbert space or X is a GT-space of cotype 2, then $SR(X,X) = R(X,X)$ (see [22] for a proof). The proof of this fact (in [22]) is however very much simplified in the context of multiplier sequences and basically follows from the following characterization of $SR(X, Y)$.

Theorem 3.16. *The space* $(SR(X, Y), \|.\|_{SR})$ *is isometrically isomorphic to the space* $(\ell_2\langle X\rangle, \mathrm{Rad}(Y))$.

Proof. Suppose $(u_n) \in SRad(X, Y)$ and $\{x_1, \cdots, x_n\} \subset X$. From [20] we know that $\|(x_i)\|_{\langle 2\rangle} = \|\sum_{i=1}^n e_i \otimes x_i\|_\wedge$ in $\ell_2\hat{\otimes}X$.

It is clear that if $(\lambda_i) \in \ell_2$ and $x \in X$ we have that $(\lambda_j u_j x) \in \mathrm{Rad}(Y)$ and $\|(\lambda_j u_j x)\|_{R_2} \leq \|(u_i)\|_{SR}\|(\lambda_i)\|_{\ell_2}\|x\|$. Hence $(0, 0, \cdots, 0, u_i x_i, 0, \cdots) = (\delta_{ij} u_j x_i)_j \in Rad(Y)$ and

$$\|(\delta_{ij} u_j x_i)_j\|_{R_2} \leq \|(u_i)\|_{SR}\|(\delta_{ij})_j\|_{\ell_2}\|x_i\| = \|(u_i)\|_{SR}\|x_i\|\|e_i\|_{\ell_2}.$$

Therefore, $(u_i x_i) = \sum_{i=1}^n (\delta_{ij} u_j x_i)_j \in \mathrm{Rad}(Y)$ and $\|(u_i x_i)\|_{R_2} \leq (\sum_{i=1}^n \|e_i\|_{\ell_2}\|x_i\|)\|(u_i)\|_{SR}$. By definition of the projective norm $\|\cdot\|_\wedge$ on $\ell_2\hat{\otimes}X$, we have

$$\|(u_i x_i)\|_{R_2} \leq \|\sum_{i=1}^n e_i \otimes x_i\|_\wedge \|(u_i)\|_{SR} = \|(x_i)\|_{\langle 2\rangle}\|(u_i)\|_{SR}.$$

This holds for all finite sets $\{x_1, \cdots, x_n\} \subset X$, showing that $(u_i) \in (\ell_2\langle X\rangle, \mathrm{Rad}(Y))$ and $\|(u_i)\|_{(\langle 2\rangle, R_2)} \leq \|(u_i)\|_{SR}$.
Conversely, suppose $(u_i) \in (\ell_2\langle X\rangle, \mathrm{Rad}(Y))$ and let $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ and $x \in X$. Then we have

$$\begin{aligned}
(\int_0^1 \|\sum_{i=1}^n r_i(t)\alpha_i u_i x\|^2 \, dt)^{\frac{1}{2}} &\leq \|(u_i)\|_{(\langle 2\rangle, R_2)}\|(\alpha_i x)\|_{\langle 2\rangle} \\
&\leq \|(u_i)\|_{(\langle 2\rangle, R_2)}(\sum_{i=1}^n |\alpha_i|^2)^{\frac{1}{2}}\|x\|.
\end{aligned}$$

Since this is true for all $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ and $x \in X$, it follows that $(u_i) \in SR(X, Y)$ and $\|(u_i)\|_{SR} \leq \|(u_i)\|_{(\langle 2\rangle, R_2)}$. □

It follows from the continuous inclusion $\ell_2\langle X\rangle \subset Rad(X)$ and Theorem 3.16, that $R(X, Y) \subseteq SR(X, Y)$ for all Banach spaces X and Y. The reader is referred to [22] (p. 380) for an example of a sequence of operators which is semi-R-bounded, but not R-bounded; indeed, the authors in [22] show that if (e_k^*) is the standard basis of $\ell_{q'}$ (where, $2 < q < \infty$) and $w = (\xi_i) \in \ell_q$ is fixed, then the uniformly bounded sequence of operators $(S_k) := (e_k^* \otimes w)$ in $\mathcal{L}(\ell_q, \ell_q)$ is not WR-bounded, whereas it is semi-R-bounded because ℓ_q has type 2.

The following proposition sheds more light on the question of when the equality $SR(X,Y) = R(X,Y)$ holds.

Proposition 3.17.

(i) *If X is a Grothendieck space of cotype 2, then $SR(X,Y) = R(X,Y)$ for all Banach spaces Y.*
(ii) *If for some Banach space Y (thus also for $Y = X$) the equality $SR(X,Y) = R(X,Y)$ holds, then X has cotype 2.*
(iii) *If X is a Hilbert space and Y is a Banach space of type 2, then $SR(X,Y) = R(X,Y)$.*

Proof. (i) This follows from Theorem 3.16 and the characterization of Grothendieck spaces of cotype 2 by $\ell_2\langle X\rangle = \ell_2\hat{\otimes}X = Rad(X)$.

(ii) We show that $SR(X,Y) = R(X,Y)$ implies that $Rad(X)$ is a linear subspace of $\ell_2(X)$. Consider $(x_i) \in Rad(X)$ and let $x_i^* \in X^*$, with $\|x_i^*\| = 1$ and $x_i^*(x_i) = \|x_i\|$. Put $u_i = x_i^* \otimes y$, where $y \in Y$ is fixed, with $\|y\| = 1$. Then, $(u_i) \in SR(X,Y) = (\ell_2\langle X\rangle, Rad(Y))$, because of

$$\begin{aligned}
\int_0^1 \|\sum_{i=1}^n u_i(z_i) r_i(t)\|^2 \, dt &= \int_0^1 \|\sum_{i=1}^n r_i(t) x_i^*(z_i) y\|^2 \, dt \\
&= \int_0^1 |\sum_{i=1}^n r_i(t) x_i^*(z_i)|^2 \, dt = \sum_{i=1}^n |x_i^*(z_i)|^2 \\
&\le \sum_{i=1}^\infty \|z_i\|^2 \le \|(z_i)\|^2_{\langle 2\rangle},
\end{aligned}$$

for all $(z_i) \in \ell_2\langle X\rangle \ \subset \ell_2(X)$. Hence, $(u_i) \in (Rad(X), Rad(Y))$. However, for all $n \in \mathbb{N}$, we also have

$$\begin{aligned}
\sum_{i=1}^n \|x_i\|^2 &= \int_0^1 |\sum_{i=1}^n r_i(t) \|x_i\||^2 \, dt \\
&= \int_0^1 \|\sum_{i=1}^n r_i(t) x_i^*(x_i) y\|^2 \, dt \\
&= \int_0^1 \|\sum_{i=1}^n r_i(t) u_i(x_i)\|^2 \, dt \,.
\end{aligned}$$

Therefore, it follows that

$$\sum_{i=1}^\infty \|x_i\|^2 \le \sup_n \int_0^1 \|\sum_{i=1}^n r_i(t) u_i(x_i)\|^2 \, dt < \infty \,,$$

showing that $Rad(X) \hookrightarrow \ell_2(X)$ is a norm ≤ 1 embedding.

(iii) Refer to Remarks 3.4 and 3.6 below, where a more general case is discussed. □

In the following few remarks, we analyse the relationship of $\ell_\infty(\mathcal{L}(X,Y))$ to the other families of multiplier sequences.

Remark 3.4. *(see for instance [12]) Let X be a Banach space of cotype q , Y be a Banach space of type p for some $1 \le p \le q \le \infty$ and r such that $1/r = 1/p - 1/q$. Then*

$$\ell_r(\mathcal{L}(X,Y)) \subset R(X,Y) \subset \ell_\infty(\mathcal{L}(X,Y)).$$

In particular, if X has cotype 2 and Y has type 2 then $R(X,Y) = \ell_\infty(\mathcal{L}(X,Y))$.

Remark 3.5. *If X and Y^* have the Orlicz property then $\ell_\infty(\mathcal{L}(X,Y)) = UR(X,Y)$.*

Proof. By Proposition 3.5 we only need to show that $\ell_\infty(\mathcal{L}(X,Y)) \subseteq UR(X,Y)$. Notice that the continuous inclusions $\ell_1^w(X) \subseteq \ell_2(X)$ and $\ell_1^w(Y^*) \subseteq \ell_2(Y^*)$correspond to the Orlicz properties of X and Y^*, respectively. Then, for $(u_n) \in \ell_\infty(\mathcal{L}(X,Y))$, we have

$$\begin{aligned} \sum_{k=1}^n |\langle u_k x_k, y_k^* \rangle| &\le \sum_{k=1}^n \|u_k\| \|x_k\| \|y_k^*\| \\ &\le (\sup_k \|u_k\|) (\sum_{k=1}^n \|x_k\|^2)^{1/2} (\sum_{k=1}^n \|y_k^*\|^2)^{1/2} \\ &\le C(\sup_k \|u_k\|) \max_{\epsilon_k = \pm 1} \| \sum_{k=1}^n \epsilon_k x_k \| \ \max_{\epsilon_k = \pm 1} \| \sum_{k=1}^n \epsilon_k y_k^* \|, \end{aligned}$$

where in the last step of the proof the existence of $C > 0$ such that the inequality holds, is a direct consequence of the inclusions mentioned in the first line of the proof.

□

Remark 3.6. *Let Y be a Banach space of type p for some $1 \le p \le 2$ and let $r \ge 1$ satisfy $1/r = 1/p - 1/2$. Then*

$$\ell^r(\mathcal{L}(X,Y)) \subset SR(X,Y) \subset \ell_\infty(\mathcal{L}(X,Y)).$$

In particular if Y has type 2, then $SR(X,Y) = \ell_\infty(\mathcal{L}(X,Y))$.

Proof. We prove the inclusion $\ell^r(\mathcal{L}(X,Y)) \subset SR(X,Y)$. There exists $C > 0$ such that

$$\begin{aligned}(\int_0^1 \|\sum_{j=1}^n u_j(x) r_j(t) a_j\|^2 \, dt)^{1/2} &\leq C(\sum_{j=1}^n \|u_j(a_j x)\|^p)^{1/p} \\ &\leq C\|x\| \|(u_j)\|_r (\sum_{j=1}^n |a_j|^2)^{1/2}.\end{aligned}$$

The other inclusion is immediate. □

Remark 3.7. *Neither $SR(X,Y) \subset WR(X,Y)$ nor $WR(X,Y) \subset SR(X,Y)$ is generally true. For instance, if Y has type 2, then $SR(X,Y) = \ell_\infty(\mathcal{L}(X,Y))$ and $WR(X,Y) = R(X,Y)$. So, $WR(X,Y) \subset SR(X,Y)$ for all X in this case. On the other hand, if we consider a GT space X space having cotype 2, then $SR(X,Y) = R(X,Y)$ for all Y (cf Proposition 3.17). So, in this case, $SR(X,Y) \subset WR(X,Y)$ for all Y.*

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