Lorentz spaces of vector-valued measures.

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Abstract

Given a non atomic, finite and complete measure space (Ω, Σ, μ) and a Banach space X, we define the modulus of continuity for a vector measure F as the function $\omega_F(t) = \sup_{\mu(E)\leq t}|F|(E)$ and introduce the space $V^{p,q}(X)$ of vector measures such that $t^{\frac{-1}{p'}}\omega_F(t) \in L^q((0,\mu(\Omega)], \frac{dt}{t})$. We show that $V^{p,q}(X)$ contains isometrically $L^{p,q}(X)$ and that $L^{p,q}(X) = V^{p,q}(X)$ if and only if X has the Radon-Nikodym property. We also prove that $V^{p,q}(X)$ coincides with the space of cone absolutely summing operators from $L^{p',q'}$ into X and the duality $V^{p,q}(X^*) = (L^{p',q'}(X))^*$ where $1/p + 1/p' = 1/q + 1/q' = 1$.

We finally identify $V^{p,q}(X)$ with the interpolation space obtained by real method $(V^1(X), V^\infty(X))_{1/p',q}$.

Spaces where the variation of F is replaced by the semivariation are also considered.

1 Introduction

Throughout this paper (Ω, Σ, μ) stands for a non atomic, finite and complete measure space, X will be a (complex or real) Banach space and X^* its topological dual space. As usual p' will denote the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Given a complex-valued measurable function f we shall denote by μ_f the distribution function of f, $\mu_f(\lambda) = \mu(E_\lambda)$ for $\lambda > 0$ where $E_\lambda = \{w \in \Omega :$

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$|f(w)| > \lambda\}$, by f^* the nonincreasing rearrangement of f, $f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}$ and we write $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

Then the Lorentz space $L^{p,q}$ consists of those measurable functions f such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left\{\dfrac{q}{p}\displaystyle\int_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & 0 < p \leq \infty,\ q = \infty. \end{cases}$$

Of course $L^{p,p} = L^p$ and if we put $\|f\|_{pq} = \|f^{**}\|^*_{pq}$ then we get an equivalent norm and $L^{p,q}$ for $1 < p \leq \infty$, $1 \leq q \leq \infty$ are Banach spaces.

Let us recall also that simple functions are dense in $L^{p,q}$ for $q \neq \infty$, and also the duality results, $(L^{p,1})^* = L^{p',\infty}$ for $1 \leq p < \infty$, as well as $(L^{p,q})^* = L^{p',q'}$ for $1 < p, q < \infty$.

The reader is referred to ([Hu], [BS], [SW], [L]) for these results and for the basic information on Lorentz spaces.

All these notions make sense also for vector-valued strongly measurable functions by replacing the modulus by the norm. This leads to the natural definition of the Lorentz-Bochner spaces $L^{p,q}(X)$, where the norm is defined by $\|f\|_{L^{p,q}(X)} = \|\|f(\cdot)\|_X\|_{pq}$.

In the vector-valued case we still have the density of simple functions, but the corresponding duality $(L^{p,q}(X))^* = L^{p',q'}(X^*)$ only holds for Banach spaces X such that X^* has the Radon-Nikodym property. The reader is referred to [DU] for a proof in the case $p = q$ or to [GU] (see Theorem 3.2 there) for a proof even in a more general case of Köthe-Bochner space $E(X)$ for certain Banach lattices including $L^{p,q}$. An identification of the dual space without assumptions on X can be achieved from some general results on the dual of $E(X)$ where E is a Banach lattice (see [B] to get a description in terms of weakly measurable functions or [GU] for a formulation in terms of vector measures).

Since $L^p(X)$ coincides with $L^{p,p}(X)$, let us first mention here that in the particular case of Lebesgue-Bochner spaces $L^p(X)$ the dual can be represented as the space of X^*-valued measures of p'-bounded variation, denoted by $V^{p'}(X^*)$ (see [D]).

Our objective will be to define a space of vector-valued measures in such a way that contains $L^{p,q}(X)$ isometrically and that coincides with $V^p(X)$ for $p = q$.

Following [GU] one could define $V^{p,q}(X)$ as the space of vector measures such that $\sup_{\pi\in D}\|\sum_{A\in\pi}\frac{F(A)}{\mu(A)}\chi_A\|_{pq}<\infty$ where the supremum is taken over the set D of all finite partitions π of Ω, but we would like to present a notion independent on the knowledge of Lorentz spaces of functions.

In this paper we shall present a natural definiton of a modulus of continuity of a vector meausure (see Definition 2.1) which will allow to define the space $V^{p,q}(X)$ independently of the notion of $L^{p,q}(X)$ and which extends the previous one for measures $dG = fd\mu$, and also will coincide with the one presented above (see Corollary 2.11).

In the case $q=\infty$, Marcinkiewicz spaces are denoted $\mathcal{V}^{p,\infty}(X)$ and $V^{p,\infty}(X)$ and defined by the existence of a constant $C>0$ for which $\|F(A)\|\leq C\mu(A)^{\frac{1}{p'}}$ or $|F|(A)\leq C\mu(A)^{\frac{1}{p'}}$ for all $A\in\Sigma$.

To deal with the case $q<\infty$, we define two different modulus of continuity for a vector measure, namely $\tilde{\omega}_F(t)=\sup_{\mu(E)\leq t}\|F(E)\|$ and $\omega_F(t)=\sup_{\mu(E)\leq t}|F|(E)$. Then we define the spaces $V^{p,q}(X)$ and $\mathcal{V}^{p,q}(X)$ consisting of vector measures such that $t^{\frac{-1}{p'}}\omega_F(t)\in L^q((0,\mu(\Omega)],\frac{dt}{t})$ and $x^*F\in V^{p,q}(\mathbb{K})$ for all $x^*\in X^*$ respectively. Also a space where ω_F is replaced by $\tilde{\omega}_F$ is considered.

The paper is divided into three sections.

In the first one, it is proved that $V^{p,q}(X)$ contains isometrically $L^{p,q}(X)$ and that $L^{p,q}(X)=V^{p,q}(X)$ if and only if X has the Radon-Nikodym property. It is also shown that $V^{p,q}(X^*)$ coincides with the dual of $L^{p',q'}(X)$. Section three deals with identification of the previous spaces of vector-valued measures as spaces of operators. In particular we show that $\mathcal{V}^{p,q}(X)$ and $V^{p,q}(X)$ can be described as spaces of bounded operators from $L^{p',q'}$ into X and cone absolutely summing ones respectively.

In the last section we describe the space as an interpolation space obtained by interpolation, using the real method, of two natural spaces of vector measures, namely $V^{p,q}(X)=(V^1(X),V^\infty(X))_{1/p',q}$ where $V^1(X)$ corresponds to the space of μ-continuous measures of bounded variation and $V^\infty(X)$ the subspace of those measures such that $\|F(A\|\leq C\mu(A)$ for all $A\in\Sigma$.

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2 Marcinkiewicz and Lorentz spaces of vector measures

Let us recall that the variation of a vector measure $F : \Sigma \to X$ at the set E is given by $|F|(E) = \sup_{\pi_E} \sum_{A\in\pi_E} \|F(A)\|$ (π_E stands for a finite partition of E and the supremum is taken over all such partitions) and the semivariation is given by $\|F\|(A) = \sup_{||x^*||=1} |x^*F|(A)$. It is worth mentioning (see [DU]) that

$$\|F\|(A) \approx \sup_{B\subset A} \|F(B)\| \tag{1}$$

Let us first introduce the following notion of "modulus of continuity of a vector measure".

Definition 2.1 *Let (Ω, Σ, μ) be a non atomic finite measure space and write $I = (0, \mu(\Omega)]$. Let X be a Banach space and let F be an X-valued measure, we define, for $t \in I$, the functions*

$$\widetilde{\omega}_F(t) = \sup_{\mu(E)\le t} \|F(E)\| \quad \textit{and} \quad \omega_F(t) = \sup_{\mu(E)\le t} |F|(E)$$

Remark 2.1 *(a) Taking into account that μ is non atomic, and using (1) one easily sees that for all $t \in I$,*

$$\omega_F(t) = \sup_{\mu(E)=t} |F|(E) \quad \textit{and} \quad \widetilde{\omega}_F(t) \approx \sup_{\mu(E)=t} \|F\|(E).$$

(b) $F \ll \mu$ if and only if $\lim_{t\to 0} \widetilde{\omega}_F(t) = 0$ and, for vector measures of bounded variation, it is also equivalent to $\lim_{t\to 0} \omega_F(t) = 0$.

(c) $\omega_F \equiv +\infty$ if and only if there exists $t \in I$ such that $\omega_F(t) = +\infty$ or, equivalently, F is not of bounded variation.

Proposition 2.2 *If $f \in L^1(X)$ and $F(E) = \int_E f d\mu$ then $tf^{**}(t) = \omega_F(t)$ for all $t \in I$.*

PROOF. The result follows easily from the facts $|F|(E) = \int_E \|f\| d\mu$ and $\int_0^t f^*(s)ds = \sup_{\mu(E)\le t} \int_E \|f\|d\mu$. ■

Proposition 2.3 *Let F be a vector measure. Then either $\omega_F \equiv +\infty$ on I or ω_F is nondecreasing, continuous and concave.*

PROOF. Let us assume $\omega_F(t) < +\infty$ for some $t \in I$. Hence F has bounded variation and clearly ω_F is nondecreasing.

Let us see first that

$$\omega_F(s+h) - \omega_F(s) \geq \omega_F(t+h) - \omega_F(t) \tag{2}$$

for all $0 < s < t < \mu(\Omega)$ and $0 < h < \mu(\Omega) - t$.

Indeed, given $\varepsilon > 0$ and $t, t+h, s \in I$, there exist measurable sets E_t, E_{t+h} and E_s for which $\mu(E_t) = t, \mu(E_{t+h}) = t+h$ and $\mu(E_s) = s$ and

$$\omega_F(t) - |F|(E_t) < \varepsilon, \qquad \omega_F(t+h) - |F|(E_{t+h}) < \varepsilon, \qquad \omega_F(s) - |F|(E_s) < \varepsilon.$$

Let A_h be a measurable set with $\mu(A_h) = h$ and $A_h \subset E_{t+h} \setminus E_s$. Then

$$\begin{aligned}
\omega_F(s+h) &\geq |F|(E_s \cup A_h) = |F|(E_s) + |F|(A_h) \\
&\geq \omega_F(s) - \varepsilon + |F|(A_h) + |F|(E_{t+h} \setminus A_h) - \omega_F(t) \\
&\geq \omega_F(s) - \varepsilon + |F|(E_{t+h}) - \omega_F(t) \\
&\geq \omega_F(s) - \varepsilon + \omega_F(t+h) - \varepsilon - \omega_F(t).
\end{aligned}$$

As long as it is valid for any $\varepsilon > 0$ we get (2).

Therefore for any $s, t \in I$ one gets

$$|\omega_F(s) - \omega_F(t)| \leq \omega_F(|s-t|^+) - \omega_F(0^+)$$

where $\omega_F(a^+) = \lim_{h \to 0^+} \omega_F(a+h)$.

Hence ω_F is uniformly continuous in I.

It is easy to see from (2) that, for $s, t \in I$,

$$\omega_F\left(\frac{s+t}{2}\right) \geq \frac{\omega_F(s) + \omega_F(t)}{2}.$$

This fact, together with the continuity, gives the concavity of ω_F. ■

A measure F is said to belong to $V^p(X)$ (respect. $\mathcal{V}^p(X)$) if

$$\|F\|_p = \sup_{\pi \in D} \left(\sum_{A \in \pi} \frac{\|F(A)\|^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p}} < \infty,$$

$$(\text{ respect. } |||F|||_p = \sup_{\pi \in D, ||x^*||=1} \left(\sum_{A \in \pi} \frac{|x^* F(A)|^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p}} < \infty),$$

where the supremum is taken over the set D of all finite partitions π of Ω.

In particular, if $F \in V^p(X)$ then $\|F(A)\| \leq C\mu(A)^{\frac{1}{p'}}$ for all measurable set A. This gives us the way to define $weak - V^p(X)$.

Definition 2.4 *Let (Ω, Σ, μ) be a measure space, X a Banach space and $1 < p \leq \infty$. Let us define by $\mathcal{V}^{p,\infty}(\mu, X)$ and $V^{p,\infty}(\mu, X)$ the spaces of X-valued measures for which there exists a constant $C > 0$ such that*

$$\|F(A)\| \leq C\mu(A)^{\frac{1}{p'}} \text{ for any } A \in \Sigma \quad (\text{equiv. } \tilde{\omega}_F(t) \leq Ct^{1/p'})$$

and

$$|F|(A) \leq C\mu(A)^{\frac{1}{p'}} \text{ for any } A \in \Sigma \quad (\text{equiv. } \omega_F(t) \leq Ct^{1/p'})$$

respectively.

We shall consider in these spaces the norms

$$\|F\|_{\mathcal{V}^{p,\infty}(X)} = \sup_{A \in \Sigma} \frac{\|F(A)\|}{\mu(A)^{\frac{1}{p'}}} = \sup_{t \in I} t^{\frac{-1}{p'}} \tilde{\omega}_F(t)$$

$$\|F\|_{V^{p,\infty}(X)} = \sup_{A \in \Sigma} \frac{|F|(A)}{\mu(A)^{\frac{1}{p'}}} = \sup_{t \in I} t^{\frac{-1}{p'}} \omega_F(t).$$

We shall use the notation $\|\cdot\|_{p,\infty}$ when the context is not ambiguous keeping the notation $\|\cdot\|_{p\infty}$ for vector-valued functions.

Remark 2.2 *(a) For $p = \infty$ we clearly have $V^{\infty,\infty}(X) = \mathcal{V}^{\infty,\infty}(X)$ and it coincides with $V^\infty(X)$ (see [D]).*

(b) Observe that, since $1 < p \leq \infty$ and $\mu(\Omega) < \infty$, measures in $\mathcal{V}^{p,\infty}(\mu, X)$ are μ-continuous, and measures in $V^{p,\infty}(\mu, X)$ are of bounded variation.

(c) It is clear that, using (1), $F \in \mathcal{V}^{p,\infty}(\mu, X)$ if and only if

$$\|F\|(A) \leq C\mu(A)^{\frac{1}{p'}} \text{ for any } A \in \Sigma$$

if and only if

$$x^*F \in V^{p,\infty}(\mathbb{K}), \text{ for all } x^* \in X^*.$$

There are many ways to define spaces of vector measures which extend the Lorentz spaces. We shall try to get a natural way to define $V^{p,q}(X)$, in such a way that the map $f \mapsto dG = fd\mu$ defines an isometric embedding from $L^{p,q}(X)$ and that the space coincides with $V^p(X)$ and $V^{p,\infty}$ for $p = q$ and $q = \infty$ respectively.

Definition 2.5 *Let X be a Banach space, $1 < p < \infty$ and $1 \leq q \leq \infty$. Let us define $V^{p,q}(\mu, X)$ and $\widetilde{\mathcal{V}}^{p,q}(\mu, X)$ as the space of vector measures such that*

$$t^{\frac{-1}{p'}} \omega_F(t) \in L^q(I, \frac{dt}{t}) \quad \text{and} \quad t^{\frac{-1}{p'}} \widetilde{\omega}_F(t) \in L^q(I, \frac{dt}{t})$$

respectively. We consider then the norms

$$\|F\|_{V^{p,q}(X)} = \Big(\int_I (t^{\frac{-1}{p'}} \omega_F(t))^q \frac{dt}{t} \Big)^{1/q}$$

and

$$\|F\|_{\widetilde{\mathcal{V}}^{p,q}(X)} = \Big(\int_I (t^{\frac{-1}{p'}} \widetilde{\omega}_F(t))^q \frac{dt}{t} \Big)^{1/q}.$$

We shall use the notation $\|\cdot\|_{p,q}$ when the context is not ambiguous keeping the notation $\|\cdot\|_{pq}$ for vector-valued functions.

Definition 2.6 *Let X be a Banach space, $1 < p < \infty$ and $1 \leq q \leq \infty$. Let us define $\mathcal{V}^{p,q}(\mu, X)$ as the space of vector measures $F : \Sigma \to X$ such that $x^*F \in V^{p,q}(\mathbb{K})$, for all $x^* \in X^*$. We consider then the norm*

$$\|F\|_{\mathcal{V}^{p,q}(X)} = \sup_{\|x^*\| \leq 1} \|x^*F\|_{V^{p,q}(\mathbb{K})}.$$

Let us now recollect some elementary results about these spaces.

Proposition 2.7 *Let $1 < p \leq \infty$ and $1 \leq q \leq \infty$. Then:*
(a) $(\mathcal{V}^{p,q}(X), \|\cdot\|_{\mathcal{V}^{p,q}(X)})$ and $(V^{p,q}(X), \|\cdot\|_{V^{p,q}(X)})$ are Banach spaces.
(b) $V^p(X) \subset V^{p,\infty}(X) \subset \mathcal{V}^{p,\infty}(X)$.
(c) $V^{p,q}(\mu, X) \subset \widetilde{\mathcal{V}}^{p,q}(X) \subset \mathcal{V}^{p,q}(X)$.
(d) $V^{p,q}(X) \subset V^{p,\infty}(X)$ and $\mathcal{V}^{p,q}(X) \subset \mathcal{V}^{p,\infty}(X)$.

PROOF. The proofs of (a), (b) and (c) are easy and left to the reader.

To see (d) note that, since ω_F is nondecreasing, integrating $s^{-q/p'-1}$ from t to $\mu(\Omega)$, we have

$$t^{-1/p'} \omega_F(t) \leq C_1 \Big(\int_t^{\mu(\Omega)} (s^{-1/p'} \omega_F(s))^q \frac{ds}{s} \Big)^{1/q} + C_2,$$

for some constants $C_1, C_2 > 0$. Similar estimate holds for $\widetilde{\omega}_F(t)$. ■

Example 2.1 *Let $1 < p, q < \infty$ and let us take $X = L^{p,q}(\mu, \Omega)$. Consider the $L^{p,q}$-valued measure F given by $F(E) = \chi_E$ for $E \in \Sigma$. Then*

(i) $\omega_F(t) = +\infty$ for $t \in I$.

(ii) $\widetilde{\omega}_F(t) = t^{\frac{1}{p}}$ for $t \in I$.

(iii) For any $x^* = \phi \in X^* = L^{p',q'}$ the measure $x^*F(E) = \int_E \phi d\mu$ for $E \in \Sigma$.

Therefore

(i') $F \notin V^{r,s}(X)$ for any r, s.

(ii') $F \in \widetilde{\mathcal{V}}^{r,s}(X)$ iff $1 \le r \le p'$ and $s = \infty$ or $1 \le r < p'$ and $1 \le s \le \infty$.

(iii') $F \in \mathcal{V}^{r,s}(X)$ iff $r = p'$ and $s \ge q'$ or $1 \le r < p'$.

In particular, there exist $F_1 \in \mathcal{V}^{p',q'}(X) \setminus \widetilde{\mathcal{V}}^{p',q'}(X)$ and $F_2 \in \mathcal{V}^{p',\infty}(X) \setminus V^{p',\infty}(X)$.

The importance of the following lemma is evident from the several references to it in the subsequent results.

Lemma 2.8 *Let $1 < p < \infty$ and $1 \le q \le \infty$. The following are equivalent:*

(a) $F \in V^{p,q}(X)$.

(b) There exists $\varphi \ge 0$, $\varphi \in L^{p,q}$ such that $|F|(A) = \int_A \varphi d\mu$ for all $A \in \Sigma$.

Moreover if $|F|(A) = \int_A \varphi d\mu$ for all $A \in \Sigma$ for some $\varphi \ge 0$ then $\|F\|_{V^{p,q}(X)} = \|\varphi\|_{pq}$.

PROOF. Assume (a) and $q = \infty$. Let us take $F \in V^{p,\infty}(X)$. Then using the Radon-Nikodým Theorem we find a nonnegative function φ such that $|F|(A) = \int_A \varphi d\mu$ for all $A \in \Sigma$. Now observe that

$$\begin{aligned}
\|F\|_{V^{p,\infty}(X)} &= \sup_{A\in\Sigma} \frac{|F|(A)}{\mu(A)^{\frac{1}{p'}}} = \sup_{t>0} \left(\sup_{\mu(A)=t} \frac{|F|(A)}{t^{\frac{1}{p'}}} \right) \\
&= \sup_{t>0} \left(t^{\frac{-1}{p'}} \sup_{\mu(A)=t} \int_A \varphi d\mu \right) = \sup_{t>0} t^{\frac{-1}{p'}} \int_0^t \varphi^*(s) ds \\
&= \sup_{t>0} t^{\frac{1}{p}} \varphi^{**}(t) = \|\varphi\|_{p\infty}.
\end{aligned}$$

For $q < \infty$ if $F \in V^{p,q}(X)$ then $F \in V^{p,\infty}(X)$. From the previous case we find a nonnegative function φ such that $|F|(A) = \int_A \varphi d\mu$ for all $A \in \Sigma$. It is plain now to see that $\|F\|_{p,q} = \|\varphi\|_{pq}$ with a look at Proposition 2.2.

The converse follows from Proposition 2.2 as well. ■

With the help of last lemma we can prove the following theorems.

Theorem 2.9 *Let X be a Banach space and $1 < p < \infty, 1 \leq q \leq \infty$. Then*
(a) $L^{p,q}(X)$ is isometrically embedded into $V^{p,q}(X)$.
(b) $L^{p,q}(X) = V^{p,q}(X)$ if and only if X has the Radon-Nikodym property.

PROOF. Let $f \in L^{p,q}(X)$ then the measure $F(E) = \int_E f d\mu$ belongs to $V^{p,q}(X)$. This follows from the fact $|F|(E) = \int_E \|f\| d\mu$ and Lemma 2.8 for $\varphi = \|f\|$.

To prove (b) let us assume that $L^{p,q}(X) = V^{p,q}(X)$ and let $T : L^1 \to X$ be a bounded operator. We need to show that T is representable (see [DU]). Note that $F(E) = T(\chi_E)$ gives a measure in $V^\infty(X)$, and then $F \in V^{p,q}(X)$. Now the assumption gives that there exists $f \in L^{p,q}(X)$ such that $F(E) = \int_E f d\mu$ for all $E \in \Sigma$. This shows that $T(\psi) = \int \psi f d\mu$ for all $\psi \in L^1$.

Conversely, let us assume that X has the Radon-Nikodym property and take $F \in V^{p,q}(X)$. Due to the fact that F is μ-continuous and with bounded variation, one gets $F(E) = \int_E f d\mu$ for all $E \in \Sigma$ for some $f \in L^1(X)$. To show that $f \in L^{p,q}(X)$ apply Lemma 2.8 for $\varphi = \|f\|$ again. ■

Lemma 2.10 *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = \infty$. If $F \in V^{p,q}(X)$ then*

$$\begin{aligned} \|F\|_{V^{p,q}(X)} &\approx \sup\{\sum_i |\alpha_i| \, \|F(A_i)\| \ : \ \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\} \\ &= \sup\{\sum_i |\alpha_i| \, |F|(A_i) \ : \ \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\}. \end{aligned}$$

PROOF. If φ is the function in Lemma 2.8, we use theorem 4.7 of [BS] (p. 220) on the dual norm in the space $L^{p,q}$, to say that $\|F\|_{p,q} = \|\varphi\|_{pq}$ is equivalent to $\sup\{\int_\Omega \varphi |f| d\mu : \ \|f\|_{p'q'} 1\}$.

Hence

$$\begin{aligned} \|F\|_{p,q} &\approx \sup\left\{\int_\Omega \varphi(\sum_i |\alpha_i| \chi_{A_i}) d\mu : f = \sum_i \alpha_i \chi_{A_i}, \ \|f\|_{p'q'} \leq 1\right\} \\ &= \sup\left\{\sum_i |\alpha_i| \int_{A_i} \varphi d\mu : \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\right\} \\ &= \sup\left\{\sum_i |\alpha_i| |F|(A_i) : \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\right\} \\ &= \sup\left\{\sum_i |\alpha_i| \|F(A_i)\| : \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\right\}. \end{aligned}$$

The equality of both suprema in the last step follows easily from the definition of variation $|F|(A_i)$. ∎

Corollary 2.11 *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Then $F \in V^{p,q}(X)$ if and only if*

$$\sup_{\pi \in D} \| \sum_{A \in \pi} \frac{F(A)}{\mu(A)} \chi_A \|_{pq} < \infty$$

where the supremum is taken over the set D of all finite partitions π of Ω.

In particular, $V^{p,p}(X) = V^p(X)$ for all $1 < p \leq \infty$.

PROOF. Let π be a partition in D. Then

$$\begin{aligned} \| \sum_{A \in \pi} \frac{F(A)}{\mu(A)} \chi_A \|_{L^{p,q}(X)} &= \| \sum_{A \in \pi} \frac{\|F(A)\|}{\mu(A)} \chi_A \|_{L^{p,q}} \\ &= \sup \{ \sum_{A \in \pi} \frac{\|F(A)\|}{\mu(A)} \int_A \psi d\mu : \|\psi\|_{p',q'} = 1 \} \\ &= \sup \{ \sum_{A \in \pi} \alpha_A \|F(A)\| : \| \sum_{A \in \pi} \alpha_A \chi_A \|_{p',q'} = 1 \}. \end{aligned}$$

We have used in the last equality the fact that $E_\pi(\psi) = \sum_{A \in \pi} \frac{\int_A \psi d\mu}{\mu(A)} \chi_A$ defines a bounded operator of norm 1 in $L^{p',q'}$. ∎

Now using the convenient results for spaces of measurable functions we obtain the following embeddings.

Corollary 2.12 *Let $1 < p < \infty$, $1 \leq q_1 \leq q_2 \leq \infty$, $1 < p_1 \leq p_2 < \infty$ and $1 \leq q, r \leq \infty$. Then*

(a) $V^{p,q_1}(X) \subset V^{p,q_2}(X), \widetilde{\mathcal{V}}^{p,q_1}(X) \subset \widetilde{\mathcal{V}}^{p,q_2}(X)$ and $\mathcal{V}^{p,q_1}(X) \subset \mathcal{V}^{p,q_2}(X)$.

(b) $V^{p_2,q}(X) \subset V^{p_1,r}(X), \widetilde{\mathcal{V}}^{p_2,q}(X) \subset \widetilde{\mathcal{V}}^{p_1,r}(X)$ and $\mathcal{V}^{p_2,q}(X) \subset \mathcal{V}^{p_1,r}(X)$.

Our next step is the description of the duality of the vector-valued Lorentz function spaces by vector measure spaces.

Theorem 2.13 *Let $1 < p < \infty$ and $1 \leq q < \infty$ or $p = q = 1$. Then $V^{p',q'}(X^*) = [L^{p,q}(X)]^*$.*

PROOF. Let $F \in V^{p',q'}(X^*)$ and ϕ_F the functional over $L^{p,q}(X)$ given as usual by $\phi_F(\sum_i x_i \chi_{A_i}) = \sum_i \langle x_i, F(A_i)\rangle$. We have quickly that

$$|\phi_F(\sum_i x_i \chi_{A_i})| \leq \sum_i \|x_i\| \|F(A_i)\|$$

for every simple function. Then Lemma 2.10 gives that $\|\phi_F\| \leq C\|F\|_{p',q'}$ for some constant $C > 0$.

Now let $\phi \in [L^{p,q}(X)]^*$. For each $A \in \Sigma$, $F_\phi(A)$ is the element of X^* such that $F_\phi(A)(x) = \phi(x\chi_A)$. Observe that F is well-defined and a countably additive measure (by continuity of ϕ). It is plain that $\phi = \phi_{F_\phi}$. Then, we can conclude since

$$\begin{aligned}
\|F_\phi\|_{p',q'} &\leq C' \sup\left\{\sum_i |\alpha_i| \; \|F(A_i)\| : \|\sum_i \alpha_i \chi_{A_i}\|_{p,q} \leq 1\right\} \\
&= C' \sup\left\{\sum_i |\alpha_i| \, |\langle x_i, F(A_i)\rangle| : \|\sum_i \alpha_i \chi_{A_i}\|_{p,q} \leq 1, \, \|x_i\| \leq 1 \; \forall i\right\} \\
&= C' \sup\left\{|\sum_i \langle x_i, F(A_i)\rangle| : f = \sum_i x_i \chi_{A_i} \in L^{p,q}(X), \; \|f\|_{p,q} \leq 1\right\} \\
&= C' \sup\left\{|\phi_F(\sum_i x_i \chi_{A_i})| : f = \sum_i x_i \chi_{A_i} \in L^{p,q}(X), \; \|f\|_{pq} \leq 1\right\} \\
&= C' \|\phi\|.
\end{aligned}$$

■

Corollary 2.14 *Let $1 < p, q < \infty$. Then $[L^{p,q}(X)]^* = L^{p',q'}(X^*)$ if and only if X^* has the Radon-Nikodym property.*

3 Vector measures and operators

We just comment that a vector measure F provides us with a linear operator T_F acting on characteristic functions as $T_F(\chi_A) = F(A)$, and it can be obviously extended by linearity to the set of step functions.

In this way we clearly identify

$$V^\infty(X) = \mathcal{V}^\infty(X) = L(L^1, X).$$

In this section we analyze the cases $\mathcal{V}^{p,q}(X)$ and $V^{p,q}(X)$.

To describe the spaces $\mathcal{V}^{p,q}(X)$ in terms of operators we need the following lemma.

Lemma 3.1 *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. If $F \in \mathcal{V}^{p,q}(X)$ then*

$$\|F\|_{\mathcal{V}^{p,q}(X)} \approx \sup\{\|\sum_i \alpha_i F(A_i)\| \ : \ \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\}.$$

PROOF. Using Lemma 2.10 for the particular case $X = \mathbb{K}$ we have

$$\begin{aligned} \|F\|_{\mathcal{V}^{p,q}(X)} &= \sup_{\|x^*\|\leq 1} \|x^* F\|_{V^{p,q}} \simeq \\ &= \sup\{\sum_i |\alpha_i||x^* F(A_i)| \ : \ \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1, \ \|x^*\| \leq 1\} \\ &= \sup\{|\langle \sum_i \alpha_i F(A_i), x^* \rangle| \ : \ \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1, \ \|x^*\| \leq 1\} \\ &= \sup\{\|\sum_i \alpha_i F(A_i)\| \ : \ \|\sum_i \alpha_i \chi_{A_i}\|_{p'q'} \leq 1\}. \end{aligned}$$

■

Therefore, the next results are straightforward corollaries.

Theorem 3.2 *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then*

$$\mathcal{V}^{p',q'}(X) = L(L^{p,q}, X).$$

Corollary 3.3 *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then*

$$\mathcal{V}^{p',q'}(X^*) = (L^{p,q} \hat{\otimes}_\pi X)^*.$$

From Theorem 3.2 we have that $V^{p',q'}(X)$ is a subspace of the space of operators from $L^{p,q}$ to X. If we want to understand the corresponding class of operators we need to recall the following notion.

Definition 3.4 *(see [S], p. 244) Let E be a Banach lattice and B a Banach space. A linear operator from E to B is said to be cone absolutely summing (c.a.s.) if there is a constant $C > 0$ such that for every $k \in \mathbb{N}$ and every family $e_1, e_2, \ldots, e_k \in E$ of positive elements we have*

$$\sum_{i=1}^k \|T(e_k)\|_B \leq C \sup_{\|e^*\|_{E^*} \leq 1} \sum_{i=1}^k |\langle e_i, e^* \rangle|.$$

We denote by $\Pi_+^1(E, B)$ the set of such operators and its norm is given by the infimum of the constants C satisfying the previous inequality.

Remark 3.1 *It is rather easy to give an equivalent defintion (see [S], p. 244) by using that if* $e_1, e_2, ..., e_k$ *are positive elements in* E *then*

$$\sup_{\|e^*\|_{E^*}\leq 1} \sum_{i=1}^{k} |\langle e_i, e^*\rangle| = \| \sum_{i=1}^{k} e_i\| \tag{3}$$

Let us mention here that $L^{p,q}$ are, clearly, Banach lattices and next theorem allows us to identify the space $\Pi^1_+(L^{p,q}, X)$.

Theorem 3.5 *Let* $1 < p < \infty$ *and* $1 \leq q < \infty$ *or* $p = q = 1$. *Then* $V^{p',q'}(X) = \Pi^1_+(L^{p,q}, X)$.

PROOF. Let $F \in V^{p',q'}(X)$ and $T_F : L^{p,q} \to X$ defined as explained in the begining of the section. Let us see that $T \in \Pi^1_+(L^{p,q}, X)$. If $N \in \mathbb{N}$ and $f_1, f_2, \ldots, f_N$ are nonnegative functions in $L^{p,q}$, then Lemma 2.8 gives the existence of $\varphi \in L^{p',q'}$ such that for all $f \in L^{p,q}$

$$\|T_F(f)\| \leq \int_\Omega |f|\varphi d\mu.$$

Therefore

$$\sum_{n=1}^{N} \|T_F(f_n)\| \leq \sum_{n=1}^{N}(\int_\Omega f_n\varphi d\mu) = \int_\Omega (\sum_{n=1}^{N} f_n)\varphi d\mu \leq \| \sum_{n=1}^{N} f_n\|_{p,q} \|F\|_{p',q'}.$$

Hence, from (3), one gets $\|T_F\|_{\Pi^1_+} \leq \|F\|_{p',q'}$.

Now if $T \in \Pi^1_+(L^{p,q}, X)$ and $F_T : \Sigma \to X$ is defined by $F_T(A) = T(\chi_A)$, it is obvious that $T = T_{F_T}$ and F_T is countably additive. Let $f = \sum \alpha_i\chi_{A_i} \in L^{p,q}$ with $\|f\|_{p,q} \leq 1$, then

$$\sum_i |\alpha_i| \|F_T(A_i)\| = \sum_i \|T(|\alpha_i|\chi_{A_i})\| \leq \|T\|_{\Pi^1_+} \| \sum_i |\alpha_i|\chi_{A_i}\|_{p,q} \leq \|T\|_{\Pi^1_+}.$$

Therefore $\|F_T\|_{p',q'} \leq C\|T\|_{\Pi^1_+}$, and the proof is over. ∎

4 Lorentz spaces and interpolation

Even though the main definitions are written down, we refer to the reader to [BS] or [BL], where a wide study of interpolation spaces is developed.

Definition 4.1 *(The K-functional) Let* (X_0, X_1) *be a compatible couple of Banach spaces. The* K*-functional can be defined for every* $f \in X_0 + X_1$ *and* $t > 0$ *by*

$$K(f, t; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum is taken from all the possible representations $f = f_0 + f_1$ *of* f *with* $f_0 \in X_0$ *and* $f_1 \in X_1$.

Theorem 4.2 *([BS], p. 298) Let* (Ω, Σ, μ) *be a totally* σ*-finite measure space, then*

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s)ds = tf^{**}(t).$$

Definition 4.3 *([BS], p. 299) Let* (X_0, X_1) *be a compatible couple and suppose* $0 < \theta < 1$, $1 \le q < \infty$ *or* $0 \le \theta \le 1$, $q = \infty$. *The space* $(X_0, X_1)_{\theta,q}$ *consists of all* f *in* $X_0 + X_1$ *for which the functional*

$$\|f\|_{\theta,q} = \begin{cases} \left\{\int_0^\infty \left[t^{-\theta} K(f,t)\right]^q \frac{dt}{t}\right\}^{\frac{1}{q}}, & 0 < \theta < 1,\ 1 \le q < \infty, \\ \sup\limits_{t>0} t^{-\theta} K(f,t), & 0 \le \theta \le 1,\ q = \infty. \end{cases}$$

is finite (here $K(f,t) := K(f, t; X_0, X_1)$*).*

Let us denote by $V^1(X)$ the space of μ- continuous vector measures of bounded variation and write $\|F\|_{V^1(X)} = |F|(\Omega)$.

Of course, $V^\infty(X) \subset V^1(X)$ and $V^{p,q}(X) \subset V^1(X)$ for all $1 < p < \infty$ and $1 \le q \le \infty$.

Theorem 4.4 *Let* $F \in V^1(X)$ *and* $t > 0$. *Then*

$$\omega_F(t) \le K(F, t; V^1(X), V^\infty(X)) \le 2\omega_F(t).$$

PROOF. Assume $F = G + H$ with $G \in V^1(X)$ and $H \in V^\infty(X)$. It follows from $|F| \le |G| + |H|$ that $\omega_F(t) \le \omega_G(t) + \omega_H(t)$.

Note that

$$\omega_G(t) = \sup_{\mu(E)\le t} |G|(E) \le |G|(\Omega) = \|G\|_{V^1(X)}$$

and

$$\omega_H(t) = \sup_{\mu(E)\le t} |H|(E) \le t \sup_{\mu(E)\le t} \frac{|H|(E)}{\mu(E)} = t\|H\|_{V^\infty(X)}$$

then we get the first estimate taking the infimum over all decompositions.

Assume now that $\omega_F(t)$ is finite. Since $|F|$ is a μ-continuous measure we can find a function $\varphi \ge 0$ such that $|F|(A) = \int_A \varphi d\mu$ for every measurable set A.

We take $E = \{w \in \Omega : \varphi(w) > \varphi^*(t)\}$. Since μ_φ y φ^* are mutually right-continuous inverse functions, then $\mu(E) \le t$.

Let us define $G(A) = F(E \cap A)$ for $A \in \Sigma$. In this case $|G|(A) = \int_A \varphi_G d\mu$ for all A, where $\varphi_G = \varphi\chi_E$. It is easy to check that

$$\varphi_G^*(s) = \begin{cases} \varphi^*(s) & 0 < s < \mu(E) \\ 0 & s \ge \mu(E) \end{cases}$$

and from here we can deduce that $\|G\|_{V^1(X)} = \|\varphi_G\|_{L^1} = \int_0^{\mu(E)} \varphi^*(s)ds$.

Defining $H(A) = F(A) - G(A) = F((\Omega \setminus E) \cap A)$ for all $A \in \Sigma$, we have that $|H|(A) = \int_A \varphi_H d\mu$ for every A, where $\varphi_H = \varphi\chi_{\Omega\setminus E}$. It is clear that

$$\mu_{\varphi_H}(\lambda) = \begin{cases} \mu_\varphi(\lambda) - \mu_\varphi(\varphi^*(t)) & 0 < \lambda \le \varphi^*(t) \\ 0 & \lambda > \varphi^*(t) \end{cases}$$

Therefore

$$\varphi_H^*(s) = \varphi^*(s + \mu_\varphi(\varphi^*(t))).$$

Hence

$$\sup_A \frac{|H|(A)}{\mu(A)} \le \sup_{s>0} \frac{\int_0^s \varphi_H^*(\theta)d\theta}{s} \le \lim_{s\to 0} \varphi_H^*(s) = \varphi^*(\mu_\varphi(\varphi^*(t))) \le \varphi^*(t).$$

And consequently $t\|H\|_{V^\infty(X)} \le t\varphi^*(t)$.

Now, using the decomposition of F and the properties of φ^*, we get

$$\|G\|_{V^1(X)} + t\|H\|_{V^\infty(X)} \le \int_0^{\mu(E)} \varphi^*(s)ds + t\varphi^*(t) \le 2\int_0^t \varphi^*(s)ds = 2\omega_F(t).$$

■

Theorem 4.5 *If* $1 < p < \infty$ *and* $1 \leq q \leq \infty$*, then*

$$V^{p,q}(X) = (V^1(X), V^\infty(X))_{\theta,q}.$$

where $\frac{1}{p} = 1 - \theta$.

Using the reiteration theorem (see [BS], page 311) and Corollary 2.11 one gets

Theorem 4.6 *If* $1 < p_1, p_2 < \infty$ *and* $1 \leq q \leq \infty$*, then*

$$V^{p,q}(X) = (V^{p_1}(X), V^{p_2}(X))_{\theta,q}.$$

where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$.

Let us mention something about the interpolation when we change also the spaces where the measures take values in. For the difference between $(V^{p_1}(X_1), V^{p_2}(X_2))_{\theta,q})$ and $V^{p,q}((X_1, X_2)_{\theta,q})$ we recall that $V^p(X) = L^p(X)$ if X has the Radon-Nikodym property and simply refer the reader to the paper by M. Cwikel (see [C]) where it is shown that for spaces of vector valued functions the equality only may happen for $q = p$ where $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$.

Even in the case $q = p$ the expected interpolation result does not hold for vector-valued measures. The reader is referred to [BX] for the difference between $(V^{p_1}(X_1), V^{p_2}(X_2))_{\theta,p}$ and $V^{p,q}((X_1, X_2)_{\theta,p})$. Although there the authors deal with interpolation between spaces of vector-valued harmonic functions, instead of vector-valued measures, they can be identified according to the results in [B1] or [B2].

References

[BL] J. Bergh , J. Löfström, *Interpolation spaces. An introduction*, Springer Verlag, Berlin, 1973.

[BS] C. Bennet, R. Sharpley, *Interpolation of operators*, Pure and applied mathematics vol. 129, Academic Press, Inc., New York, 1988.

[B1] O. Blasco, Boundary values of functions in vector-valued Hardy spaces and Geometry of Banach spaces, *J. Funct. Anal..,* Vol 78 (1988) 346-364.

[B2] O. Blasco, Boundary values of vector valued functions in Orlicz-Hardy classes, *Arch. Math.,* Vol. 49 (1987) 434-439

[BX] O. Blasco, Q. Xu, Interpolation between vector-valued Hardy spaces,*J. Funct. Anal.,* Vol 102 (1991), 331-359.

[B] A.V. Bukhvalov, The analytic representation of operators with abstract norm (russian), *Izv. Vyss. Ucebn. Zadev. Matematika* Vol 11 (1975),21-32.

[C] M. Cwikel, On $(L^{p_o}(A_0), L^{p_1}(A_1))_{\theta,q}$,*Proc. Amer. Math. Soc.* Vol 44 (1974) , 286-292.

[DU] J. Diestel, J.J. Uhl,*Vector measures,* American Mathematical Society, Mathematical Surveys, Number 15, R. I. (1977).

[D] N. Dinculeanu, *Vector measures,* International Series of Monographs in Pure and Applied Mathematics, Volume 95, Pergamon Press (1967).

[GU] N.E. Gretsky, J.J. Uhl, Bounded linear operators on Banach function spaces of vector-valued functions, *Trans. Amer. Math. Soc.,* Vol 167 (1972), 263-277.

[Ha] P.R. Halmos, Measure theory, Van Nostrand, New York (1950)

[Hu] R.A. Hunt, On $L(p,q)$ spaces, *L'Enseignement mathèm., t. XII, fasc. 4* (1966) 249-276

[LTII] J. Lindenstrauss, L. Tzafriri,*Classical Banach spaces II. Function spaces,* Springer-Verlag, Berlin, (1979).

[L] G.G. Lorentz, Some new functional spaces, *Ann. Math.,* Vol 51 (1950), 37-55.

[S] H.H. Schaefer, *Banach lattices and positive operators,* Springer-Verlag, New York (1974)

[SW] E.M. Stein, G. Weiss, *Introduction to Fourier analysis on euclidean spaces* Princeton University Press, Princeton (1971)

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