A note on Mobius transformations and Bogolubov coefficients[1]

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Abstract

We analyze the creation of scalar massless particles in two dimensions under the action of conformal transformations. We focus our attention to Mobius transformation and clarify an apparent tension between the results obtained with the Bogolubov coefficients and those obtained within the conformal field theory approach.

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One of the basic ingredients of quantum field theory in curved spacetime [1] are the Bogolubov transformations. These reflect the absence, in general, of a privileged vacuum state, in parallel to the absence of global inertial frames. This framework is general and can be applied to a large number of physical situations, including flat spacetime (like the Unruh-Fulling effect [1]). On the other hand, of particular physical interest are those field theories possessing the spacetime conformal symmetry $SO(d, 2)$, where d is the dimension of the Lorentzian spacetime. This symmetry is especially powerful in two dimensions, where the group $SO(2, 2)$ can be enlarged to an infinite-dimensional group [2]. However, this $SO(2, 2)$ subgroup, which includes dilatations, Poincaré and special conformal transformations, still plays an important role because it leaves the vacuum invariant [2]. From the point of view of Bogolubov transformations this should imply that the β coefficients associated to them vanish. This is obvious for Poincaré and dilatations: they do not produce any mixing of positive and negative frequencies. For special conformal transformations the β coefficients (calculated for a massless scalar field) are vanishing only if one considers the two branches of the transformation. If we consider only one branch we get a nonvanishing result [3, 1], which has been interpreted with the fact that the absence of energy (always true irrespective of the presence of one or two branches) does not imply absence of quanta too. The purpose of this note is to clarify this apparent tension between both approaches.

Let us first briefly review the definition of the Bogolubov coefficients for the two-dimensional massless scalar field f satisfying the wave equation

$$\nabla^2 f = 0 \ . \tag{1}$$

In conformal gauge $ds^2 = -e^{2\rho}dx^+dx^-$ we can decompose the field into positive and negative frequencies using the standard mode solutions:

$$f = \int_0^\infty \frac{dw}{\sqrt{4\pi w}} (\overrightarrow{a}_w e^{-iwx^-} + \overleftarrow{a}_w e^{-iwx^+} + \overrightarrow{a}^\dagger_w e^{iwx^-} + \overleftarrow{a}^\dagger_w e^{iwx^+}) \ . \tag{2}$$

These modes form an orthonormal basis under the scalar product

$$(f_1, f_2) = -i \int_\Sigma d\Sigma^\mu \left(f_1 \partial_\mu f_2^* - \partial_\mu f_1 f_2^*\right) \ , \tag{3}$$

where Σ is an appropiate Cauchy hypersurface. One can construct the Fock space from the commutation relations

$$[\overrightarrow{a}_w, \overrightarrow{a}^\dagger_{w'}] = \delta(w - w') \ , \tag{4}$$

$$[\overleftarrow{a}_w, \overleftarrow{a}^\dagger_{w'}] = \delta(w - w') \ . \tag{5}$$

The vacuum state $|0_x\rangle$ is defined by

$$\overrightarrow{a}_w|0_x\rangle = 0, \quad \overleftarrow{a}_w|0_x\rangle = 0 \ , \tag{6}$$

and the excited states can be obtained by the application of creation operators $\overrightarrow{a}^\dagger_w$, $\overleftarrow{a}^\dagger_w$ out of the vacuum. We can perform an arbitrary conformal transformation

$$x^\pm \to y^\pm = y^\pm(x^\pm) \ , \tag{7}$$

and consider the expansion

$$f = \int_0^\infty \frac{dw}{\sqrt{4\pi w}}(\overrightarrow{b}_w e^{-iwy^-} + \overleftarrow{b}_w e^{-iwy^+} + \overrightarrow{b}^\dagger_w e^{iwy^-} + \overleftarrow{b}^\dagger_w e^{iwy^+}) \ . \tag{8}$$

As both sets of modes are complete, the new modes $\frac{1}{\sqrt{4\pi w}}e^{-iwy^-}$, $\frac{1}{\sqrt{4\pi w}}e^{-iwy^+}$ can be expanded in terms of the old ones:

$$\frac{e^{-iwy^\pm}}{\sqrt{4\pi w}} = \int_0^\infty \frac{dw'}{\sqrt{4\pi w'}}\left(\alpha^\pm_{ww'}e^{-iw'x^\pm} + \beta^\pm_{ww'}e^{iw'x^\pm}\right), \tag{9}$$

where $\alpha^\pm_{ww'}$ and $\beta^\pm_{ww'}$ are called the Bogolubov coefficients. These coefficients can be evaluated by the following scalar products

$$\alpha^\pm_{ww'} \quad = \quad (\bar{\phi}^\pm_w, \phi^\pm_{w'}) \tag{10}$$

$$\beta^\pm_{ww'} \quad = \quad -(\bar{\phi}^\pm_w, \phi^{\pm *}_{w'}) \tag{11}$$

where $\phi^\pm_w = \frac{e^{-iwx^\pm}}{\sqrt{4\pi w}}, \bar{\phi}^\pm_w = \frac{e^{-iwy^\pm}}{\sqrt{4\pi w}}$. The results are:

$$\alpha^\pm_{ww'} = \frac{1}{2\pi}\sqrt{\frac{w}{w'}}\int dx^\pm \left(\frac{dy^\pm}{dx^\pm}\right) e^{-iwy^\pm(x^\pm)+iw'x^\pm} \ , \tag{12}$$

$$\beta^\pm_{ww'} = -\frac{1}{2\pi}\sqrt{\frac{w}{w'}}\int dx^\pm \left(\frac{dy^\pm}{dx^\pm}\right) e^{-iwy^\pm(x^\pm)-iw'x^\pm} \ . \tag{13}$$

The relation between creation and annhilation operators in the two basis is

$$\vec{b}_w = \int_0^\infty dw' \left(\alpha^{-*}_{ww'} \vec{a}_{w'} - \beta^{-*}_{ww'} \vec{a}^{\dagger} \right) \ , \tag{14}$$

along with the corresponding one for $\vec{b}^{\dagger}_w$. Similar equations hold for the left movers. Therefore the expectation value of the (right mover sector) particle number operator $\vec{N}_w \equiv \vec{b}^{\dagger}_w \vec{b}_w$ is given by the expression

$$\langle 0_x | \vec{N}_w | 0_x \rangle = \int_0^\infty dw' |\beta^{-}_{ww'}|^2 \ . \tag{15}$$

Let us now discuss how to obtain an expression for $\langle 0_x | \vec{N}_w | 0_x \rangle$ within the framework of conformal field theory. The two-point correlation function $\langle 0_x | f(x) f(x') | 0_x \rangle$ is ill-defined due to the infrared divergence of the scalar field in two dimensions. This can be cured by introducing a frequency cut-off λ. In doing so one gets

$$\langle 0_x | f(x) f(x') | 0_x \rangle = -\frac{1}{4\pi} \left(2\gamma + \ln \lambda^2 (x - x')^2 \right) \ , \tag{16}$$

where γ is the Euler constant. The ambiguity inherent to the cut-off disappears when one considers, instead of the two-point function for the field f, the correlations for the derivatives $\partial_\pm f$. We have then

$$\langle 0_x | \partial_\pm f(x^\pm) \partial_\pm f(x'^\pm) | 0_x \rangle = -\frac{1}{4\pi} \frac{1}{(x^\pm - x'^\pm)^2} \ . \tag{17}$$

Under conformal transformations $x^\pm \to y^\pm = y^\pm(x^\pm)$, the above correlation functions transform according to the rule for primary fields:

$$\langle 0_x | \partial_\pm f(y^\pm) \partial_\pm f(y'^\pm) | 0_x \rangle = -\frac{1}{4\pi} \left(\frac{dx^\pm(y^\pm)}{dy^\pm} \right) \left(\frac{dx^\pm(y'^\pm)}{dy^\pm} \right) \frac{1}{(x^\pm(y^\pm) - x'^\pm(y^\pm))^2} . \tag{18}$$

These relations are fundamental to construct the normal ordered stress tensor $: T_{\pm\pm} :$ as well as the particle number operator. In the coordinates $\{x^\pm\}$ and the choice of modes $\phi^\pm_w$, the normal ordered stress tensor operator can be defined via point-splitting

$$: T_{\pm\pm}(x^\pm) := \lim_{x^\pm \to x'^\pm} \partial_\pm f(x^\pm) \partial_\pm f(x'^\pm) + \frac{1}{4\pi} \frac{1}{(x^\pm - x'^\pm)^2} . \tag{19}$$

Similar relations hold in the coordinates $\{y^{\pm}\}$ and with the choice of the modes $\bar{\phi}^{\pm}_{w}$. It is easy to relate $:T_{\pm\pm}(y^{\pm}):$ with $:T_{\pm\pm}(x^{\pm}):$ and the result is

$$:T_{\pm\pm}(y^{\pm}):=\left(\frac{dx^{\pm}}{dy^{\pm}}\right)^{2}:T_{\pm\pm}(x^{\pm}):-\frac{1}{24\pi}\{x^{\pm},y^{\pm}\}\ , \tag{20}$$

where

$$\{x^{\pm},y^{\pm}\}=\frac{d^{3}x^{\pm}}{dy^{\pm 3}}/\frac{dx^{\pm}}{dy^{\pm}}-\frac{3}{2}\left(\frac{d^{2}x^{\pm}}{dy^{\pm 2}}/\frac{dx^{\pm}}{dy^{\pm}}\right)^{2} \tag{21}$$

is the Schwarzian derivative. The two-point correlation function $\partial_{\pm}f(y^{\pm})\partial_{\pm}f(y'^{\pm})$ also serves to construct the particle number operator. We start from the explicit form of the normal ordered operator $:\partial_{\pm}f(x^{\pm})\partial_{\pm}f(x'^{\pm}):$ in terms of the creation and annihilation operators (for simplicity we shall consider only the right mover sector)

$$\langle 0_x|:\partial_{-}f(y^{-})\partial_{-}f(y'^{-}):|0_x\rangle=\int_{0}^{\infty}dw\int_{0}^{\infty}dw'\frac{\sqrt{ww'}}{4\pi}\{\langle 0_x|\overrightarrow{b}^{\dagger}_{w}\overrightarrow{b}_{w'}|0_x\rangle(e^{iwy^{-}-iw'y'^{-}}+ e^{-iw'y^{-}+iwy'^{-}})-\langle 0_x|\overrightarrow{b}_{w}\overrightarrow{b}_{w'}|0_x\rangle e^{-iwy^{-}-iw'y'^{-}}-\langle 0_x|\overrightarrow{b}^{\dagger}_{w}\overrightarrow{b}^{\dagger}_{w'}|0_x\rangle e^{iwy^{-}+iw'y'^{-}}\}\ . \tag{22}$$

Now, instead of taking the limit $x^{\pm}\to x'^{\pm}$, as in the construction of the stress tensor, we shall take the following Fourier transform

$$\frac{1}{(2\pi)^{2}}\int_{-\infty}^{+\infty}dy^{-}dy'^{-}\langle 0_x|:\partial_{-}f(y^{-})\partial_{-}f(y'^{-}):|0_x\rangle e^{-i\tilde{w}y^{-}+i\tilde{w}'y'^{-})}=\frac{\sqrt{\tilde{w}\tilde{w}'}}{4\pi}\langle 0_x|\overrightarrow{b}^{\dagger}_{\tilde{w}}\overrightarrow{b}_{\tilde{w}'}|0_x\rangle\ . \tag{23}$$

Note that to obtain this last equation it is crucial to integrate over all range in the coordinates y^{-}, y'^{-}. We then immediately get an expression for the expectation value of the particle number operator $\overrightarrow{N}_{w}=\overrightarrow{b}^{\dagger}_{w}\overrightarrow{b}_{w}$ of frequency w :

$$\langle 0_x|\overrightarrow{N}_{w}|0_x\rangle=\frac{1}{\pi w}\int_{-\infty}^{+\infty}dy^{-}dy'^{-}\langle 0_x|:\partial_{-}f(y^{-})\partial_{-}f(y'^{-}):|0_x\rangle e^{-iw(y^{-}-y'^{-})}\ . \tag{24}$$

From the above considerations we obtain

$$\langle 0_x|\overrightarrow{N}_{w}|0_x\rangle=-\frac{1}{4\pi^{2}w}\int_{-\infty}^{+\infty}dy^{-}dy'^{-}e^{-iw(y^{-}-y'^{-})}\left[\left(\frac{dx^{-}(y^{-})}{dy^{-}}\right)\left(\frac{dx^{-}(y'^{-})}{dy^{-}}\right)\frac{1}{(x^{-}-x'^{-})^{2}}-\frac{1}{(y^{-}-y'^{-})^{2}}\right]\ . \tag{25}$$

We mention that the integral $\int_0^\infty dww\langle 0_x|\overrightarrow{N}_w|0_x\rangle$ gives the integrated flux $\int dy^-\langle 0_x| : T_{--}(y^-) : |0_x\rangle$.

As an illustrative example we shall show how the CFT approach reproduces the thermal properties associated to the conformal transformation

$$x^\pm = \pm\kappa^{-1}e^{\pm\kappa y^\pm}. \tag{26}$$

We can think of this transformation as relating the Minkowskian $x^\pm$ and Rindler $y^\pm$ null coordinates, where κ is the acceleration parameter (the same relation holds for the Schwarzschild black hole between the Kruskal and Eddington-Finkelstein null coordinates with $\kappa = 1/4M$). Since we are using plane waves instead of wave-packets we first work out an expression for $\langle 0_x|\overrightarrow{b}^\dagger_w \overrightarrow{b}_{w'}|0_x\rangle$

$$\begin{aligned}\langle 0_x|\overrightarrow{b}^\dagger_w \overrightarrow{b}_{w'}|0_x\rangle &= -\frac{1}{4\pi^2\sqrt{ww'}}\int_{-\infty}^{+\infty} dy^- dy'^-[\frac{dx^-}{dy^-}(y^-)\frac{dx^-}{dy^-}(y'^-)\frac{1}{(x^- - x'^-)^2} \\ &- \frac{1}{(y^- - y'^-)^2}]e^{-iwy^- + iw'y'^-} \ . \end{aligned} \tag{27}$$

Substitution of the relations (26) leads to

$$\langle 0_x|\overrightarrow{b}^\dagger_w \overrightarrow{b}_{w'}|0_x\rangle = -\frac{1}{2\pi w}\delta(w-w')\int_{-\infty}^{+\infty} dz\left[\frac{\kappa^2 e^{-\kappa z}}{(1-e^{-\kappa z})^2} - \frac{1}{z^2}\right]e^{-iwz} \ , \tag{28}$$

where $z = y^- - y'^-$. Evaluation of the integral gives

$$\langle 0_x|\overrightarrow{b}^\dagger_w \overrightarrow{b}_{w'}|0_x\rangle = \delta(w-w')\frac{1}{e^{\frac{2\pi w}{\kappa}} - 1} \ , \tag{29}$$

leading to the number of particles emitted per unit time of

$$\langle 0_x|\overrightarrow{N}_w|0_x\rangle = \frac{1}{e^{\frac{2\pi w}{\kappa}} - 1}, \tag{30}$$

which corresponds to the Planckian spectrum of radiation at the temperature $T = \frac{\kappa}{2\pi}$. Similar results hold for the left mover sector. Evaluation of the expectation value of the stress tensor using (20), taking into account that $\langle 0_x| : T_{\pm\pm}(x^\pm) : |0_x\rangle = 0$, gives

$$\langle 0_x| : T_{\pm\pm}(y^\pm) : |0_x\rangle = \frac{\kappa^2}{48\pi} = \frac{\pi T^2}{12} \ . \tag{31}$$

This is nothing else but the stress tensor corresponding to a two dimensional thermal bath of radiation at the temperature T.

We shall now analyse the case associated to the Mobius transformations

$$x^{\pm} \to y^{\pm} = \frac{a^{\pm}x^{\pm} + b^{\pm}}{c^{\pm}x^{\pm} + d^{\pm}} \tag{32}$$

where $a^{\pm}d^{\pm} - b^{\pm}c^{\pm} = 1$. These form the so called global conformal group $((SL(2,R) \otimes SL(2,R))/Z_2 \approx SO(2,2))$ and have the property of giving a vanishing Schwarzian derivative. Therefore, under the action of the Mobius transformations the flux of radiation in the vacuum $|0_x\rangle$ for the observer $\{y^{\pm}\}$ vanishes

$$\langle 0_x| : T_{\pm\pm}(y^{\pm}) : |0_x\rangle = 0 \ . \tag{33}$$

Moreover, since the two-point function (17) is invariant under (32) it is clear from (25) that the expectation value of the particle number operator also vanishes

$$\langle 0_x|\overrightarrow{N}_w|0_x\rangle = 0 = \langle 0_x|\overleftarrow{N}_w|0_x\rangle \ . \tag{34}$$

This is what we expect in the context of Conformal Field Theory, since the vacuum is invariant under Mobius transformations. However, the conclusion is different in the approach of the Bogolubov coefficients. For those Mobius transformations which are not dilatations nor Poincaré such as

$$x^{-} = -\frac{1}{a^2 y^{-}} \ , \tag{35}$$

where a is an arbitrary constant, the Bogolubov coefficients are

$$\begin{aligned}
\alpha_{ww'} &= \frac{1}{2\pi}\sqrt{\frac{w}{w'}}\left(\int_0^{+\infty} dy^- e^{-iwy^- - iw'/a^2y^-} + \int_{-\infty}^0 dy^- e^{-iwy^- - iw'/a^2y^-}\right) \ , \\
\beta_{ww'} &= -\frac{1}{2\pi}\sqrt{\frac{w}{w'}}\left(\int_0^{+\infty} dy^- e^{-iwy^- + iw'/a^2y^-} + \int_{-\infty}^0 dy^- e^{-iwy^- + iw'/a^2y^-}\right)
\end{aligned} \tag{36}$$

If one restricts only to one branch (for instance $0 < y^- < +\infty$) the results, given in [3, 1, 4], are

$$\begin{aligned}
\alpha_{ww'} &= \frac{1}{a\pi}K_1(2i\sqrt{ww'/a^2}) \ , \\
\beta_{ww'} &= \frac{i}{a\pi}K_1(2\sqrt{ww'/a^2}) \ ,
\end{aligned} \tag{37}$$

where K_1 is a modified Bessel function. Therefore confining to just one branch would seem to give paradoxical results, namely a vanishing flux but a production of quanta due to the nonvanishing β coefficient

$$\langle 0_x | \overrightarrow{N}_w | 0_x \rangle \neq 0 \ . \tag{38}$$

We mention that the transformation (35) originally appeared in the moving mirror model of Davies and Fulling [3] and more recently in the analysis of extremal black holes [5, 6, 7] and in the late-time behaviour of evaporating near-extremal Reissner-Nordstrom black holes [8]. On the other hand, (38), based on (37), does not only give a nonvanishing particle number, but also exhibits a logarithmic infrared divergence. It has been suggested in [5] that such divergence can be cured, as usual, by using wave packets instead of plane waves. However, it has been recently pointed out in [6] that the integrals (12), (13) defining the Bogolubov coefficients are not well defined for the transformation (35). Indeed, the results (37) are obtained by means of an unjustified Wick rotation. This problem cannot be cured by using wave packets [6].

The results (38), (37) and (34) are in apparent contradiction. The result (34) is well established, since the Mobius invariance of the vacuum is almost an axiom of CFT. It cannot be eluded if one wants to maintain the conformal invariance at the quantum level. However, the full Mobius transformation (35), on which the CFT results are based, cannot be restricted only to one branch. Therefore in order to make the comparison one has to consider also the second integral, from $-\infty$ to 0, in the formulas (36) leading to

$$\begin{aligned} \alpha_{ww'} &= Re\left[\frac{2}{a\pi}K_1(2i\sqrt{ww'/a^2})\right] \ , \\ \beta_{ww'} &= 0 \ . \end{aligned} \tag{39}$$

This result restores compatibility with the CFT, as zero β coefficient implies that no quanta are produced, in agreement with the fact that the energy flux is zero. This result of the cancellation of the β coefficient should be general and independent of the prescription one could use to properly define the integral (36).

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