Post-Newtonian constraints on $f(R)$ cosmologies in metric and Palatini formalism

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(Received 31 May 2005; revised manuscript received 8 September 2005; published 7 October 2005)

We compute the complete post-Newtonian limit of both the metric and Palatini formulations of $f(R)$ gravities using a scalar-tensor representation. By comparing the predictions of these theories with laboratory and solar system experiments, we find a set of inequalities that any lagrangian $f(R)$ must satisfy. The constraints imposed by those inequalities allow us to find explicit bounds to the possible nonlinear terms of the lagrangian. We conclude that in both formalisms the lagrangian $f(R)$ must be almost linear in R and that corrections that grow at low curvatures are incompatible with observations. This result shows that modifications of gravity at very low cosmic densities cannot be responsible for the observed cosmic speed-up.

DOI: 10.1103/PhysRevD.72.083505 PACS numbers: 98.80.Es, 04.25.Nx, 04.50.+h

I. INTRODUCTION

It is now generally accepted that the universe is undergoing a period of accelerated expansion [1,2], which cannot be justified by the description provided by the equations of motion of General Relativity (GR) and a universe filled with standard sources of matter and energy. It has been suggested that this effect could have its origin in, among other possibilities, corrections to the equations of motion of GR generated by nonlinear contributions of the scalar curvature in the gravity lagrangian $f(R)$ [3–5] (see also [6], and [7,8] for lagrangians that do not fit the $f(R)$ form). Reasons for considering nonlinear curvature terms in the gravity lagrangian can be found in quantum effects in curved space [9] or in certain low-energy limits of string/M-theories [10]. The nonlinearity of the lagrangian can also be related to the existence of scalar degrees of freedom in the gravitational interaction [11]. In any case, the fact that certain $f(R)$ lagrangians naturally lead to early-time inflationary behaviors is the main motivation to study possible new gravitational effects in the late-time cosmic expansion.

Once a nonlinear lagrangian $f(R)$ has been proposed, the equations of motion for the metric can be derived in two inequivalent ways. On the one hand, one can follow the standard metric formalism, in which variation of the action with respect to the metric leads to a system of fourth-order equations. On the other hand, one may assume that metric and connection are independent fields and then take variations of the action with respect to the metric and with respect to the connection. In this case, the resulting equations of motion for the metric are second-order. Only when the function $f(R)$ is linear in R, GR and GR plus cosmological constant, metric and Palatini formalisms lead to the same equations of motion. In this work we will analyze and compare in detail the two formulations of $f(R)$ gravities.

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Though much work has been carried out in the last few years with regard to $f(R)$ gravities in the cosmological regime, very little is known about the form that the gravity lagrangian should have in order to be compatible with the cosmological observations [12,13]. The main reason for this seems to be the fact that the precision of the supernovae luminosity distance data and other currently available tests supporting the late-time cosmic speed-up is not enough to discriminate with confidence between one model or another. It would be thus desirable to have a new arena where to test these theories with higher precision. In our opinion, the solar system represents a scenario more suitable than the cosmological one to study the possible constraints on the lagrangian $f(R)$. In fact, if in addition to modified gravitational dynamics, sources of dark energy were acting in the cosmic expansion, it would be very difficult to distinguish their effect from a purely gravitational one. In the solar system, however, it is ordinary matter which dominates the gravitational dynamics, being the contribution of dark sources negligible. Therefore, we should see the solar system as a more suitable laboratory to impose the first useful constraints on $f(R)$ cosmologies.

In order to confront the predictions of a given gravity theory with experiment in the solar system, it is necessary to compute its weak-field, slow-motion (or post-Newtonian) limit. This limit has been computed for many metric theories of gravity and put in a standardized form [14], which depends on a set of parameters that change from theory to theory (Parametrized Post-Newtonian [PPN] formalism). However, for $f(R)$ gravities this limit has not yet been computed in detail. The Newtonian limit of these theories was recently studied in the metric formalism in [15] (see also [16]) and in the Palatini formalism in [17,18] (see also [19]). Our aim is to compute the complete post-Newtonian limit of $f(R)$ gravities and investigate the possible observational constraints on the lagrangian $f(R)$. To do so we rewrite the equations of motion of $f(R)$

gravities in the form of Brans-Dicke-like scalar-tensor theories [20]. In this form, metric and Palatini formalisms can be identified with the cases $\omega = 0$ and $\omega = -3/2$ of such theories, respectively, which clarifies the interpretation of the nonlinear terms of the lagrangian and simplifies the computations.

Unlike in the original Brans-Dicke theory [21], the scalar field associated to $f(R)$ gravities has self-interactions due to the nontrivial potential to be defined below in Eq. (13). Moreover, in order to constraint the theory, our task is not to determine the value of the parameter ω, which now is fixed, but to determine the admissible forms that the potential $V(\phi)$ may take. Since the potential is intimately related to the form of the function $f(R)$, the constraints on $V(\phi)$ will also condition the form of the lagrangian $f(R)$. Brans-Dicke theories have been extensively studied in the literature and their post-Newtonian limit is well known [14,22]. Those results were used in [23] to show that the Carroll *et al.* model [4], $f(R) = R - \mu^4/R$, in metric formalism could be ruled out according to solar system experiments. That conclusion was based on the fact that the scalar field had a small effective mass, which was computed in terms of the second derivative of the potential. However, that prescription is usually derived under the assumption that the potential and its first derivative vanish (see for instance [22,24]), conditions that, in general, cannot be imposed on $f(R)$ theories (see section III A). On the other hand, the case $\omega = -3/2$ of the original Brans-Dicke theory is a pathological exception of the general case $\omega =$ constant (see [25] for a discussion of the limit $\omega \rightarrow -3/2$). For this reason one can neither follow the philosophy of [23] and naively apply the results of the literature to investigate the post-Newtonian predictions for this case. As we will see, the existence of a nontrivial potential associated to the nonlinear terms of the lagrangian $f(R)$ cures the pathological aspects of the $\omega = -3/2$ theories. We are thus forced to study the post-Newtonian limit of $f(R)$ gravities having in mind the central role played by the potential of the scalar field.

In this work we will use a Brans-Dicke-like scalar-tensor representation to compute the post-Newtonian limit of $f(R)$ gravities taking into account all the terms associated to the potential of the scalar field. We will not make any assumption or simplification about the form of the function $f(R)$ that defines the lagrangian. In other words, rather than proposing a particular function $f(R)$ and comparing its predictions with the experimental and observational data, we want to see how those data constraint the form of the lagrangian. For the metric formalism, we will actually compute the post-Newtonian limit corresponding to Brans-Dicke-like scalar-tensor theories with arbitrary potential and a generic constant value of ω and will then particularize to the case $\omega = 0$, which corresponds to the metric form of $f(R)$ gravities. In this manner we generalize the results of the literature so as to include all the terms that are relevant for our discussion. The Palatini form, ($\omega = -3/2$), is still an exception of the general case $\omega =$ constant and must be computed independently (see Sec. IV).

The resulting post-Newtonian metrics will allow us to confront the predictions of these theories with the observational data. In this way, we will find a series of constraints for the lagrangian, which is *a priori* completely unknown. Those constraints turn out to be so strong that the lagrangians compatible with observations are bounded by well defined functions that forbid the growing of the nonlinear terms at low curvatures. This result, valid for both metric and Palatini formalisms, will be enough to invalidate the arguments supporting the cosmic speed-up as due to new gravitational effects at low curvatures.

The paper is organized as follows. Firstly we derive the equations of motion in the original $f(R)$ form and show how to obtain the scalar-tensor representation out of them. Secondly we compute the post-Newtonian limit of the metric formalism and discuss the observational constraints on the lagrangian. Then we follow the same scheme to study the Palatini formalism. Finally we summarize the results and compare the two formalisms.

II. EQUATIONS OF MOTION

The action that defines $f(R)$ gravities has the generic form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m[g_{\mu\nu}, \psi_m] \tag{1}$$

where $S_m[g, \psi_m]$ represents the matter action, which depends on the metric $g_{\mu\nu}$ and the matter fields ψ_m. For notational purposes, we remark that the scalar R is defined as the contraction $R = g^{\mu\nu} R_{\mu\nu}$, where $R_{\mu\nu}$ has the form of the Ricci tensor

$$R_{\mu\nu} = -\partial_\mu \Gamma^\lambda_{\lambda\nu} + \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} \tag{2}$$

and $\Gamma^\alpha_{\beta\gamma}$ is the connection.

In the metric formalism the connection is given in terms of the metric as follows

$$\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\lambda}}{2}(\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}) \tag{3}$$

In this case, since the metric is the basic geometrical object, we use the notation $R_{\mu\nu} \rightarrow R_{\mu\nu}(g)$ and $R \rightarrow R(g)$. In the Palatini formalism, where the connection is independent of the metric, we use the notation $R_{\mu\nu} \rightarrow R_{\mu\nu}(\Gamma)$ and $R \rightarrow R(\Gamma)$. We will see below that, in general, $R_{\mu\nu}(g) \neq R_{\mu\nu}(\Gamma)$ and, therefore, $R(g) \neq R(\Gamma)$.

A. Metric formalism

Varying Eq. (1) with respect to the metric, we obtain the following equations of motion:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \kappa^2 T_{\mu\nu} \tag{4}$$

where $f'(R) \equiv df/dR$. According to Eq. (4), we see that, in general, the metric satisfies a system of fourth-order partial differential equations. The higher-order derivative terms $\nabla_\mu\nabla_\nu f'$ and $\Box f'$ stem from integration by parts of the derivatives of the metric present in $\delta R_{\mu\nu}$. Only when $f(R)$ is a linear function of the scalar curvature, $f(R) = a + bR$, the equations of motion are second-order. The trace of Eq. (4) takes the form

$$3\Box f' + f'R - 2f = \kappa^2 T \tag{5}$$

B. Palatini formalism

Varying Eq. (1) with respect to the metric we obtain

$$f'(R)R_{\mu\nu}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{6}$$

where $f'(R) \equiv df/dR$. Note that the trace of Eq. (6)

$$f'(R)R - 2f(R) = \kappa^2 T, \tag{7}$$

implies an algebraic relation between $R \equiv R(\Gamma)$ and the trace T. The solution to this algebraic equation will be denoted by $R(\Gamma) = \mathcal{R}(T)$. The variation with respect to the connection must vanish independently of Eq. (6) and gives

$$\nabla_\lambda\left[\sqrt{-g}\left(\delta^\lambda_\alpha f' g^{\beta\gamma} - \frac{1}{2}\delta^\beta_\alpha f' g^{\lambda\gamma} - \frac{1}{2}\delta^\gamma_\alpha f' g^{\beta\lambda}\right)\right] = 0 \tag{8}$$

where $f' = f'(\mathcal{R}(T))$ is also a function of the matter terms. Using an auxiliary tensor $t_{\mu\nu} \equiv f' g_{\mu\nu}$, Eq. (8) can be readily solved [26]. The solution states the compatibility between the connection $\Gamma^\alpha_{\beta\gamma}$ and the *metric* $t_{\mu\nu}$. In other words, $\Gamma^\alpha_{\beta\gamma}$ can be written as the Levi-Civita connection of $t_{\mu\nu}$

$$\Gamma^\alpha_{\beta\gamma} = \frac{t^{\alpha\lambda}}{2}(\partial_\beta t_{\lambda\gamma} + \partial_\gamma t_{\lambda\beta} - \partial_\lambda t_{\beta\gamma}) \tag{9}$$

Inserting this solution for $\Gamma^\alpha_{\beta\gamma}$, written in terms of $g_{\mu\nu}$ and $f'(\mathcal{R}(T))$, in Eq. (6) we obtain

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = {} & \frac{\kappa^2}{f'}T_{\mu\nu} - \frac{\mathcal{R}(T)f' - f}{2f'}g_{\mu\nu} \\ & - \frac{3}{2(f')^2}\left[\partial_\mu f'\partial_\nu f' - \frac{1}{2}g_{\mu\nu}(\partial f')^2\right] \\ & + \frac{1}{f'}[\nabla_\mu\nabla_\nu f' - g_{\mu\nu}\Box f'] \end{aligned} \tag{10}$$

where $R_{\mu\nu}(g)$ and $R(g)$ are computed in terms of the Levi-Civita connection of the metric $g_{\mu\nu}$, i.e., they represent the usual Ricci tensor and scalar curvature. To make our notation clearer, since $t_{\mu\nu}$ and $g_{\mu\nu}$ are conformally related, it follows that $\mathcal{R}(T) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ and $R(g) = g^{\mu\nu}R_{\mu\nu}(g)$ are related by

$$\mathcal{R}(T) = R(g) + \frac{3}{2f'}\partial_\lambda f'\partial^\lambda f' - \frac{3}{f'}\Box f' \tag{11}$$

where, recall, $f' = f'(\mathcal{R}(T))$ is a function of T.

C. Scalar-tensor representation

The equations of motion derived above can be rewritten in a more compact and illuminating form introducing the following definitions

$$\phi \equiv f' \tag{12}$$

$$V(\phi) \equiv R(\phi)f' - f(R(\phi)) \tag{13}$$

where ϕ represents a scalar field and $V(\phi)$ is its potential. Note that in Eq. (13) we have assumed invertible the relation between R and $f'(R)$ to obtain $R(f')$. The equations of motion for the metric can then be expressed as follows

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = {} & \frac{\kappa^2}{\phi}T_{\mu\nu} - \frac{1}{2\phi}g_{\mu\nu}V(\phi) \\ & + \frac{\omega}{\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right] \\ & + \frac{1}{\phi}[\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\phi] \end{aligned} \tag{14}$$

where ω is a constant parameter. It is straightforward to verify that the metric formalism [see Eq. (4)] can be identified with the case $\omega = 0$ and the Palatini formalism [see Eq. (10)] with the case $\omega = -3/2$. The equation of motion for the scalar field ϕ is provided by the trace Eqs. (5) and (7) and can be expressed as

$$(3 + 2\omega)\Box\phi + 2V(\phi) - \phi\frac{dV}{d\phi} = \kappa^2 T \tag{15}$$

It is worth noting that in the metric case, $\omega = 0$, the scalar field is dynamical (it satisfies a second-order differential equation) whereas in the Palatini case, $\omega = -3/2$, the field is nondynamical (it satisfies an algebraic equation). In this latter case, the scalar field can be algebraically solved as $\phi = \phi(T)$. Thus, the effect of the nonlinear lagrangian in the Palatini formalism is rather different from its effect in the metric formalism. In the metric case, the compatibility between metric and connection gives rise to additional gravitational degrees of freedom in the theory, which manifest in a dynamical scalar field. In the Palatini formalism, however, the independent connection retains the second order of the equations of motion and modifies the way matter generates the space-time curvature associated to the metric. In other words, when $\omega = -3/2 \Leftrightarrow \phi = \phi(T)$, the right hand side of Eq. (14) represents a generalized energy-momentum tensor of matter in which the trace T

plays an enhanced role by means of the terms $\phi = \phi(T)$ and its derivatives.

It is remarkable the fact that the equations of motion (14) and (15) can be derived from the following action

$$S[g_{\mu\nu}, \phi, \psi_m] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[\phi R(g) - \frac{\omega}{\phi}(\partial_\mu\phi\partial^\mu\phi) - V(\phi)] + S_m[g_{\mu\nu}, \psi_m] \quad (16)$$

which represents a Brans-Dicke-like scalar-tensor theory (in the original Brans-Dicke theory $V(\phi) = 0$).

III. METRIC FORMALISM

In this section we will compute and analyze the post-Newtonian limit of the metric form of $f(R)$ gravities using their Brans-Dicke-like representation ($\omega = 0$). Since this computation is only slightly simpler than the general case $\omega \neq -3/2$, we consider the general case and then particularize to $\omega = 0$. In this manner we generalize the results of the literature regarding the post-Newtonian limit of Brans-Dicke theories so as to include all the terms that are relevant for our discussion.

A. Boundary conditions and coordinates

In order to obtain the metric in the solar system we will follow the basic guidelines outlined in chapter 4 of Will's book [14]. First thing to do is to identify the dynamical variables and set appropriate boundary conditions, which are provided by the value of those magnitudes far from the local system, i.e., in the cosmic regime. In the metric formalism ($\omega = 0$), and in general for $\omega \neq -3/2$ theories, the dynamical variables are the metric and the scalar field. In the cosmic regime, the high degree of homogeneity and isotropy allows to express the metric as a Friedman-Robertson-Walker metric

$$ds^2 = g^B_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 dx_i dx^i \quad (17)$$

The same argument indicates that for dynamical scalar fields the cosmic solution must be of the form $\phi = \phi^B(t)$, which only depends on the cosmic time t. Once the boundary conditions have been established, which requires a complete cosmic solution, we need to note that at smaller scales homogeneity and isotropy are lost and local deviations from the background values $g^B_{\mu\nu}$ and ϕ^B may appear.

In our computations we will use coordinates $(\bar{t}, \bar{x}^j)$ in which the outer regions of the local system are in free fall with respect to the surrounding cosmological model. Neglecting second-order corrections, the local and background coordinates are simply related by $\bar{t}(t_0, x_0; t, x) \approx (t - t_0)$ and $\bar{x}^j(t_0, x_0; t, x) \approx a_0(x - x_0)^j$. From now on we will omit the bar on the local coordinates and will denote ϕ_0, $\dot{\phi}_0$ the asymptotic boundary values of the scalar field at the cosmic time t_0, i.e., $\phi_0 = \phi^B(t_0)$ and $\dot{\phi}_0 = \dot{\phi}^B(t_0)$.

For approximately static solutions, corresponding to gravitating masses such as the Sun or Earth, to lowest-order, we can drop the terms involving time derivatives from the equations of motion. In our local coordinate system, the metric can be expanded about its Minkowskian value as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The solution for the scalar field can be expressed in the form $\phi = \phi_0 + \varphi(t, x)$, where $\varphi(t, x)$ vanishes far from the local system and represents the local deviation from ϕ_0. We want to remark that since ϕ_0 and $\dot{\phi}_0$ depend on R_0 and $\dot{R}_0$, the metric of the local post-Newtonian system will also depend on the background cosmic values R_0 and $\dot{R}_0$. The dependence on these background quantities will make the metric change adiabatically in a cosmic timescale. This adiabatic evolution could make a theory be compatible with the current experimental tests during some cosmic era but fail in other periods. Because of the relevance of this issue in our discussion, we will give below some examples to illustrate this effect.

With regard to the potential defined for the scalar field, see Eq. (13), it is easy to see that $dV/d\phi = R$ [20]. Since, the curvature can be expressed as $R = R_0 + \sigma(t, x)$, where $\sigma(t, x)$ denotes the local deviation from the background cosmic curvature R_0, it is clear that the scalar field will not, in general, satisfy the extremum condition $dV/d\phi = 0$. This is to be contrasted with the results of the literature regarding the post-Newtonian limit of Brans-Dicke-like theories, where it is generally assumed that the field is near an extremum [22,24]. We thus see that for $f(R)$ gravities it is necessary to consider all the terms associated to the potential.

B. Second-order corrections

As we advanced above, we will expand the equations of motion around the background values of the metric and the scalar field. In particular, we will take $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$, $\phi = \phi_0 + \varphi(t, x)$ and $V(\phi) \approx V_0 + \varphi V'_0 + \varphi^2 V''_0/2 + \ldots$ The complete post-Newtonian limit needs the different components of the metric and the scalar field evaluated to the following orders $g_{00} \sim O(2) + O(4)$, $g_{0j} \sim O(3)$, $g_{ij} \sim O(2)$ and $\phi \sim O(2) + O(4)$ (see [14]). The details of the calculations and the complete post-Newtonian limit for the theories defined in Eq. (16) can be found in Appendix A. For convenience, we will discuss here only the lowest-order corrections, $g_{00} \sim O(2)$, $g_{ij} \sim O(2)$ and $\phi \sim O(2)$, of the case we are interested in, namely, $\omega = 0$. The order of approximation will be denoted by a superindex. This approximation will be enough to place tight constraints on the gravity lagrangian. To this order, the metric satisfies the following equations

$$-\frac{1}{2}\nabla^2\left[h^{(2)}_{00} - \frac{\varphi^{(2)}}{\phi_0}\right] = \frac{\kappa^2\rho}{2\phi_0} + \left(\frac{3}{2}\frac{\ddot{\phi}_0}{\phi_0} - \frac{V_0}{2\phi_0}\right) \quad (18)$$

$$-\frac{1}{2}\nabla^2\left[h_{ij}^{(2)} + \delta_{ij}\frac{\varphi^{(2)}}{\phi_0}\right] = \delta_{ij}\left[\frac{\kappa^2\rho}{2\phi_0} - \frac{\ddot{\phi}_0}{2\phi_0} + \frac{V_0}{2\phi_0}\right] \quad (19)$$

where the gauge condition $h^{\mu}_{k,\mu} - \frac{1}{2}h^{\mu}_{\mu,k} = \partial_k\varphi^{(2)}/\phi_0$ has been used. In eliminating the zeroth-order terms in the field equation for φ, corresponding to the cosmological solution for ϕ_0, the equation for the scalar field to this order boils down to

$$[\nabla^2 - m_\varphi^2]\varphi^{(2)}(t, x) = -\frac{\kappa^2\rho}{3} \quad (20)$$

where m_φ^2 is a slowly-varying function of the cosmological time given by

$$m_\varphi^2 \equiv \frac{\phi_0 V_0'' - V_0'}{3} \quad (21)$$

Note that, despite our notation, there is no *a priori* restriction on the sign of m_φ^2. The equations of above can be easily integrated to give

$$\varphi^{(2)}(t, x) = \frac{\kappa^2}{3}\frac{1}{4\pi}\int d^3x' \frac{\rho(t, x')}{|x - x'|}F(|x - x'|) \quad (22)$$

$$h_{00}^{(2)}(t, x) = \frac{\kappa^2}{\phi_0}\frac{1}{4\pi}\int d^3x' \frac{\rho(t, x')}{|x - x'|}\left[1 + \frac{F(|x - x'|)}{3}\right] - \left(\frac{3}{2}\frac{\ddot{\phi}_0}{\phi_0} - \frac{V_0}{2\phi_0}\right)\frac{|x - x_c|^2}{3} \quad (23)$$

$$h_{ij}^{(2)}(t, x) = \left(\frac{\kappa^2}{\phi_0}\frac{1}{4\pi}\int d^3x' \frac{\rho(t, x')}{|x - x'|}\left[1 - \frac{F(|x - x'|)}{3}\right] + \left[\frac{\ddot{\phi}_0}{2\phi_0} - \frac{V_0}{2\phi_0}\right]\frac{|x - x_c|^2}{3}\right)\delta_{ij} \quad (24)$$

where x_c is an arbitrary constant vector and the function $F(|x - x'|)$ is given by

$$F(|x - x'|) = \begin{cases} e^{-m_\varphi|x-x'|} & \text{if } m_\varphi^2 > 0 \\ \cos(m_\varphi|x - x'|) & \text{if } m_\varphi^2 < 0 \end{cases} \quad (25)$$

Note that the term $\dot{\phi}_0$ does not appear in Eqs. (22)–(24) and, therefore, the fact that $\dot{R}_0$ may not be strictly zero affects the Newtonian limit very weakly. In the post-Newtonian limit it contributes to $h_{00}^{(4)}$ (see the Appendix). In any case, since to all effects ϕ_0 is almost constant, we can neglect the contributions due to $\dot{\phi}_0$ and $\ddot{\phi}_0$.

Since in the solar system the Sun represents the main contribution to the metric, we can approximate the expressions of above far from the sources by

$$h_{00}^{(2)} \approx 2G\frac{M_\odot}{r} + \frac{V_0}{6\phi_0}r^2 \quad (26)$$

$$h_{ij}^{(2)} \approx \delta_{ij}\left[2\gamma G\frac{M_\odot}{r} - \frac{V_0}{6\phi_0}r^2\right] \quad (27)$$

where $M_\odot = \int d^3x' \rho_{\rm sun}(t, x')$ is the Newtonian mass of the Sun. We have defined the effective Newton's constant G as

$$G = \frac{\kappa^2}{8\pi\phi_0}\left[1 + \frac{F(r)}{3}\right] \quad (28)$$

and the effective PPN parameter γ as

$$\gamma = \frac{3 - F(r)}{3 + F(r)} \quad (29)$$

We shall show now that the oscillatory solutions, $m_\varphi^2 < 0 \rightarrow F(r) = \cos(m_\varphi r)$, are always unphysical. For this case, the inverse-square law gets modified as follows

$$\frac{M_\odot}{r^2} \rightarrow \left(1 + \frac{\cos(m_\varphi r) + (m_\varphi r)\sin(m_\varphi r)}{2}\right)\frac{M_\odot}{r^2} \quad (30)$$

For very light fields, which represent long-range interactions, the argument of the sinus and the cosinus is very small in solar system scales ($m_\varphi r \ll 1$). We can thus approximate $\cos(m_\varphi r) \approx 1$ and $\sin(m_\varphi r) \approx 0$ and recover the usual Newtonian limit up to an irrelevant redefinition of Newton's constant. However, these approximations also lead to $\gamma \approx 1/2$, which is observationally unacceptable since $\gamma_{\rm obs} \approx 1$ [27]. If the scalar interaction were short- or midrange, the Newtonian limit would get dramatically modified. In fact, the leading-order term is then oscillating, $\sin(m_\varphi r)M_\odot/r$, and is clearly incompatible with observations. We are thus led to consider only the damped solutions $F(r) = e^{-m_\varphi r}$.

The Yukawa-type correction in the Newtonian potential has not been observed over distances that range from meters to planetary scales. In addition, since the post-Newtonian parameter γ is observationally very close to unity, we see that the effective mass in Eqs. (29) and (30) must satisfy the constraint $m_\varphi^2 L^2 \gg 1$, where L represents a typical experimental length scale. This inequality indicates that the scalar field must be heavy or, equivalently, that the scalar interaction is short range.

We mention that when ω is not fixed (see Appendix A), there is also the possibility of having a very light (long-range) field that yields almost space-independent values of G and γ. In that case, the theory behaves as a Brans-Dicke theory with γ given by

$$\gamma = \frac{1 + \omega}{2 + \omega} \quad (31)$$

and it takes $\omega > 40000$ to satisfy the observational constraints [28].

The cosmological constant term $(V_0/6\phi_0)r^2$ appearing in Eqs. (26) and (27) also imposes constraints on particular models. This contribution, related to the scalar energy-density, must be very small in order not to modify the gravitational dynamics of local systems ranging from the solar system to clusters of galaxies. In the terminology of $f(R)$ gravities, the constraint from this term is

$$|\frac{f_0 - R_0 f_0'}{f_0'}| L_L^2 \ll 1 \tag{32}$$

where L_L may represent a (relatively large) length scale the same order or greater than the solar system. The constraint $m_\varphi^2 L_S^2 \gg 1$ associated to the effective mass of the scalar field can be reexpressed as

$$\left(\frac{f_0' - R_0 f_0''}{f_0''}\right) L_S^2 \gg 1 \tag{33}$$

where L_S represents a (relatively short) length scale that can range from meters to planetary scales, depending on the particular test used to verify the theory. It is worth noting that a generic lagrangian of the form

$$f(R) = R + \lambda h(R) \tag{34}$$

with λ a suitable small parameter, satisfies the two constraints of above if $h(R)$, $h'(R)$ or $h''(R)$ are finite or vanish as the universe expands. General Relativity, which can be seen as the limit $\lambda \longrightarrow 0$, saturates those constraints. We will consider in the next section some examples of theories with the form proposed in Eq. (34). In Sec. III D we will analyze in detail the implications of the constraint of Eq. (33).

Before concluding this section, we shall briefly discuss some simplifications that may be carried out from the above considerations in the complete post-Newtonian metric given in Appendix A. First of all, it is worth noting that with a tiny V_0 we can eliminate part of the cosmological constant terms. This fact together with our definition for G leads to the PPN parameter $\beta = 1$, which coincides with the one corresponding to GR. On the other hand, a massive field would allow us to neglect the exponential terms and the $\varphi^{(2)}$ contributions. Further simplifications could be achieved from the observational evidence supporting the constancy of Newton's constant. Assuming a massive field, it follows that $\dot{G}/G \approx -\dot{\phi}_0/\phi_0$. This relation provides a justification to argue that $\dot{\phi}_0/\phi_0$ and $\ddot{\phi}_0/\phi_0$ are small, if nonzero. With these simplifications we recover the post-Newtonian limit of GR, where $\dot{G}/G = 0$. We thus see that measurements of a change in G with time and of Yukawa-type corrections in the inverse-square law could be due to the presence of nonlinear elements in the gravity lagrangian.

C. Adiabatic evolution of m_φ and V_0. Examples

We shall now illustrate with some simple examples how the parameters that characterize the post-Newtonian metric, such as the scalar energy density V_0 or the interaction range m_φ^{-1}, are subject to a slow adiabatic evolution due to the cosmic expansion. Since V_0 and m_φ may change with time, the gravitational dynamics of local systems may undergo dramatic changes driven by the cosmic expansion. For this reason, a given $f(R)$ model could pass the current observational constraints at a given cosmic time but fail at other times. The aim of this section is to point out that the dynamical properties of a local gravitating system at a given time may not be completely determined by its own internal characteristics, but may be affected by the state of the universe as a whole at that moment. Therefore, the discussion of the post-Newtonian limit of the metric form of a given $f(R)$ model requires the knowledge of the cosmic evolution of that theory.

1. Positive powers of R

Following the structure of the ansatz proposed in Eq. (34), we can consider the family of models defined by $f(R) = R + R^n/M^{2n-2}$, where M represents a very large mass scale. We will only consider the cases $n \geq 2$. These models are characterized by

$$\phi \equiv f' = 1 + n\left(\frac{R}{M^2}\right)^{n-1} \tag{35}$$

$$V(\phi) = M^2(n-1)\left(\frac{R}{M^2}\right)^n = M^2(n-1)\left(\frac{\phi-1}{n}\right)^{n/n-1} \tag{36}$$

With Eq. (36) at hand, we can compute the effective mass m_φ^2 that characterizes the post-Newtonian metric. It is given by

$$\begin{aligned} m_\varphi^2 &= \frac{R_0}{3(n-1)}\left[\frac{1}{n}\left(\frac{M^2}{R_0}\right)^{n-1} - (n-2)\right] \\ &= \frac{M^2}{3n}\left(\frac{n}{\phi_0 - 1}\right)^{n-2/n-1}\left[1 - \frac{(n-2)}{(n-1)}\phi_0\right] \end{aligned} \tag{37}$$

where $\phi_0 \equiv f'(R_0)$ and R_0 represent the cosmological values of ϕ and R at the moment t_0. The time-time component of Eq. (4) can be used to extract some information about the cosmological evolution of R. This will help us to understand the adiabatic change in the post-Newtonian metric. The expansion factor satisfies the following equation

$$\begin{aligned} 3\left(\frac{\dot{a}}{a}\right)^2 = \kappa^2\rho - \left(\frac{R}{M^2}\right)^{n-1}\Big[3n\left(\frac{\dot{a}}{a}\right)^2 - \frac{(n-1)}{2}R \\ + 3n(n-1)\frac{\dot{a}}{a}\frac{\dot{R}}{R}\Big] \end{aligned} \tag{38}$$

Inserting $a(t) = a_0 e^{\gamma t}$ in Eq. (38) and taking $\rho = 0$ for simplicity, it follows that at early-times the evolution is dominated by the $(R/M^2)^n$ contribution with $\gamma^{2(n-1)} = (M^2/12)^{n-1}/(n-2)$. After the early-time inflation predicted by these expansion factors, as the curvature decays below the scale defined by M^2, the $(R/M^2)^n$ effect is suppressed and the subsequent evolution is governed by GR, with $a(t) = a_0 t^s$ and $s = 1/2$ during the radiation dominated era, and $s = 2/3$ during the matter dominated era. Thus, at all times after the inflationary period, we have $M^2/R \gg 1$, or equivalently $(\phi - 1) \rightarrow 0$. This leads to a very large effective mass for the scalar field and a tiny cosmological constant term $V_0/\phi_0 \to 0$. In consequence, this family of models yields an acceptable weak-field limit. In fact, it seems reasonable to think that these theories are compatible with GR in all astrophysical applications, since the curvature is expected to be much smaller than M^2 in all situations except at the very early universe.

2. Negative powers of R

A well-known example of this type is the Carroll *et al.* model [4], defined by $f(R) = R - \mu^4/R$, where μ represents a tiny mass scale of order 10^{-33} eV. The reason for the minus sign in front of μ^4 is intriguing, since this definition leads to a negative effective mass

$$m_\varphi^2 = -\frac{R}{6\mu^4}(R^2 + 3\mu^4) \tag{39}$$

which we have shown to be in conflict with the post-Newtonian limit (see (30)). An improved formulation of the theory could be obtained by changing the sign in front of μ^4 in the definition of $f(R)$. In this way, we can easily extend the results of the examples of above to the models $f(R) = R + \mu^{2n+2}/R^n$. A direct consequence of the positive sign in front of μ^{2n+2} is the loss of exponential solutions for $a(t)$ at late-times, since the relation between γ and μ turns into $\gamma^{2(n+1)} = -(n+2)(\mu^2/12)^{n+1}$. These models are characterized by

$$\phi \equiv f' = 1 - n\left(\frac{\mu^2}{R}\right)^{n+1} \tag{40}$$

$$V(\phi) = -\mu^2(n-1)\left(\frac{\mu^2}{R}\right)^n = \mu^2(n+1)\left(\frac{n}{1-\phi}\right)^{n/n+1} \tag{41}$$

The effective mass of the scalar field takes the form

$$\begin{aligned} m_\varphi^2 &= \frac{R_0}{3(n+1)}\left[\frac{1}{n}\left(\frac{R_0}{\mu^2}\right)^{n+1} - (n+2)\right] \\ &= \frac{\mu^2}{3n}\left(\frac{n}{1-\phi_0}\right)^{n+2/n+1}\left[\frac{(n+2)}{(n+1)}\phi_0 - 1\right] \end{aligned} \tag{42}$$

We will restrict our discussion to the cases with $n \geq 1$. The cosmological evolution of these models during the radiation dominated era requires a complete solution of the model, since a simple power law expansion is ill-defined. We will just concentrate on the matter dominated era, $a(t) = a_0 t^{2/3}$, and beyond, $a(t) = \tilde{a}_0 t^{s_n}$ with $s_n = (2n+1)(n+1)/(n+2)$. These solutions imply that the curvature decays with the cosmic time as $R = 6s_n(2s_n - 1)/t^2$. One can numerically check that the transition from the matter dominated era, $s = 2/3$, to its final value s_n is smooth (we took $\kappa^2\rho_{m_0}/\mu^2 = 3/7$). During the matter dominated era, $\mu^2/R \to 0$ and $\phi \approx 1$, Eqs. (42) and (41) indicate that m_φ^2 is very large and V_0/ϕ_0 very small. In consequence, these models yield a valid post-Newtonian limit. However, as the universe expands and the curvature approaches the critical value $(R_c/\mu^2)^{n+1} = n(n+2)$, in which $m_\varphi^2 = 0$, the effective mass is small and the post-Newtonian limit tends to that of a Brans-Dicke theory with $\omega = 0$, which is ruled out by observations. At later times, m_φ^2 becomes negative and the weak-field approximation is ill-defined, as we discussed above. We can, thus, conclude that these theories do not represent a good alternative to explain the late-time cosmic speed-up, since they have an unacceptable weak-field limit at the present time.

D. Constrained lagrangian

We have just seen that the post-Newtonian metric of $f(R)$ theories may be very sensitive to the evolution of the cosmic boundary values. In particular, we have shown that when the lagrangian contains negative powers of R the scalar interaction goes from a short-range interaction during the matter dominated era to a long-range interaction at later times. This"methamorphosis", due to the running of the background boundary values with the cosmic expansion, can make a theory initially viable invalid at late times. A qualitative analysis of the constraint given in Eq. (33) can be used to argue that, in general, $f(R)$ gravities with terms that become dominant at low cosmic curvatures are not viable theories in solar system scales and, therefore, cannot represent an acceptable mechanism for the cosmic expansion. Roughly speaking, Eq. (33) says that the smaller the term f_0'', with $f_0'' > 0$ to guarantee $m_\varphi^2 > 0$, the heavier the scalar field[1]. In other words, the smaller f_0'', the shorter the interaction range of the field. In the limit $f_0'' \to 0$, corresponding to GR, the scalar degree of freedom is completely suppressed. Thus, if the nonlinearity of the gravity lagrangian had become dominant in the last few billions of years, the scalar field interaction range would have increased accordingly. In consequence, gravitating systems such as the solar system, globular clusters, galaxies, ...would have experienced (or will experience) observable changes in their gravitational dynamics. Since there is no experimental evidence supporting such a change and all currently available solar system gravitational experiments are compatible with GR, it seems unlikely that the nonlinear corrections may be dominant at the current epoch.

Let us analyze now in a more quantitative way the constraint $m_\varphi^2 L_S^2 \gg 1$ given in Eq. (33). That equation can be rewritten as follows

$$R_0\left[\frac{f'(R_0)}{R_0 f''(R_0)} - 1\right]L_S^2 \gg 1 \tag{43}$$

We are interested in the form of the lagrangian at intermediate and low cosmic curvatures R_0 (matter dominated

[1]Note that $\phi \equiv f'$ must be positive in order to have a well-posed theory.

and vacuum dominated eras), i.e., when the environmental conditions allow for the existence of planetary and stellar systems. We have shown that the scalar interaction today must be short range (recall that there is no evidence for Yukawa-type corrections to the inverse-square law and that $\gamma \approx 1$). We shall thus demand that the interaction range of the scalar field remains as short as it is today or decreases with time so as to avoid dramatic modifications of the gravitational dynamics in post-Newtonian systems with the cosmic expansion. This can be implemented imposing

$$\left[\frac{f'(R)}{Rf''(R)} - 1\right] \geq \frac{1}{l^2 R} \tag{44}$$

as R decreases with the expansion, where $l^2 \ll L_S^2$ represents a bound to the current interaction range of the scalar field. Manipulating this expression, we obtain

$$\frac{d\log[f'(R)]}{dR} \leq \frac{l^2}{1 + l^2 R} \tag{45}$$

which can be integrated twice to give the following inequality

$$f(R) \leq A + B\left(R + \frac{l^2 R^2}{2}\right) \tag{46}$$

where B is a positive constant, which can be set to unity without loss of generality. Since f' and f'' are positive, the lagrangian is also bounded from below, i.e., $f(R) \geq A$. In addition, according to the cosmological data, $A \equiv -2\Lambda$ must be of order a cosmological constant $2\Lambda \sim 10^{-53}$ m^2. We thus conclude that the gravity lagrangian at curvatures $R \ll l^{-2}$ is bounded by

$$-2\Lambda \leq f(R) \leq R - 2\Lambda + \frac{l^2 R^2}{2} \tag{47}$$

This result shows that the lagrangian must be almost linear in R and that the nonlinear corrections are bounded quadratically. Since the nonlinearities are relevant only at curvatures $R \gtrsim l^{-2}$, taking $l \sim 1$ m as a rough upper bound for the scalar interaction range and using $R = (8\pi G/c^2)\rho$ we see that the nonlinear effects appear at very high densities, $\rho \sim 10^{22}$ g/cm^3. Therefore, Eq. (47) holds over the entire matter dominated and vacuum dominated eras ($R \ll l^{-2}$). Obviously, at very high curvatures the post-Newtonian constraints may not make sense and higher powers of R may be allowed in the lagrangian. Equation (47) should thus be seen as valid at relatively low curvatures and with $l^2 R^2$ representing the leading order of the possible nonlinear corrections.

We have thus confirmed, as we argued above, that $f(R)$ gravities with nonlinear terms that grow with the expansion of the universe are incompatible with observations and cannot represent a valid mechanism to justify the cosmic speed-up. In the viable models the nonlinearities represent a short-range scalar interaction, whose effect in the late-time cosmic dynamics reduces to that of a cosmological constant and, therefore, do not substantially modify the description provided by General Relativity with a cosmological constant. As a final remark, we want to point out that the Starobinsky model $f(R) = R + l^2 R^2$ [29] besides leading to early-time inflation and satisfying the solar system observational constraints, also seems compatible with CMBR observations [30].

IV. PALATINI FORMALISM

In this section we will compute and analyze the post-Newtonian limit of the Palatini form of $f(R)$ gravities using their Brans-Dicke-like scalar-tensor representation ($\omega = -3/2$). Unlike in the original Brans-Dicke theory, where $V(\phi) = 0$, now the potential associated to the scalar field is nontrivial and the theory is well defined, which allows to evaluate the post-Newtonian limit. In this manner, we complete the computation of the post-Newtonian limit of Brans-Dicke-like theories and fill a gap present in the literature. The details of the computations can be found in Appendix B.

A. Boundary conditions and coordinates

Since in this case the scalar field is nondynamical (see Eq. (15) and take $\omega = -3/2$), the metric represents the only dynamical field . Using the same coordinates as introduced in section III A, the background Friedman-Robertson-Walker metric can be made Minkowskian and the contributions to the post-Newtonian metric can be found perturbatively as $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$. This choice of coordinates fixes all the boundary conditions needed for this problem. Note that no boundary conditions are needed for $\phi = \phi(T)$ since it is determined locally by the value of the trace T.

Once a solution $\phi = \phi(T)$ has been obtained, it could be expanded to different orders of approximation in the post-Newtonian expansion using the fact that for a perfect fluid $T = -\rho(1 + \Pi - 3P/\rho) \approx -\rho + \rho O(v^2/c^2)$, where ρ is the rest-mass density, Π is the specific energy density (ratio of energy density to rest-mass energy), and P is the pressure (see chapter 4 of [14]). In this way one would obtain an expansion of the form $\phi(T) \approx \phi(-\rho) + \partial_T \phi|_{-\rho} O(v^2/c^2) + \ldots$ However, this is an unnecessary notational complication and, therefore, we will keep $\phi(T)$ exact in our calculations. Note that this expansion in post-Newtonian orders is different from an expansion around the vacuum $\phi(T) \approx \phi(0) + \partial_T \phi(0) T + \ldots$ such as the one apparently considered in [17,18] using the original $f(R)$ representation. In their calculations they expanded the function $f(R)$ around a de Sitter background characterized by a constant curvature R_0. The fact that in the Palatini approach $R(\Gamma) = \mathcal{R}(T)$ (do not confuse $R(\Gamma)$ with $R(g)$), implies that $R_0 = \mathcal{R}(T = 0)$. Thus, an expansion of $f(R)$ around R_0 actually represents an expansion around $T \approx 0$, which is an expansion around the vacuum, not an expansion in post-Newtonian orders. Since the functional dependence of $f(R)$ with T is *a priori* unknown, there is no guarantee that such expansion around the vacuum can be valid in the range from $T = 0$ up to the typical densities inside planets, stars or laboratory-size bodies. In fact, the weak-field slow-motion limit does not require low densities but not too high matter concentrations and low matter velocities, $v^2/c^2 \ll 1$. Thus, the conclusions regarding the Newtonian limit obtained in [17,18] could not be valid. This point will be clarified below in detail. In our description in terms of a scalar field, $\phi(T) \equiv f'[\mathcal{R}(T)]$, the role of the $\phi(T)$ terms is clear from the very beginning: they represent new contributions of the matter sources to the equations of motion of the metric. In consequence, they must be treated as matter terms and expanded in post-Newtonian orders, not around the vacuum.

B. Second-order corrections

For convenience, we introduce a dimensionless quantity $\tilde{\phi} = \phi/\phi_0$, where $\phi_0 \equiv \phi(0)$ is the vacuum reference value, and define $\Omega(T) \equiv \log(\tilde{\phi})$. In order to get h_{ij} diagonal and to respect the perturbative description, we find that $\Omega(T)$ must be seen, at least, of order $O(v^2)$. We will indicate with a superindex the order of approximation of each quantity when necessary. To second order, the metric satisfies the equations

$$-\frac{1}{2}\nabla^2[h_{00}^{(2)} - \Omega^{(2)}] = \frac{\kappa^2\rho - V(\phi)}{2\phi} \tag{48}$$

$$-\frac{1}{2}\nabla^2[h_{ij}^{(2)} + \delta_{ij}\Omega^{(2)}] = \left[\frac{\kappa^2\rho + V(\phi)}{2\phi}\right]\delta_{ij} \tag{49}$$

where we have used the gauge condition[2] $h^{\mu}_{k,\mu} - \frac{1}{2}h^{\mu}_{\mu,k} = \partial_k\Omega$. These equations admit the following solutions

$$h_{00}^{(2)}(t,x) = \frac{\kappa^2}{4\pi\phi_0}\int d^3x' \frac{[\rho(t,x') - V(\phi)/\kappa^2]}{\tilde{\phi}|x - x'|} + \Omega^{(2)} \tag{50}$$

$$h_{ij}^{(2)}(t,x) = \left[\frac{\kappa^2}{4\pi\phi_0}\int d^3x' \frac{[\rho(t,x') + V(\phi)/\kappa^2]}{\tilde{\phi}|x - x'|} - \Omega^{(2)}\right]\delta_{ij} \tag{51}$$

In these equations, the local term $\Omega^{(2)} = \log[\tilde{\phi}(-\rho)]$ represents a new effect that is not present in the general Brans-Dicke-like case $\omega \neq -3/2$. The contribution due to Ω is identically zero only if $f(R)$ is linear (GR and GR plus cosmological constant) and, therefore, its presence would imply the nonlinearity of the gravity lagrangian. It is worth noting that rather than an integrated quantity (cumulative effect), it is directly related to the local matter density. In consequence, an isolated body will contribute to the exterior space-time metric by means of the integral terms of Eqs. (50) and (51) only. If we now put an object in orbit around the first one, the metric at the position of this new body will be modified by the local term Ω and by the self-gravity of the body. For the moment, we will concentrate on the integral terms of Eqs. (50) and (51) (isolated body within a completely empty environment).

Assuming that the main contribution to the metric in the solar system is due to the sun, we can express Eqs. (50) and (51) outside the Sun as follows

$$h_{00}^{(2)}(t,x) = 2G\frac{M_\odot}{r} - \frac{V_0}{\phi_0}\frac{r^2}{6} \tag{52}$$

$$h_{ij}^{(2)}(t,x) = \left[2\gamma G\frac{M_\odot}{r} + \frac{V_0}{\phi_0}\frac{r^2}{6}\right]\delta_{ij} \tag{53}$$

In these expressions, G and γ are defined as

$$G = \frac{\kappa^2}{8\pi\phi_0}\left(1 + \frac{M_V}{M_\odot}\right) \tag{54}$$

$$\gamma = \frac{M_\odot - M_V}{M_\odot + M_V} \tag{55}$$

where $M_\odot \equiv \int d^3x'\rho(t,x')/\tilde{\phi}$, $M_V \equiv \kappa^{-2}\int d^3x'[V_0 - V(\phi)/\tilde{\phi}]$ and $V_0 = V(\phi_0)$. Since the cosmological constant term V_0/ϕ_0 must be negligible in solar system scales in order not to affect the local dynamics, we find a constraint on the function $f(R)$. We need to note that the value ϕ_0 is solution of Eq. (15) with $\omega = -3/2$ and $T = 0$. Using that equation and the definition of $V(\phi)$ in terms of $f(R)$, it follows that $V_0 = f(R_0)$, where R_0 is solution of Eq. (7) outside the Sun, i.e., $R_0 = \mathcal{R}(T = 0)$. From these considerations it follows that

$$\left|\frac{f(R_0)}{f'(R_0)}\right| L_L^2 \ll 1 \tag{56}$$

where L_L represents a (Large) length scale the same order or greater than the solar system and R_0 presumably is of order the cosmological constant $\Lambda \sim 10^{-53}\ \mathrm{m}^{-2}$.

Let us consider now the observational constraints on G and γ. It is well known (see Sec. III) that in dynamical scalar-tensor theories the effective constants G and γ depend on two cosmic parameters, namely, the state of the field, ϕ_0, and the range m_φ^{-1} of its interaction, which are the same for all bodies at a given cosmic time [14,22]. In the nondynamical situation discussed here, G and γ are not universal quantities, i.e., they are not the same for all bodies. According to the definitions given above after Eq. (55), two bodies with the same $M_\odot$ do not necessarily have the same value M_V and, therefore, may lead to different values of G and γ. This is due to the fact that $M_\odot$ and M_V are defined as integrals over quantities related to $\phi = \phi(T)$, whose values depend on the structure and composi-

[2] This condition was already used in [19], and also in [18]. In their notation, $-\partial_k\Omega = b_k$.

tion of the body. Obviously, the experimental evidence supporting the universality of G and the measurements of $\gamma \approx 1$ [27] indicate that $|M_V/M_\odot| \ll 1$. The only cases in which $M_V = 0$ correspond to GR and GR plus cosmological constant, i.e., those cases in which the lagrangian $f(R)$ is linear, or $V = V_0 =$ constant. All nonlinear lagrangians predict a nontrivial potential $V(\phi)$ and, therefore, a non vanishing M_V, which may give rise to the effects discussed above. Unfortunately, the fact that M_V is given as an integrated quantity does not allow us to place any explicit constraint on the form of the function $f(R)$. On the other hand, it is quite disturbing the fact that a body with Newtonian mass $M_N \equiv \int d^3x' \rho(t, x')$ may yield different values of $M_\odot$, G and γ depending on its internal properties. Stated another way, a given amount of Newtonian mass may lead to gravitational fields of different strengths and dynamical properties. Since, as far as we know, effects of this type have not been observed in laboratory, we expect a very weak dependence of $M_\odot$ on ϕ. This is equivalent to saying that ϕ cannot change too much with the density.

It is worth noting that with the definitions given above for G and γ and neglecting the cosmological constant term, we can write for an isolated body $h_{00}^{(2)} \equiv 2U$, where U represents the Newtonian potential. It is thus easy to see that the term $(h_{00}^{(2)})^2/2$ of the complete post-Newtonian limit (see the Appendix) leads to the PPN parameter $\beta = 1$, like in GR. The remaining higher-order terms of the metric are all affected by ϕ.

C. $\Omega^{(2)}$ contribution

We will now analyze the effect of the term $\Omega^{(2)}$ that we omitted above in the case of an isolated massive body. As we pointed out, this term must be taken into account when a test body is placed within the gravitational field of another body. Thus, it must be present in any physical situation. Neglecting the cosmological constant contribution for simplicity, we can write the metric as follows

$$h_{00}^{(2)}(t, x) = 2U(r) + \Omega^{(2)}(T) \tag{57}$$

$$h_{ij}^{(2)}(t, x) = [2\gamma U(r) - \Omega^{(2)}(T)]\delta_{ij} \tag{58}$$

where $U(r) \equiv GM_\odot/r$ is the Newtonian potential generated by the massive body and $\Omega^{(2)}(T) = \log[\phi(T)/\phi_0]$ is a local term that depends on the matter density $T = -\rho$ at the point (t, x), where our test body is located. First thing we need to note is that $\Omega^{(2)}(T)$ is a perturbative quantity of order $O(v^2/c^2) \ll 1$. The only manner to respect the perturbative approach is accepting that $\phi(T)$ depends very weakly on ρ, i.e., that $\phi(T)$ must be almost constant over a wide range of densities and can be well approximated by $\phi(T) = \phi_0 + (\partial\phi/\partial T)|_{T=0}T + \ldots,$ with $\phi_0^{-1}(\partial\phi/\partial T)|_{T=0}T \ll 1$ from $T = 0$ up to nuclear densities ($T \sim 10^{14}$ g/cm3) at least. The need for this expansion about ϕ_0 indicates that the lagrangian must be almost linear in R (recall that $\phi \equiv df/dR$). Furthermore, if $\phi(T)$ had a stronger dependence on T, individual atoms could experience strong accelerations due to sudden changes in $\Omega^{(2)}$ when going from outside atoms to inside atoms. Those individual microscopic gravitational effects would manifest in the macroscopic, averaged, description of matter. Since such effects have not been observed, they must be very small, if they actually exist. Thus, the weak dependence of ϕ on T within this wide density interval confirms that the contribution of the nonlinear terms to the lagrangian $f(R)$ must be very small, if any. This conclusion agrees with our previous claims regarding the weak dependence of $M_\odot$, G and γ on $\phi(T)$.

Let us analyze in detail the dependence of ϕ on T. We will consider, as an illustration, the Newtonian limit of the conservation equations $\nabla_\mu T^{\mu\nu} = 0$ of a perfect fluid. These equations lead to

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{v}) = 0 \tag{59}$$

$$\rho\frac{d\vec{v}}{dt} = \rho\nabla\left(\frac{h_{00}^{(2)}}{2}\right) - \nabla P \tag{60}$$

where a modification with respect to the classical Euler equations is introduced by the term Ω contained in $h_{00}^{(2)}$. This modification is given by

$$\frac{\rho}{2}\nabla\Omega^{(2)} = -\frac{\rho}{2}\frac{(\partial\phi/\partial T)}{\phi}\nabla\rho \tag{61}$$

and requires that the condition

$$\left|\frac{\rho(\partial\phi/\partial T)}{\phi}\right| \ll 1 \tag{62}$$

be satisfied over the wide range of densities mentioned above in order to guarantee the validity of the macroscopic classical Euler equations. Note that Eq. (62) must be true in general, since the contribution of $\Omega^{(2)}$ to the acceleration of a body is given in terms of $\nabla\Omega^{(2)}$. This constraint can be rewritten using Eq. (15) to evaluate $\partial\phi/\partial T$ as follows

$$\left|\frac{(\kappa^2\rho/\phi)}{(\phi V'' - V')}\right| \ll 1 \tag{63}$$

It is remarkable the fact that the denominator $[\phi V'' - V']$ in Eq. (63) is the counterpart of the effective square mass $m_\varphi^2 \equiv [\phi_0 V_0'' - V_0']/[3 + 2\omega]$ associated to dynamical Brans-Dicke-like fields with $\omega \neq -3/2$ (see Appendix A). For our discussion it will be more convenient to see this effective mass as an inverse length defining the interaction range of the scalar field. We can thus interpret Eq. (63) as the quotient of two lengthscales, one associated to the scalar field over another related to the mass density, $L^{-2}(\rho) \equiv (\kappa^2\rho c/\phi_0)$. Equation (63) can then be seen as the counterpart of the condition $m_\varphi^2 L^2 \gg 1$ that corresponds to $\omega \neq -3/2$ theories. Written in terms of the lagrangian $f(R)$, Eq. (63) turns into

$$R\tilde{f}'(R)\left|\frac{\tilde{f}'(R)}{R\tilde{f}''(R)}-1\right|L^2(\rho)\gg 1 \tag{64}$$

where $\tilde{f}'\equiv f'/f'_0=\phi/\phi_0$. According to our interpretation of the denominator of Eq. (63), we must demand that the multiplicative factor in front of $L^2(\rho)$ in Eq. (64) satisfies

$$\left|\frac{\tilde{f}'(R)}{R\tilde{f}''(R)}-1\right|\geq\frac{1}{l^2R\tilde{f}'} \tag{65}$$

where l^2 represents a lengthscale much smaller than $L^2(\rho)$ at nuclear densities, which is the shortest $L^2(\rho)$ that we can associate to ordinary matter. This inequality will allow us to find out which lagrangians satisfy the condition given in Eq. (64) at densities such that $L^2(\rho)\gg l^2$. Notice that the modulus of Eq. (65) may lead to more than one solution. Let us consider first the case

$$\frac{\tilde{f}'(R)}{R\tilde{f}''(R)}-1\geq\frac{1}{l^2R\tilde{f}'} \tag{66}$$

Defining $A\equiv 1/(R\tilde{f}')>0$, Eq. (66) turns into

$$-\left[\frac{2+\frac{R}{A}\frac{dA}{dR}}{1+\frac{R}{A}\frac{dA}{dR}}\right]\geq\frac{A}{l^2} \tag{67}$$

Since the right hand side of this equation is positive, the left hand side must also be positive. This can only happen if the denominator is negative and the numerator is positive. In fact, if we denote $-\varepsilon\equiv 1+\frac{R}{A}\frac{dA}{dR}$, Eq. (67) can be written as $(1-\varepsilon)/\varepsilon\geq A/l^2$. Thus, only if $\varepsilon<1$ the condition of the sign can be satisfied. In addition, for a highly linear lagrangian we expect $A/l^2\gg 1$. This leads to $\varepsilon\ll 1$, which is compatible with $\tilde{f}'\approx 1$. The sign and magnitude of the denominator indicate that $\tilde{f}''$ must be very small and positive. A little algebra is enough to show that

$$d\log[A(A+2l^2)]\geq d\log\left[\frac{1}{R^2}\right] \tag{68}$$

Once integrated, the new inequality can be written as

$$(A-A_+)(A-A_-)\geq 0 \tag{69}$$

where an integration constant, c_0^2, appears in

$$A_\pm=\frac{-l^2R\pm\sqrt{c_0^2+(l^2R)^2}}{R} \tag{70}$$

Since A must be positive ($\phi\equiv f'>0$ to have a well-posed theory), the only valid solution to Eq. (69) is $A\geq A_+$, which implies

$$0<\frac{df}{dR}\leq\sqrt{(f'_0)^2+(l^2R)^2}+l^2R \tag{71}$$

where we have fixed $c_0=f'_0$ to eliminate the tilde from $\tilde{f}'$. We can finally integrate this last inequality to obtain

$$f\leq\alpha+\frac{l^2R^2}{2}+\frac{R}{2}\sqrt{(f'_0)^2+(l^2R)^2}+\frac{f'_0}{2l^2}\log\left[l^2R+\sqrt{1+(l^2R)^2}\right] \tag{72}$$

Before commenting this result, let us consider the second inequality that follows from Eq. (65)

$$\frac{\tilde{f}'(R)}{R\tilde{f}''(R)}-1\leq-\frac{1}{l^2R\tilde{f}'} \tag{73}$$

Using again the function $A\equiv 1/(R\tilde{f}')$, we obtain

$$\left[\frac{2+\frac{R}{A}\frac{dA}{dR}}{1+\frac{R}{A}\frac{dA}{dR}}\right]\geq\frac{A}{l^2} \tag{74}$$

This inequality can only be satisfied if $(1+\frac{R}{A}\frac{dA}{dR})>0$, which corresponds to $\tilde{f}''<0$. Simple algebraic manipulations lead to

$$\frac{df}{dR}\geq\sqrt{(f'_0)^2+(l^2R)^2}-l^2R \tag{75}$$

which integrated gives

$$f\geq\alpha-\frac{l^2R^2}{2}+\frac{R}{2}\sqrt{(f'_0)^2+(l^2R)^2}+\frac{f'_0}{2l^2}\log\left[l^2R+\sqrt{1+(l^2R)^2}\right] \tag{76}$$

Let us discuss now the significance of Eqs. (72) and (76). The constant l^2 was introduced in Eq. (65) to represent the length scale over which the nonlinear contributions of the gravity lagrangian were relevant. For a given l^2, the nonlinear effects will begin to be important about a certain high density scale at which $l^2/L^2(\rho)\sim 1$. If, for instance, we take $l^2=0$, the nonlinear effects would be completely suppressed, since then Eq. (64) would be satisfied at all densities. The choice $l^2=0$ forces the lagrangian to be linear, which can be seen from Eqs. (72) and (76) in the limit $l^2\to 0$. This limit also indicates that the constant f'_0 can be naturally set to unity. On the other hand, if the nonlinear terms were relevant at low cosmic curvatures, the lengthscale l would be of order the radius of the universe and, therefore, the nonlinear effects would dominate the gravitational dynamics at all scales. This fact is obviously in contradiction with our experience, as we have discussed in detail throughout this section.

Though the inequalities derived above are only strictly valid in the limit of relatively low curvatures, $l^2R\ll 1$ (far from the early-time inflationary period), Eqs. (72) and (76) not only estimate the leading order of the possible nonlinear corrections, but give precise functions that bound the nonlinearities of the gravity lagrangian in this limit. Expanding around $l^2R\ll 1$ we find

$$\alpha+R-\frac{l^2R^2}{2}\leq f(R)\leq\alpha+R+\frac{l^2R^2}{2} \tag{77}$$

which confirms that the lagrangian is almost linear in R and that the leading-order corrections can grow, as much, quadratically in R.

D. On the Newtonian limit

Before concluding, we will briefly discuss the Newtonian limit obtained in [17,18]. As we mentioned above, the expansion around the vacuum carried out in those papers is not valid *a priori*. We have shown, however, that a viable theory must admit such expansion because of the experimental evidence supporting the weak dependence of ϕ on T. This conclusion, in our case, came out after analyzing the predictions of the theory. In [17,18], the expansion was due to an apparent failure to correctly identify the matter terms and the geometrical terms. In any case, expanding $\Omega^{(2)}(T)$ around $T = 0$ we reproduce the term $A\rho(x)$ obtained in [18]. This term, however, is not present in [17]. On the other hand, a Yukawa-type exponential correction in the Newtonian potential was found in [17,18], which is not present in our calculations. In the case of a dynamical field satisfying a second-order differential equation, a term of this type is expected to be related to the interaction range of the field. In the Palatini case, the field is nondynamical and, therefore, there is no reason for such a term. Moreover, assuming spherical symmetry, the Palatini equations admit exact Schwarzschild-de Sitter solutions $ds^2 = -A(r)dt^2 + dr^2/A(r) + r^2 d\Omega^2$ with $A(R) = 1 - \alpha/r + \Lambda r^2/3$. The effect of the asymptotic background curvature is given by the $\Lambda r^2/3$ term, which has the same form as the cosmological constant term $(V_0/\phi_0)r^2$ that appears in our Eqs. (50) and (51). Thus, there is no reason to expect an exponential correction related to the background curvature. The error seems to be due to a failure in the identification of the leading-order contribution of the term $(Rf' - f)g_{\mu\nu}/f'$. In our case, this term is represented by $(V/\phi)g_{\mu\nu}$, and its leading order is $(V/\phi)\eta_{\mu\nu}$ not $(V/\phi)h_{\mu\nu}$, which could justify the Yukawa-type correction for $V/\phi =$ constant.

V. SUMMARY AND CONCLUSIONS

In this work we have computed the complete post-Newtonian limit of the metric and Palatini forms of $f(R)$ gravities and have analyzed the constraints that laboratory and solar system experiments impose on the function $f(R)$. To do so we have rewritten the equations of motion of these theories in a more compact form which can be identified with particular cases of Brans-Dicke-like theories, characterized by a parameter ω and a potential $V(\phi)$. In this representation, the scalar field is identified with the derivative of the function f [see Eqs. (12) and (13)]. Because of the deep relation between the function $f(R)$ and the potential $V(\phi)$, we were forced to extend the results of the literature regarding the post-Newtonian limit of Brans-Dicke theories ($V(\phi) = 0$) so as to include all the potential terms, which were crucial to our discussion.

In the new scalar-tensor representation, metric and Palatini formalisms, $\omega = 0$ and $\omega = -3/2$ respectively, appear as theories with completely different properties. The case $\omega = 0$ behaves as a generic $\omega =$ constant theory in which the scalar field satisfies a second-order differential equation. The Palatini case ($\omega = -3/2$), however, is an exception of the general case $\omega =$ constant: the field is nondynamical and can be expressed as an algebraic function of the trace T. The fact the the scalar field in the metric case is dynamical implies that local systems may be affected by the exterior (the rest of the Universe) through the boundary values of the field. Since these background values change in a cosmic timescale, the post-Newtonian dynamics of those local systems may also evolve adiabatically with the cosmic expansion. The most dramatic effect is the modification of the interaction range of the scalar field. We have shown that $f(R)$ theories in which nonlinear terms grow at low curvatures exhibit an increment of the scalar interaction range that strongly modifies the local dynamics. Such modification, which affects the cosmic expansion rate leading to late-time acceleration, is incompatible with laboratory experiments and solar system observations. Therefore, since the same set of equations that explain the cosmology must also describe the gravitational dynamics of local systems, it follows that the gravity lagrangian cannot contain nonlinear terms that grow at low curvatures.

If in the metric formalism the nonlinear terms are constrained by the sensitivity of the local system to the external conditions, in the Palatini formalism the constraints stem from the sensitivity of the system to its own internal properties. We have seen that the effective constants G and γ, and even the gravitational mass $M_\odot$ of a system, may take different values depending on the composition and structure of the system. This unexpected and unusual behavior is more evident when the nonlinearities of the lagrangian dominate, which indicates that the lagrangian must be almost linear in R. On the other hand, we have found that the Newtonian limit can only be recovered if the nonlinear effects are restricted to very short length scales. Since the length scale in this case is determined by the matter density as $L^2(\rho) \sim \rho^{-1}$ [see the discussion following Eq. (63)], it follows that they can only be important at very high densities, which opposes the philosophy of nonlinear terms dominant at low cosmic densities.

In summary, in order to find agreement between the predictions of $f(R)$ gravities and laboratory and solar system experiments, the gravity lagrangian $f(R)$ must be almost linear in R, with the leading order of the possible nonlinear corrections bounded by R^2 terms.

ACKNOWLEDGMENTS

I am grateful to Professor Leonard Parker and Professor José Navarro-Salas for their always bright comments and insights. I also thank O. Varela, M. Nebot and J. A. Sans for useful discussions and H.Sanchis for his patience. Special thanks go to Shani M. Lizardi and Luis López for their hospitality during my stay in Milwaukee. This work has been supported by the Regional Government of Valencia (Spain) and the research grant BFM2002-04031-C02-01 from the Ministerio de Educación y Ciencia (Spain), and could not have been carried out without the understanding of Sonia G. B.

APPENDIX A: METRIC FORMALISM

We will take $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$ and $\phi = \phi_0 + \varphi(t, x)$. For convenience, we will rewrite the equations of motion corresponding to the action of Eq. (16) in the following form

$$R_{\mu\nu} = \frac{\kappa^2}{\phi}\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right] + \frac{\omega}{\phi^2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{\phi}\nabla_\mu\nabla_\nu\phi + \frac{1}{2\phi}g_{\mu\nu}[\Box\phi + V(\phi)] \tag{A1}$$

We keep the term with ω because at the same price we can compute the post-Newtonian limit of any Brans-Dicke-like theory. At the end of the calculations we can particularize to the case $\omega = 0$ to obtain the desired result. It is also useful to keep the term with ω to check that, when the potential terms are neglected, we recover the expected limit of Brans-Dicke theories.

The expansion of the Ricci tensor around the Minkowski metric can be written as follows

$$R_{ij} = -\frac{1}{2}\nabla^2 h_{ij}^{(2)} + \frac{1}{2}\partial_i\left[h^\mu_{j,\mu} - \frac{1}{2}h^\mu_{\mu,j}\right] + \frac{1}{2}\partial_j\left[h^\mu_{i,\mu} - \frac{1}{2}h^\mu_{\mu,i}\right] \tag{A2}$$

$$R_{0j} = -\frac{1}{2}\nabla^2 h_{0j}^{(3)} + \frac{1}{2}\partial_j\left[h^\mu_{0,\mu} - \frac{1}{2}h^\mu_{\mu,0}\right] + \frac{1}{2}\partial_0\left[h^\mu_{j,\mu} - \frac{1}{2}h^\mu_{\mu,j}\right] \tag{A3}$$

$$R_{00} = -\frac{1}{2}\nabla^2\left[h_{00}^{(4)} + \frac{(h_{00}^{(2)})^2}{2}\right] + \partial_0[h^\mu_{0,\mu} - \frac{1}{2}h^\mu_{\mu,0} + \frac{1}{2}h^{(2)}_{00,0}] + \frac{1}{2}\left[h^\mu_{j,\mu} - \frac{1}{2}h^\mu_{\mu,j}\right]\partial^j h_{00}^{(2)} + \frac{1}{2}h_{00}^{(2)}\nabla^2 h_{00}^{(2)} + \frac{1}{2}h^{(2)ij}\partial_i\partial_j h_{00}^{(2)} \tag{A4}$$

where all the indices are raised and lowered with the Minkowski metric. Assuming a perfect fluid, the elements on the right hand side of Eq. (A1) are given, up to the necessary order, by

$$\tau_{ij} = \frac{\kappa^2\rho}{2\phi_0}\delta_{ij} + \rho O(v^2) \tag{A5}$$

$$\tau_{0j} = -\frac{\kappa^2}{\phi_0}\rho v_j + \rho O(v^3) \tag{A6}$$

$$\tau_{00} = \frac{\kappa^2\rho}{2\phi_0}\left[1 + \Pi + 2v^2 - \left(h_{00}^{(2)} + \frac{\varphi^{(2)}}{\phi_0}\right) + \frac{3P}{\rho}\right] + \rho O(v^4) \tag{A7}$$

where

$$\tau_{\mu\nu} \equiv \frac{\kappa^2}{\phi}\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right] \tag{A8}$$

We can also define the contribution due to the scalar field as

$$\tau^\phi_{\mu\nu} \equiv \frac{\omega}{\phi^2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{\phi}\nabla_\mu\nabla_\nu\phi + \frac{1}{2\phi}g_{\mu\nu}[\phi + V(\phi)] \tag{A9}$$

Its components are

$$\tau^\phi_{ij} = \partial_i\partial_j\left(\frac{\varphi^{(2)}}{\phi_0}\right) + \frac{\delta_{ij}}{2\phi_0}[V_0 - \ddot{\phi}_0 + \nabla^2\varphi^{(2)}] \tag{A10}$$

$$\tau^\phi_{0j} = \frac{1}{2}\partial_j\left[2\omega\frac{\dot{\phi}_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot{\phi}_0}{\phi_0}h_{00}^{(2)} + \frac{\dot{\varphi}^{(2)}}{\phi_0}\right] + \frac{1}{2}\partial_0\partial_j\left(\frac{\varphi^{(2)}}{\phi_0}\right) + \frac{1}{2}\frac{\dot{\phi}_0}{\phi_0}\partial_j\left(\frac{\varphi^{(2)}}{\phi_0}\right) \tag{A11}$$

$$\tau^\phi_{00} = \partial_0\left[2\omega\frac{\dot{\phi}_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot{\phi}_0}{\phi_0}h_{00}^{(2)} + \frac{\dot{\varphi}^{(2)}}{\phi_0}\right] + \frac{1}{2}\frac{\ddot{\varphi}^{(2)}}{\phi_0} + \frac{1}{2}\partial^k h_{00}^{(2)}\partial_k\left(\frac{\varphi^{(2)}}{\phi_0}\right) + \omega\left(\frac{\dot{\phi}_0}{\phi_0}\right)^2\left[1 + 2\left(\frac{\varphi^{(2)}}{\phi_0}\right)\right] + \left(\frac{\ddot{\phi}_0}{\phi_0}\right)\left[\frac{3}{2} - \left(\frac{3}{2} + 2\omega\right)\times\left(\frac{\varphi^{(2)}}{\phi_0}\right) - h_{00}^{(2)}\right] + \left(\frac{\dot{\phi}_0}{\phi_0}\right)\left[\left(\frac{\dot{\phi}_0}{\phi_0}\right)h_{00}^{(2)} + \left(\frac{\dot{\varphi}^{(2)}}{\phi_0}\right) - h_{00,0}^{(2)} - \frac{1}{2}\left(h^\mu_{0,\mu} - \frac{1}{2}h^\mu_{\mu,0}\right)\right] - \frac{1}{2\phi_0}\left[V_0\left(1 - h_{00}^{(2)} - \frac{\varphi^{(2)}}{\phi_0}\right) + \varphi^{(2)}V_0'\right] + \frac{1}{2}\left[h^\mu_{k,\mu} - \frac{1}{2}h^\mu_{\mu,k}\right]\partial^k\left(\frac{\varphi^{(2)}}{\phi_0}\right) + \frac{1}{2}\left[h_{00}^{(2)} + h_{[ij]}^{(2)} + \frac{\varphi^{(2)}}{\phi_0} - 1\right]\nabla^2\left(\frac{\varphi^{(2)}}{\phi_0}\right) \tag{A12}$$

Using the gauge conditions

$$h^{\mu}_{k,\mu} - \frac{1}{2}h^{\mu}_{\mu,k} = \frac{\partial_k \varphi^{(2)}}{\phi_0} \tag{A13}$$

$$h^{\mu}_{0,\mu} - \frac{1}{2}h^{\mu}_{\mu,0} = \left(2\omega\frac{\dot\phi_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot\phi_0}{\phi_0}h^{(2)}_{00} + \frac{\dot\varphi^{(2)}}{\phi_0}\right) - \frac{1}{2}h^{(2)}_{00,0} \tag{A14}$$

the equations of motion boil down to

$$-\frac{1}{2}\nabla^2\left[h^{(2)}_{ij} + \delta_{ij}\frac{\varphi^{(2)}}{\phi_0}\right] = \frac{\delta_{ij}}{2\phi_0}[\kappa^2\rho + V_0 - \ddot\phi_0] \tag{A15}$$

$$-\frac{1}{2}\nabla^2 h^{(3)}_{0j} - \frac{1}{4}h^{(2)}_{00,0j} = -\frac{\kappa^2}{\phi_0}\rho v_j \tag{A16}$$

$$\begin{aligned}-\frac{1}{2}\nabla^2\left[h^{(4)}_{00} - \frac{\varphi^{(4)}}{\phi_0} + \frac{(h^{(2)}_{00})^2}{2} + \frac{1}{2}\left(\frac{\varphi^{(2)}}{\phi_0}\right)^2\right] &= \frac{\kappa^2\rho}{2\phi_0}[1 + \Pi + 2v^2 + h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0} + \frac{3P}{\rho}] + \frac{\ddot\varphi^{(2)}}{\phi_0} + \omega\left(\frac{\dot\phi_0}{\phi_0}\right)^2[1 + h^{(2)}_{00} + h^{(2)}_{[ij]}] \\ &\quad - \frac{1}{2\phi_0}\left[V_0\left(1 + h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0}\right) + \varphi^{(2)}V_0'\right] + \frac{\ddot\phi_0}{\phi_0}\left[\frac{3}{2}\left(1 + h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0}\right) + \frac{h^{(2)}_{00}}{2}\right. \\ &\quad \left. - 2\omega\frac{\varphi^{(2)}}{\phi_0}\right] + \frac{\dot\phi_0}{2\phi_0}\left[2\omega\frac{\dot\phi_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot\phi_0}{\phi_0}h^{(2)}_{00} + \frac{\dot\varphi^{(2)}}{\phi_0} - \frac{1}{2}h^{(2)}_{00,0}\right]\end{aligned} \tag{A17}$$

where $h_{[ij]}$ simply states the relation $h_{ij} = \delta_{ij}h_{[ij]}$. The equation for the scalar field is given by

$$\begin{aligned}(\nabla^2 - m^2_\varphi)\left[\varphi^{(4)} - \frac{1}{2}\frac{(\varphi^{(2)})^2}{\phi_0}\right] &= -\frac{\kappa^2\rho}{3+2\omega}\left[1 + \Pi - \frac{3P}{\rho} + h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0}\right] + \ddot\varphi^{(2)} \\ &\quad + m^2_\varphi\varphi^{(2)}h^{(2)}_{[ij]} - \dot\phi_0\left[2\omega\frac{\dot\phi_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot\phi_0}{\phi_0}h^{(2)}_{00} + \frac{\dot\varphi^{(2)}}{\phi_0}\right. \\ &\quad \left. - \frac{1}{2}h^{(2)}_{00,0}\right] + \frac{(\varphi^{(2)})^2}{2\phi_0}\left[\frac{\phi_0^2 V_0'''}{3+2\omega} - \frac{m^2_\varphi}{2}\right]\end{aligned} \tag{A18}$$

where we have defined

$$m^2_\varphi = \frac{\phi_0 V_0'' - V_0'}{3+2\omega} \tag{A19}$$

The solutions are formally given by

$$\frac{\varphi^{(2)}(t,x)}{\phi_0} = \frac{\kappa^2}{4\pi\phi_0}\int d^3x'\frac{\rho(t,x')}{|x-x'|}\frac{F(|x-x'|)}{(3+2\omega)} \tag{A20}$$

$$h^{(2)}_{00}(t,x) = \frac{\kappa^2}{4\pi\phi_0}\int d^3x'\frac{\rho(t,x')}{|x-x'|}\left[1 + \frac{F(|x-x'|)}{(3+2\omega)}\right] + \left[V_0 - 3\frac{\ddot\phi_0}{\phi_0} - 2\omega\left(\frac{\dot\phi_0}{\phi_0}\right)^2\right]\frac{|x-x_c|^2}{6} \tag{A21}$$

$$h^{(2)}_{ij}(t,x) = \left(\frac{\kappa^2}{4\pi\phi_0}\int d^3x'\frac{\rho(t,x')}{|x-x'|}\left[1 - \frac{F(|x-x'|)}{(3+2\omega)}\right] - \frac{(V_0 - \ddot\phi_0)}{\phi_0}\frac{|x-x_c|^2}{6}\right)\delta_{ij} \tag{A22}$$

$$h^{(3)}_{0j}(t,x) = -\frac{\kappa^2}{4\pi\phi_0}\int d^3x'\frac{2\rho(t,x')v_j'}{|x-x'|} + \frac{1}{4\pi}\times\int d^3x'\frac{h^{(2)}_{00,0j}}{2|x-x'|} \tag{A23}$$

$$\begin{aligned}h^{(4)}_{00}(t,x) &= \frac{1}{4\pi}\int d^3x'\frac{1}{|x-x'|}\left[\frac{\ddot\varphi^{(2)}}{\phi_0} + \frac{\dot\phi_0}{\phi_0}\left(2\omega\frac{\dot\phi_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot\phi_0}{\phi_0}h^{(2)}_{00} + \frac{\dot\varphi^{(2)}}{\phi_0} - \frac{1}{2}h^{(2)}_{00,0}\right)\right] + \frac{\kappa^2}{4\pi\phi_0}\int d^3x'\frac{\rho(t,x')}{|x-x'|}\left[\left(1 + \Pi\right.\right. \\ &\quad \left. + h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0}\right)\left(1 + \frac{F(|x-x'|)}{3+2\omega}\right) - \frac{3P}{\rho}\left(1 - \frac{F(|x-x'|)}{3+2\omega}\right) + 2v^2] + \frac{\omega}{\pi}\left(\frac{\dot\phi_0}{\phi_0}\right)^2\int d^3x'\frac{[h^{(2)}_{00} + h^{(2)}_{[ij]}]}{|x-x'|} + \frac{(h^{(2)}_{00})^2}{2} \\ &\quad - \frac{1}{4\pi\phi_0}\int d^3x'\frac{[V_0(h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0}) + \varphi^{(2)}V_0']}{|x-x'|} + \frac{1}{4\pi}\int d^3x'\frac{(\frac{\ddot\phi_0}{\phi_0})}{|x-x'|}\left[3\left(h^{(2)}_{[ij]} - \frac{\varphi^{(2)}}{\phi_0}\right) + h^{(2)}_{00} - 4\omega\frac{\varphi^{(2)}}{\phi_0}\right] \\ &\quad - \frac{1}{4\pi}\int d^3x'\frac{F(|x-x'|)}{|x-x'|}\left[m^2_\varphi\varphi^{(2)}h^{(2)}_{[ij]} + \frac{(\varphi^{(2)})^2}{2\phi_0}\left(\frac{\phi_0^2V_0'''}{3+2\omega} - \frac{m^2_\varphi}{2}\right)\right]\end{aligned} \tag{A24}$$

$$\frac{\varphi^{(4)}(t,x)}{\phi_0} = \frac{1}{2}\left(\frac{\varphi^{(2)}}{\phi_0}\right)^2 + \frac{\kappa^2}{4\pi\phi_0}\int d^3x' \frac{\rho(t,x')}{|x-x'|}\frac{F(|x-x'|)}{(3+2\omega)}\left[1+\Pi-\frac{3P}{\rho}+h^{(2)}_{[ij]}-\frac{\varphi^{(2)}}{\phi_0}\right] - \frac{1}{4\pi}\int d^3x' \frac{F(|x-x'|)}{|x-x'|} \times\left[\frac{\ddot{\varphi}^{(2)}}{\phi_0} - \frac{\dot{\phi}_0}{\phi_0}\left(2\omega\frac{\dot{\phi}_0}{\phi_0}\frac{\varphi^{(2)}}{\phi_0} + \frac{\dot{\phi}_0}{\phi_0}h^{(2)}_{00} + \frac{\dot{\varphi}^{(2)}}{\phi_0} - \frac{1}{2}h^{(2)}_{00,0}\right)\right] - \frac{1}{4\pi}\int d^3x' \frac{F(|x-x'|)}{|x-x'|}\left[m_\varphi^2\frac{\varphi^{(2)}}{\phi_0}h^{(2)}_{[ij]} + \frac{1}{2}\left(\frac{\varphi^{(2)}}{\phi_0}\right)^2 \times\left(\frac{\phi_0^2 V_0'''}{3+2\omega} - \frac{m_\varphi^2}{2}\right)\right] \tag{A25}$$

where the function $F(|x-x'|)$ denotes

$$F(|x-x'|) = \begin{cases} e^{-m_\varphi|x-x'|} & \text{if } m_\varphi^2 > 0 \\ \cos(m_\varphi|x-x'|) & \text{if } m_\varphi^2 < 0 \end{cases} \tag{A26}$$

APPENDIX B: PALATINI FORMALISM

We will just remark here that in the case $\omega = -3/2$ the scalar field cannot be expanded in the same manner as in the general case $\omega \neq -3/2$, since now the field is non-dynamical and is completely determined by the matter distribution of the local system, $\phi = \phi(T)$. In the general case, however, the field is a dynamical entity whose state is determined by the Universe as a whole. The post-Newtonian system only contributes with local fluctuations from the background asymptotic state. In the $\omega = -3/2$ case, due to the fact that $T = -\rho(1 + \Pi - 3P/\rho) \approx -\rho + \rho O(v^2)$, we could expand $\phi(T) \approx \phi(-\rho) + \partial_T\phi\rho O(v^2)$, though this seems an unnecessary complication. We will keep all ϕ terms exact in our calculations and will expand them at the end up to the necessary order. According to this, $\tau_{\mu\nu} \equiv \frac{\kappa^2}{\phi}[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T]$ is given by

$$\tau_{ij} = \frac{\kappa^2\rho}{2\phi}\delta_{ij} + \rho O(v^2) \tag{B1}$$

$$\tau_{0j} = -\frac{\kappa^2}{\phi}\rho v_j + \rho O(v^3) \tag{B2}$$

$$\tau_{00} = \frac{\kappa^2\rho}{2\phi}\left[1 + \Pi + 2v^2 - h^{(2)}_{00} + \frac{3P}{\rho}\right] + \rho O(v^4) \tag{B3}$$

The contribution coming from the scalar field terms can be written as

$$\tau^\phi_{\mu\nu} = (\omega+1)\partial_\mu\Omega\partial_\nu\Omega + \nabla_\mu\nabla_\nu\Omega + \frac{1}{2}g_{\mu\nu}\left[\frac{V}{\phi} + \Box\Omega + \partial_\lambda\Omega\partial^\lambda\Omega\right] \tag{B4}$$

where $\Omega \equiv \log(\phi/\phi_0)$ and ϕ_0 is an arbitrary constant that may be fixed as $\phi_0 = \phi(T=0)$. The components of $\tau^\phi_{\mu\nu}$ are

$$\tau^\phi_{ij} = \partial_i\partial_j\Omega + \frac{\delta_{ij}}{2}\left[\frac{V}{\phi} + \nabla^2\Omega\right] \tag{B5}$$

$$\tau^\phi_{0j} = \partial_0\partial_j\Omega \tag{B6}$$

$$\tau^\phi_{00} = \frac{3}{2}\ddot{\Omega} + \frac{1}{2}\partial_k\Omega\partial^k h^{(2)}_{00} - \frac{1}{2}\nabla^2\Omega + \frac{1}{2}\left[h^\mu_{k,\mu} - \frac{1}{2}h^\mu_{\mu,k} - \partial_k\Omega\right]\partial^k\Omega - \frac{V}{2\phi}(1 - h^{(2)}_{00}) + \frac{h^{ij}}{2}\partial_i\partial_j\Omega + \frac{1}{2}h^{(2)}_{00}\nabla^2\Omega \tag{B7}$$

where we assume $\Omega \sim O(v^2)$ at least to guarantee h_{ij} diagonal and a consistent post-Newtonian expansion.

Equating the left hand side of Eq. (A1) to its right hand side, a little algebra leads to

$$-\frac{1}{2}\nabla^2[h^{(2)}_{ij} + \delta_{ij}\Omega] + \frac{1}{2}\partial_i\left[h^\mu_{j,\mu} - \frac{1}{2}h^\mu_{\mu,j} - \partial_j\Omega\right] + \frac{1}{2}\partial_j\left[h^\mu_{i,\mu} - \frac{1}{2}h^\mu_{\mu,i} - \partial_i\Omega\right] = \frac{\delta_{ij}}{2\phi}[\kappa^2\rho + V(\phi)] \tag{B8}$$

$$-\frac{1}{2}\nabla^2 h^{(3)}_{0j} + \frac{1}{2}\partial_j\left[h^\mu_{0,\mu} - \frac{1}{2}h^\mu_{\mu,0} - \partial_0\Omega\right] + \frac{1}{2}\partial_0\left[h^\mu_{j,\mu} - \frac{1}{2}h^\mu_{\mu,j} - \partial_j\Omega\right] = -\frac{\kappa^2}{\phi}\rho v_j \tag{B9}$$

$$-\frac{1}{2}\nabla^2\left[h^{(4)}_{00} + \frac{(h^{(2)}_{00})^2}{2}\right] + \frac{1}{2}\nabla^2\Omega + \frac{1}{2}\partial_0\left[h^\mu_{0,\mu} - \frac{1}{2}h^\mu_{\mu,0} - \partial_0\Omega + \frac{1}{2}h^{(2)}_{00,0}\right] + \frac{1}{2}\left[h^\mu_{k,\mu} - \frac{1}{2}h^\mu_{\mu,k}\right]\partial^k h^{(2)}_{00} + \frac{1}{2}h^{(2)}_{00}\nabla^2[h^{(2)}_{00} - \Omega] = \frac{\kappa^2\rho}{2\phi}\left[1 + \Pi + 2v^2 - h^{(2)}_{00} + \frac{3P}{\rho}\right] + \frac{1}{2}\left[h^\mu_{k,\mu} - \frac{1}{2}h^\mu_{\mu,k} - \partial_k\Omega\right]\partial^k\Omega + \frac{\ddot{\Omega}}{2} + \frac{1}{2}\partial_k\Omega\partial^k h^{(2)}_{00} - \frac{V}{2\phi}(1 - h^{(2)}_{00}) + \frac{h^{ij}}{2}\partial_i\partial_j[h^{(2)}_{00} - \Omega] \tag{B10}$$

Using the gauge conditions

$$h^{\mu}_{k,\mu} - \frac{1}{2}h^{\mu}_{\mu,k} = \partial_k \Omega \tag{B11}$$

$$h^{\mu}_{0,\mu} - \frac{1}{2}h^{\mu}_{\mu,0} = \partial_0 \Omega - \frac{1}{2}h^{(2)}_{00,0} \tag{B12}$$

the equations of above become

$$-\frac{1}{2}\nabla^2[h^{(2)}_{ij} + \delta_{ij}\Omega] = \frac{\delta_{ij}}{2\phi}[\kappa^2\rho + V(\phi)] \tag{B13}$$

$$-\frac{1}{2}\nabla^2 h^{(3)}_{0j} - \frac{1}{4}h^{(2)}_{00,0j} = -\frac{\kappa^2}{\phi}\rho v_j \tag{B14}$$

$$-\frac{1}{2}\nabla^2\left[h^{(4)}_{00} - \Omega + \frac{(h^{(2)}_{00})^2}{2}\right] = \frac{\kappa^2\rho}{2\phi}\left[1 + \Pi + 2v^2 + h^{(2)}_{[ij]} + \frac{3P}{\rho}\right] + \frac{\ddot{\Omega}}{2} - \frac{V}{2\phi}(1 + h^{(2)}_{[ij]}) \tag{B15}$$

where $h_{[ij]}$ denotes the ij-component of $h^{(2)}_{ij}$ and we have used that, to second order, $h^{(2)}_{00}$ satisfies

$$-\frac{1}{2}\nabla^2[h^{(2)}_{00} - \Omega] = \frac{1}{2\phi}[\kappa^2\rho - V(\phi)] \tag{B16}$$

The post-Newtonian corrections to the metric are thus given by (we denote $\tilde{\phi} \equiv \phi/\phi_0$)

$$h^{(2)}_{ij}(t,x) = \left[\frac{\kappa^2}{4\pi\phi_0}\int d^3x' \frac{[\rho(t,x') + V(\phi)/\kappa^2]}{\tilde{\phi}|x - x'|} - \Omega^{(2)}\right]\delta_{ij} \tag{B17}$$

$$h^{(2)}_{00}(t,x) = \frac{\kappa^2}{4\pi\phi_0}\int d^3x' \frac{[\rho(t,x') - V(\phi)/\kappa^2]}{\tilde{\phi}|x - x'|} + \Omega^{(2)} \tag{B18}$$

$$h^{(3)}_{0j}(t,x) = -\frac{\kappa^2}{2\pi\phi_0}\int d^3x' \frac{\rho(t,x')v'_j}{\tilde{\phi}|x - x'|} + \frac{1}{8\pi}\int d^3x' \frac{h^{(2)}_{00,0j}(t,x')}{|x - x'|} \tag{B19}$$

$$h^{(4)}_{00}(t,x) = \frac{\kappa^2}{4\pi\phi_0}\int d^3x' \frac{\rho(t,x')}{\tilde{\phi}|x - x'|}[1 + \Pi + 2v^2 + h^{(2)}_{[ij]} + 3P/\rho] - \frac{\kappa^2}{4\pi\phi_0} \times \int d^3x' \frac{V(\phi)[1 + h^{(2)}_{[ij]}] + \ddot{\Omega}^{(2)}/2}{\tilde{\phi}|x - x'|} - \frac{(h^{(2)}_{00})^2}{2} + \Omega^{(4)} \tag{B20}$$

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