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Static quantum corrections to the Schwarzschild spacetime

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Abstract. We study static quantum corrections of the Schwarzschild metric in the Boulware vacuum state. Due to the absence of a complete analytic expression for the full semiclassical Einstein equations we approach the problem by considering the s-wave approximation and solve numerically the associated backreaction equations. The solution, including quantum effects due to pure vacuum polarization, is similar to the classical Schwarzschild solution up to the vicinity of the classical horizon. However, the radial function has a minimum at a time-like surface close to the location of the classical event horizon. There the g_{00} component of the metric reaches a very small but non-zero value. The analysis unravels how a curvature singularity emerges beyond this bouncing point. We briefly discuss the physical consequences of these results by extrapolating them to a dynamical collapsing scenario.

1. Introduction

The most important results on quantum properties of black holes [1, 2, 3] (see also [4, 5, 6]) are obtained in the so-called fixed background approximation. This means that the spacetime background is assumed to be fixed and not modified by the quantum behavior of matter. There are two reasons for doing this. First, one expects that the inclusion of backreaction effects will not modify essentially the physical results obtained in the fixed background approximation, at least until reaching the Planck scale. Second, to go beyond this approximation requires to solve the semiclassical Einstein equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle \Psi | T_{\mu\nu}(g_{\alpha\beta}) | \Psi \rangle \tag{1}$$

and this is a very difficult task. Solving these equations requires to know an explicit expression for the expectation values of the quantum stress-energy tensor for a large family of metrics, necessarily including those that could potentially be the solution of the semiclassical equations. Moreover, the quantities $\langle \Psi | T_{\mu\nu}(g_{\alpha\beta}) | \Psi \rangle$ depend also on the quantum state of the matter $|\Psi\rangle$, and the way one fixes this dependence is a non-trivial issue.

Due to the static character of the classical Schwarzschild spacetime, a natural state to consider in the fixed background approximation is the one defined with respect to the Schwarzschild time "t". This is the so-called Boulware vacuum state $|B\rangle$ [7] and is the state that most closely reproduces the familiar notion of Minkowski vacuum at infinity. The evaluation of the expectation values $\langle B|T_{\mu\nu}|B\rangle$ shows that they vanish at infinity but are, for a free-falling observer, highly divergent when $r \to r_S \equiv 2GM/c^2$ [8]. This "drawback" of the state $|B\rangle$ has a natural interpretation. It describes the (vacuum polarization) exterior to a static star but it cannot describe the exterior of a collapsing body producing a black hole. To eliminate this divergence we need to replace $|B\rangle$ with another state. However, the consequence of cancelling the divergence at the horizon is the emergence of a non-vanishing thermal flux (with a particular temperature) in the late-time asymptotic future. This flux is associated to the Hawking emission (see [9] for a discussion based on the equivalence principle).

However, the fact that $\langle B|T_{\mu\nu}|B\rangle$ gets divergent at the horizon means that the semiclassical corrections to the Schwarzschild metric would be very large when approaching the surface $r = r_S$. This opens the question about the geometry of the spacetime in the vicinity of r_S once backreaction effects are properly included. One usually disregards this question arguing, as we have already stressed, that the Boulware state is not the appropriate one to describe a collapsing star. However, one could expect that the type of divergence of $\langle B|T_{\mu\nu}|B\rangle$ is, in some way, related to the late-time radiation properties of the black hole. This is indeed what happens in the fixed background approximation. Moreover, in addition to this there is another motivation to study this problem. It appears in an apparently different scenario, namely in braneworld models in Anti-de Sitter space. There is an intriguing holographic relation between the quest of static black hole solutions in AdS branworlds and the problem of finding consistent solutions to the Einstein semiclassical equations in the Boulware state. This is so according to an extension, applied to the Randall-Sundrum model [10], of the Maldacena AdS/CFT duality [11]. One expects that "4D black holes localized on the brane found by solving the classical bulk equations in $AdS(5)$ are quantum corrected black holes (in the Boulware state) and not classical ones" [12, 13]. Significative evidence for this conjecture has recently been given in [14] through a numerical computation of $\langle B|T_{\mu\nu}|B\rangle$ at large r. More details can be found in the contribution [15] in this Conference.

To solve the backreaction equations requires an exact analytical expression of $\langle B|T_{\mu\nu}|B\rangle$ for a generic geometry. Since no such expression exists in the literature (for analytic approximations see [16, 5]) we resort to the so-called s-wave approximation. This means to assume spherical symmetry for the background and keep only the s-wave contribution of the matter prior to quantization. In this situation a generic expression for $\langle B|T_{\mu\nu}|B\rangle$ can be worked out [17, 6] generalizing the well-known results in two-dimensional spacetimes [18, 4]. In section 2 we briefly present this approximation scheme and, in section 3, we focus on the Boulware vacuum. Our results are exposed in section 4. We pay attention to describe how the non-perturbative solution to the backreaction equations prevents the formation of an event horizon and how a singularity emerges after a bouncing point for the radial function. Finally, in section 5 we discuss on these results.

2. S-wave approximation and semiclassical equations

The Hilbert-Einstein action coupled to a massless scalar field

$$S^{(4)} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g^{(4)}}R^{(4)} - \frac{1}{8\pi}\int d^4x\sqrt{-g^{(4)}}(\nabla f)^2 \tag{2}$$

can be reduced, under the assumption of spherical symmetry $ds^2_{(4)} = ds^2_{(2)} + r^2 d\Omega^2$ and keeping only the s-wave component of the expansion for the matter field

$$f = f(x^a) \equiv \frac{f_{l=0}}{r}Y_{00}, \tag{3}$$

to the following effective two-dimensional theory

$$S=\frac{c^3}{4G}\int d^2x\sqrt{-g}\left[r^2R+2\left(1+|\nabla r|^2\right)-\frac{1}{2}r^2(\nabla f)^2\right] . \tag{4}$$

The equations of motion obtained by varying directly the action S are

$$\begin{aligned}\frac{c^4}{4G}\left[-2r\nabla_a\nabla_b r+g_{ab}\left(2r\nabla^2 r-1+|\nabla r|^2\right)\right] &= T_{ab} ,\\ \frac{c^3}{G}\left[rR-2\nabla^2 r\right] &= -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta r} ,\end{aligned} \tag{5}$$

where T_{ab} is the two-dimensional stress-energy tensor and S_m the matter sector of the action

$$T_{ab}\equiv-\frac{2c}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{ab}} . \tag{6}$$

The quantities in the right hand side of Eqs. (5) are related to the four-dimensional stress-energy tensor as follows: $T^{(4)}_{ab}=\frac{T_{ab}}{4\pi r^2}, T^{(4)}_{\theta\theta}=T^{(4)}_{\varphi\varphi}\sin^{-2}\theta=-\frac{r}{8\pi\sqrt{-g^{(2)}}}\frac{\delta S_m}{\delta r}$.

To construct the semiclassical theory we need an expression for the expectation values $\langle T_{ab}\rangle$ and $\langle\frac{\delta S_m}{\delta r}\rangle$. Remarkably this can be done in a very simple way by working in the conformal gauge ($ds^2=-e^{2\rho}dx^+dx^-$) for the two-dimensional part of the metric. The following natural conditions:

(i) the four-dimensional covariant conservation law, which in our two-dimensional language reads
$$\nabla^a\langle T_{ab}\rangle=\nabla_b r\frac{1}{\sqrt{-g}}\langle\frac{\delta S_m}{\delta r}\rangle , \tag{7}$$

(ii) at an arbitrary point X of the spacetime manifold the expectation values of the quantum stress-energy tensor $\langle T_{\pm\pm}(x^\pm(X))\rangle$ reduce to the normal ordering ones $\langle : T_{\pm\pm}(x^\pm(X)) :\rangle$ when using a locally inertial frame ξ^α_X based on that point
$$\langle T_{\pm\pm}(\xi^\alpha_X(X))\rangle=\langle : T_{\pm\pm}(\xi^\alpha_X(X)) :\rangle, \tag{8}$$

related to basic ingredients of general relativity: i) covariance and ii) equivalence principle, are enough to provide an expression for $\langle\Psi|T_{\pm\pm}|\Psi\rangle$, $\langle\Psi|T_{+-}|\Psi\rangle$ and $\langle\Psi|\frac{\delta S_m}{\delta r}|\Psi\rangle$ (here we write explicitly the quantum state $|\Psi\rangle$)

$$\begin{aligned}\langle\Psi|T_{\pm\pm}(x^\pm)|\Psi\rangle &= -\frac{\hbar}{12\pi}(\partial_\pm\rho\partial_\pm\rho-\partial^2_\pm\rho)+\frac{\hbar}{2\pi}(\partial_\pm\rho\partial_\pm\phi+\rho(\partial_\pm\phi)^2)\\ &+ \langle\Psi| : T_{\pm\pm}(x^\pm) : |\Psi\rangle , \qquad (9)\\ \langle\Psi|T_{+-}(x^\pm)|\Psi\rangle &= -\frac{\hbar}{12\pi}(\partial_+\partial_-\rho+3\partial_+\phi\partial_-\phi-3\partial_+\partial_-\phi) , \qquad (10)\\ \langle\Psi|\frac{\delta S_m}{\delta\phi}|\Psi\rangle &= -\frac{\hbar}{2\pi}(\partial_+\partial_-\rho+\partial_+\rho\partial_-\phi+\partial_-\rho\partial_+\phi+2\rho\partial_+\partial_-\phi)\\ &+ \langle\Psi|\frac{\delta S_m}{\delta\phi}|\Psi\rangle_{\rho=0} , \qquad (11)\end{aligned}$$

where we have introduced, not to break with tradition in this area, the dilaton field ϕ defined as $r=r_0e^{-\phi}$ (r_0 is an arbitrary length scale). The dependence on the quantum state is all contained in the three functions $\langle\Psi| : T_{\pm\pm} : |\Psi\rangle \equiv \langle\Psi|T_{\pm\pm}|\Psi\rangle_{\rho=0}$ and $\langle\Psi|\frac{\delta S_m}{\delta\phi}|\Psi\rangle_{\rho=0}$. These functions are not independent and verify the following relations

$$\partial_\mp\langle\Psi| : T_{\pm\pm} : |\Psi\rangle + \partial_\pm\phi\langle\Psi|\frac{\delta S_m}{\delta\phi}|\Psi\rangle_{\rho=0} - \frac{\hbar}{4\pi}\partial_\pm(\partial_+\phi\partial_-\phi - \partial_+\partial_-\phi) = 0 \ . \tag{12}$$

We note that the non-vanishing of $\langle T_{+-}\rangle$ implies the existence of a trace anomaly, absent in the classical theory. The specific value of $\langle T_{+-}\rangle$ is related to the anomalous transformation law of $\langle\Psi| : T_{\pm\pm} : |\Psi\rangle$ under a conformal rescaling of coordinates. Moreover, the expressions (9) reduce, when the value of ϕ is fixed, to the well-known expressions of a conformal scalar field in two-dimensions [18, 4, 19, 20, 21]: $\langle\Psi|T_{\pm\pm}|\Psi\rangle = \langle\Psi|T_{\pm\pm}|\Psi\rangle_{\rho=0} - \frac{\hbar}{12\pi}(\partial_\pm\rho\partial_\pm\rho - \partial^2_\pm\rho)$, or equivalently, $\langle\Psi|T_{\pm\pm}|\Psi\rangle = -\frac{\hbar}{12\pi}(\partial_\pm\rho\partial_\pm\rho - \partial^2_\pm\rho + t_\pm)$, with the identification $-\frac{\hbar}{12\pi}t_\pm = \langle\Psi| : T_{\pm\pm} : |\Psi\rangle$.

3. Semiclassical equations in the Boulware state

Since we are interested in the state which more naturally mimics the Minkowski vacuum in flat space (i.e., the Boulware state) we shall define it by using, in quantizing the matter, the time coordinate "t" respect to which the metric takes the static form. Therefore it is natural to impose that

$$\langle B| : T_{\pm\pm}(t,x) : |B\rangle = \langle B|T_{\pm\pm}(t,x)|B\rangle_{\rho=0} = 0 \ . \tag{13}$$

This definition assumes that the semiclassical background metric is static: $\rho = \rho(x)$ and $\phi = \phi(x)$, where $x = (x^+ - x^-)/2$. A straightforward consequence of the above equation is that the expectation value $\langle\Psi|\frac{\delta S_m}{\delta\phi}|\Psi\rangle_{\rho=0}$ can be determined immediately

$$\langle B|\frac{\delta S_m}{\delta\phi}|B\rangle_{\rho=0} = -\frac{\hbar}{16\pi}\frac{(\phi_x^2 - \phi_{xx})_x}{\phi_x} \ , \tag{14}$$

where the subindex x means derivative with respect to the coordinate x. Therefore we have all ingredients to write down the semiclassical Einstein equations in the s-wave approximation in the Boulware vacuum. Fixing, for simplicity, the value of the parameter r_0 as $r_0 \equiv \sqrt{\lambda} = \sqrt{\frac{\hbar G}{12\pi c^3}} \equiv \sqrt{\frac{l^2_{Planck}}{12\pi}}$, the final result is

$$\begin{aligned}
\phi_{xx} - \phi_x^2 - 2\rho_x\phi_x &= e^{2\phi}\left[\rho_{xx} - \rho_x^2 + 6\rho_x\phi_x + 6\rho\phi_x^2\right] \ , && (15)\\
\phi_{xx} - 2\phi_x^2 + \frac{e^{2(\phi+\rho)}}{\lambda} &= e^{2\phi}\left[\rho_{xx} - 3(\phi_{xx} - \phi_x^2)\right] \ , && (16)\\
\phi_{xx} - \phi_x^2 - \rho_{xx} &= e^{2\phi}\left[3\rho_{xx} + 6\rho_x\phi_x + 6\rho\phi_{xx} + \frac{3}{2}\frac{(\phi_{xx} - \phi_x^2)_x}{\phi_x}\right] . && (17)
\end{aligned}$$

When the right hand side is zero, the solution describes the Schwarzschild geometry

$$\begin{aligned}
\rho &= \frac{1}{2}\ln(1 - \frac{2GM}{c^2 r}) && (18)\\
r^* \equiv x &= r + \frac{2GM}{c^2}\ln(1 - \frac{2GM}{c^2 r}) \ , && (19)
\end{aligned}$$

with mass M. In the quantum theory the solution deviates from the above expressions but we shall impose that, for large r, the semiclassical solution approaches the classical one.

4. Exact Semiclassical solutions

Our task is to investigate how the classical relations (18–19) for $\rho = \rho(r)$ and $r = r(x)$ are modified by pure vacuum polarization effects. The main properties of the classical solutions are

- The function $r = r(\rho)$ is monotonic, with

$$\frac{dr}{d\rho} > 0 \ , \tag{20}$$

  reaching $\rho = -\infty$ at the finite value $r = r_S$.
- The function $r(x)$ is monotonic, with

$$\frac{dr}{dx} > 0 \ , \tag{21}$$

  reaching $x = -\infty$ at $r = r_S$.

The numerical solution for the semiclassical equations violates the above properties and unravels the following features (the details of the integration can be found in [22]):

- **Existence of a bouncing point for the radial function.** The function $r = r(\rho)$ is very similar to the classical solution till very close to r_S. Just before r_S the function $r = r(\rho)$ has a minimum. This bouncing point $r = r_B > r_S$ for the radial function

$$\frac{dr}{d\rho}(r_B) = 0 \ , \tag{22}$$

  takes place at a very small but non-zero value of g_{00}. In the vicinity of this point the metric can be approximated by

$$ds^2 \approx -e^{2\rho(r_B)}c^2dt^2 + \alpha\frac{dr^2}{(1-\frac{r_B}{r})} + r^2d\Omega^2 \ , \tag{23}$$

  where α is a numerical constant.
- **Existence of a branching point for the radial function**. The function $r = r(x)$ is very similar to the classical solution till very close to r_S. Just before r_S the function $r = r(x)$ has a minimum, corresponding to the bouncing point $r = r_B$ described above.

$$\frac{dr}{dx}(r_B) = 0 \ . \tag{24}$$

  Moreover, just after it we encounter a minimum for the "tortoise" coordinate x

$$\frac{dx}{dr}(x_M) = 0 \ , \tag{25}$$

  at a finite value $x = x_M$. The dependence of the radial function around this point is

$$r \approx r_M - \beta(x - x_M)^{2/3} \tag{26}$$

  where β is a positive numerical constant. The radial function has a branching point at $x = x_M$, which turns out to be the minimal possible value for the coordinate x. Although all components of the metric are finite at $x = x_M$ the branching (26) generates a curvature singularity for the metric.

5. Final comments

The most important result of the investigation presented in this contribution is the emergence of a bouncing point for the radial function in the static quantum corrected Schwarzschild geometry. It would be interesting to see if this feature is also reproduced in the static solutions in braneworld models in 5-dimensional Anti-de Sitter space.

More explicitly, the classical solution for the tortoise coordinate

$$r^* \equiv x = r + r_S \ln \frac{|r - r_S|}{r_S} \tag{27}$$

is modified, in the vicinity of $r_B \approx r_S$, by

$$x \approx x_B + e^{-\rho(r_B)} \sqrt{\frac{r_B}{\alpha}(r - r_B)} \ . \tag{28}$$

Naive use of the standard formulaes to derive the emitted radiation would convert the thermal Hawking luminosity $L \propto T_H^2$, where T_H is the Hawking temperature, into

$$L \propto \frac{1}{(x^- - x_B^-)^2} \ , \tag{29}$$

which is unbounded when the retarded time x_B^- corresponding to the bounce point is reached. This seems to indicate that backreaction effects can produce significative changes to the standard view of the evaporation process.

References

[1] Hawking S W 1975 *Comm. Math. Phys.* **43** 199
[2] Parker L 1975 *Phys. Rev.* D **12** 1519
[3] Wald R M 1975 *Comm. Math. Phys.* **45** 9
[4] Birrel N D and Davies P C W 1982 *Quantum fields in curved spaces* (Cambridge: Cambridge University Press)
[5] Frolov V P and Novikov I D 1996 *Black hole physics* (Dordrecht: Kluwer Academic Publisher)
[6] Fabbri A and Navarro-Salas J 2005 *Modeling black hole evaporation* (London: Imperial College Press-World Scientific)
[7] Boulware D 1975 *Phys. Rev.* D **11** 1404
[8] Christensen S M and Fulling S A 1977 *Phys. Rev.* D **15** 2088
[9] Fabbri A and Navarro-Salas J 2005 *Int. J. Mod. Phys.* D (to appear); gr-qc/0505110
[10] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
[11] Aharony O Gubser S Maldacena J and Ooguri H 2000 *Phys. Rep.* **323** 183
[12] Emparan R, Fabbri A and Kaloper N 2002 JHEP **08** 043
[13] Tanaka T 2003 *Prog. Theor. Phys. Suppl.* **148** 307
[14] Anderson P R, Balbinot B and Fabbri A 2005 *Phys. Rev. Lett.* **94** 061301
[15] Fabbri A *Journal of Physics: Conference Series*, Quantum black holes and holography, Proceedings of the Conference: *Constrained dynamics and quantum gravity, 2005*, Cala Gonone, Italy
[16] Anderson A R, Hiscock W A and Samuel D A 1995 *Phys. Rev.* **D51** 4337
[17] Fabbri A, Farese S and Navarro-Salas J 2003 *Phys. Lett.* B **574** 309
[18] Davies P C W, Fulling S A and Unruh W G 1976 *Phys. Rev.* D **13** 2720
[19] Giddings S G 1994 "Quantum mechanics of black holes", hep-th/0412138
[20] Strominger A 1995 "Les Houches lectures on black holes", hep-th/9501071 (1995)
[21] Thorlacious L 1995 *Nucl. Phys. Proc. Suppl.* **41** 245
[22] Fabbri A, Farese S, Navarro-Salas J, Olmo G J and Helios-Alepuz S "Quantum corrections to the Schwazschild spacetime: backreaction in the Boulware state", to appear