Malliavin Calculus:
Computation of Greeks for European Options under Black-Scholes Model

Sergio de Diego

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Supervisor: Eva Ferreira
University of the Basque Country UPV/EHU

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Abstract

This article pretends to investigate the application of the Malliavin calculus for Greeks computation in the case of European options assuming the Black-Scholes model dynamics for the underlying asset. We calculate all the main Greeks formulae for vanilla and digital European options employing the integration by parts formula developed in the theory of Malliavin calculus. Then, we apply the localization technique presented by Founié et al. for variance reduction in Monte Carlo simulation. This two methods are compared to the previous ones existing, such as finite-difference, pathwise derivative and likelihood ratio methods.

1 Introduction

As it is known, in finance the adecuate hedging of financial products is as important as their correct valuation. For that purpose, in the case of derivatives the computation of their sensitivities, called Greeks, with respect to various of the parameters of the problem becomes essential for the hedging. In this sense, we focus on the case of the computation of Greeks for European options. In this case, the price can be expressed as the updated expected value of the payoff

$$V_t = \mathbb{E}[e^{-r(T-t)}\Phi(S_T)]$$

The payoff Φ is a functional of the underlying asset at maturity and satisfies the condition $\mathbb{E}_Q[\Phi(S_T)^2] < \infty$. The underlying asset is given by a markovian process $\{S_t; t \in [0,T]\}$ with values in $\mathbb{R}^n$ being the solution of the stochastic differential equation

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

where $\{W_t; t \in [0,T]\}$ is a brownian motion with values in $\mathbb{R}^n$. The coefficients $\mu(t, S_t)$ and $\sigma(t, S_t)$ are Lipschitz ensuring the existence and uniqueness of a solution for the differential equation. For this study we consider the Black-Scholes model assuming a geometric brownian motion for the underlying asset.

In most cases, there is not an analytical solution for the derivative price because the density function of the payoff is unkown. Then, we can simulate trajectories for the underlying asset emplying Monte Carlo techniques and, hence, also compute the price for the European options by calculating the expected value for all the final states of the trajectories. Then, for the hedging of the options we compute the differentials with respect to some parameters of the problem, called Greeks. The most common and important ones are Delta, Gamma, Vega, Theta, Rho and Lambda.

There are several alternatives to compute these differentials. The first and simplest one is to compute these quantities using finite-difference method based on the definition of the derivative

$$\frac{\partial V_0(\alpha)}{\partial \alpha} \simeq \frac{V_0(\alpha+h) - V_0(\alpha)}{h}$$

However, this method can produce large biases and variances, particularly with the incresing of the derivative order. The second method is the pathwise derivative method which interchanges the derivative and expectative operators

$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\mathbb{E}_Q[\Phi(S(\alpha))] = \mathbb{E}_Q\left[\frac{\partial}{\partial \alpha}\Phi(S(\alpha))\right]$$

Nevertheless, this technique is not always executable, mainly when the payoff is not smooth. The last alternative, until now, is the likelihood ratio method, which can be seen as a complementary method of the parthwise derivative. In this case, instead of derivate the payoff function, we apply the derivative operator to the density function of the payoff

$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\mathbb{E}_Q[\Phi(S)] = \mathbb{E}_Q\left[\Phi(S)\frac{\partial log(f(S;\alpha))}{\partial \alpha}\right]$$

where $f(S;\alpha)$ is the density function of the payoff depending on the parameter α. Therefore, this method is applicable even with discontinuos payoffs but provides the higher variance of them all and we are limited by the knowledge of the density function of the payoff, which rarely happens. So, in practice is the less operative method.
In the appendix, in Section 7, we resume the basic points and development of these methods applied to the case of European options. For more information see Glasserman [4].

In this article we apply the Malliavin calculus as an alternative for the computation of Greeks for European options showing that the sensitivities can be expressed by a formula of the type

$$\frac{\partial V_0}{\partial \alpha} = \mathbb{E}_Q\left[\Phi'(S(\alpha))\frac{\partial S(\alpha)}{\partial \alpha}\right] = \mathbb{E}_Q\left[\Phi(S(\alpha))H\left(S(\alpha), \frac{\partial S(\alpha)}{\partial \alpha}\right)\right]$$

where H is a random function to be determinated. As we see, this method involves the payoff and a weight, which will be different for each Greek.

We complete the computation for all the main Greeks, following the work started by Nualart [8], for European call vanilla options. Then we make use of the localization technique developed by Fournié et al. [2] for variance reduction, extended to all the Greeks. Finally, we extend the study to discontinuous payoffs, for European call digital options. We will see that this technique makes the Malliavin calculus more competitive and efficient with respect to the rest of the methods mentioned before.

The article scheme is the following. First of all, we present in Section 2 the basic results of Malliavin calculus for our purpose. In Section 3 we obtain the formulae for the main Greeks of European options and, then in Section 4 we used the localization technique introduced by Fournie et al. [2] for variance reduction which improves the Monte Carlo simulation for Greeks computation. In Section 5 we present some simulations and numerical results, comparing the results of this technique with the most common ones for Greeks computation. Finally, in Section 6 we resume the most relevant conclusions observed in the study. Moreover, in Section 7 we include an appendix explaining briefly the other computational techniques employed in the computation of the financial derivatives sensitivities.

2 Fundamentals of Malliavin Calculus

In this section we present briefly the main results of Malliavin calculus on developing the integration by parts formula, which relates the derivative operator on the Wiener space and the divergence operator. Our goal is to define the Malliavin derivative of a square integrable random variable $F : \Omega \to \mathbb{R}$ with respect to the parameter $\omega \in \Omega$, and the diverge operator which is the dual of the derivative operator. Then, with these tools we are able to obtain the integration by parts formula. For more information see Nualart [8].

Consider a multidimensional brownian motion $\{W_t; t \in [0,T]\}$ with values in $\mathbb{R}^n$ defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with $\{\mathfrak{F}_t\}$ the filtration generated by the brownian motion. Let $\mathfrak{C}$ be the set of smooth random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)) \qquad f \in \mathfrak{C}_p^\infty(\mathbb{R}^n) \tag{1}$$

with

$$W(h_i) = \int_0^\infty h_i(t) dW_t, \quad \forall i = 1, \ldots, n \tag{2}$$

where $\mathfrak{C}_p^\infty(\mathbb{R}^n)$ is the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ that are infinitely differentiable and its partial derivatives have polynomial growth.

Definition 2.1 *The derivative of a smooth random variable F of the form previosly defined in* (1) *is the random variable H-valued given by*

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1) \ldots W(h_n)) h_i \tag{3}$$

Proposition 2.1 *The second derivative of a smooth random variable F of the form previosly defined in* (1) *is the random variable $H \otimes H$-valued given by*

$$D_t^2 F = \sum_{i,j=1}^{n} \partial_{ij}^2 f(W(h_1) \dots W(h_n))(h_i \otimes h_j) \tag{4}$$

In general, the k-th derivative of F, $D_t^k F$, for any $k \geq 1$ is the $H^{\otimes k}$-valued random variable obtained iterating the derivative operator k times.

Proposition 2.2 *For any $p \geq 1$ and $k \geq 1$ natural numbers, the derivative operator D_t^k is closable from $\mathfrak{C}$ to $L^p(\Omega; H^{\otimes k})$*

Proposition 2.3 *The domain of the derivative operator D_t^k is the space $\mathbb{D}^{k,p}$ defined as the completion of $\mathfrak{C}$ with respect to the norm*

$$||F||_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{i=1}^{k} \mathbb{E}[||D^i F||_{H^{\otimes k}}^p] \right)^{\frac{1}{p}} \tag{5}$$

Proposition 2.4 *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function with bounded partial derivatives, and fix $p \geq 1$. Suppose that $F = (F^1 \dots F^n)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Then $\varphi(F) \in \mathbb{D}^{1,p}$, and*

$$D_t(\varphi(F)) = \sum_{i=1}^{n} \partial_i \varphi(F) D_t F^i \tag{6}$$

Definition 2.2 *We denote by δ the adjoint operator of the derivative operator. This means, δ is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ such that:*

1. *The domain of δ, denoted by $Dom(\delta)$, is the set of $H -$ valued square integrable random variables $u \in L^2(\Omega; H)$ such that*

$$|\mathbb{E}[\langle D_t F, u \rangle_H]| \leq c||F||_2 \qquad \forall F \in \mathbb{D}^{1,2}, c \text{ constant depending on } u \tag{7}$$

2. *If u belongs to $Dom(\delta)$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by*

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle D_t F, u \rangle_H] \qquad \text{for any} \quad F \in \mathbb{D}^{1,2} \tag{8}$$

Proposition 2.5 *Let $u \in Dom(\delta)$ and $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega; H)$. Then $Fu \in Dom(\delta)$ and satisfies*

$$\delta(Fu) = F\delta(u) - \langle D_t F, u \rangle_H \tag{9}$$

Now we present the result known as the integration by parts formula, which will be applied for the computation of Greeks.

Proposition 2.6 *Let F,G be two random variables such that $F \in \mathbb{D}^{1,2}$. Consider an H-valued random variable u such that $\langle D_t F, u \rangle_H \neq 0$ a.s. and $Gu\langle D_t F, u \rangle_H^{-1} \in Dom(\delta)$. Then, for any continuously differentiable function f with bounded derivative we have*

$$\mathbb{E}[f'(F)G] = \mathbb{E}[f(F)H(F,G)] \tag{10}$$

where $H(F,G) = \delta(Gu\langle D_t F, u \rangle_H^{-1})$

3 Greeks for European options

We apply the integration by parts formula to the case of European options considering the Black-Scholes model. Thus, the dynamics of the underlying asset, under the risk-neutral probability measure, is described by the geometric brownian motion

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t} \tag{11}$$

The Black-Scholes model is complete, which means that any marketable asset is replicable. We can create a self-financing replicating portfolio with a derivative and its underlying asset, and value the derivative solving the partial differential equation we obtain with a dynamic hedging strategy. In our case, we suppose that there is no analytical solution for the differential equation and we use the Monte Carlo simulation for pricing.

A Greek is a partial derivative of a financial derivative with respect to any of the model parameters. This derivatives measure the sensitivity of the derivative price under variations of any parameter. We consider an European option with the payoff depending on the price of the underlying asset at maturity $t = T$, $\Phi(S_T)$ such that $\mathbb{E}_Q[\Phi(S_T)^2] < \infty$, whose price at time t is given by $V_t = \mathbb{E}_Q[e^{-r(T-t)}\Phi(S_T)]$. Then, the partial derivative with respet to a parameter α of the problem at time $t = 0$ will be given by

$$\frac{\partial V_0}{\partial \alpha} = \mathbb{E}_Q\left[e^{-r(T-t)}\Phi'(S_T)\frac{\partial S_T}{\partial \alpha}\right] \tag{12}$$

Making use of the integration by parts formula we obtain

$$\frac{\partial V_0}{\partial \alpha} = \mathbb{E}_Q\left[e^{-r(T-t)}\Phi(S_T)H\left(S_T, \frac{\partial S_T}{\partial \alpha}\right)\right] \tag{13}$$

where

$$H\left(S_T, \frac{\partial S_T}{\partial \alpha}\right) = \delta\left(\frac{\partial S_T}{\partial \alpha}u\langle D_s S_T, u\rangle_H^{-1}\right) \tag{14}$$

Employing this general result we obtain the specific formulae for the main Greeks. First of all, we calculate the scalar product $\langle D_t S_T, u\rangle_H$ for $u = 1$, which will be used in the following calculations for each Greek

$$\langle D_t S_T, 1\rangle_H = \int_T D_t S_T dt = \int_0^T \sigma S_T dt = \sigma T S_T \tag{15}$$

The Delta is the first partial derivative of the option value with respect to the underlying asset price, and measures the rate change of the option value with respect to changes of the underlying asset price

$$\Delta = \frac{\partial V_0}{\partial S_0} = \mathbb{E}_Q\left[e^{-rT}\Phi'(S_T)\frac{\partial S_T}{\partial S_0}\right] = \mathbb{E}_Q\left[e^{-rT}\Phi'(S_T)\frac{S_T}{S_0}\right] =$$

$$= \frac{e^{-rT}}{S_0}\mathbb{E}_Q[\Phi'(S_T)S_T] \tag{16}$$

Hence, we have for $F = S_T$, $G = S_T$ and $u = 1$

$$\delta\Big(S_T(\sigma T S_T)^{-1}\Big) = \delta\Big(\frac{1}{\sigma T}\Big) = \frac{1}{\sigma T}\int_0^T dW_t = \frac{W_T}{\sigma T} \tag{17}$$

Replacing the result in the integration by parts formula we obtain for the Delta

$$\Delta = \frac{e^{-rT}}{S_0}\mathbb{E}_Q\Big[\Phi(S_T)\frac{W_T}{\sigma T}\Big] = \frac{e^{-rT}}{S_0\sigma T}\mathbb{E}_Q[\Phi(S_T)W_T] \tag{18}$$

The Gamma is the second partial derivative of the option value with respect to the underlying asset price, and measures the rate change of the option Delta with respect to changes of the underlying asset price

$$\begin{aligned}\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \mathbb{E}_Q\Big[e^{-rT}\Phi''(S_T)\Big(\frac{\partial S_T}{\partial S_0}\Big)^2\Big] = \mathbb{E}_Q\Big[e^{-rT}\Phi''(S_T)\Big(\frac{S_T}{S_0}\Big)^2\Big] = \\ = \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q[\Phi''(S_T)S_T^2]\end{aligned} \tag{19}$$

Hence, we have for $F = S_T$, $G = S_T^2$ and $u = 1$

$$\begin{aligned}\delta\Big(S_T^2(\sigma T S_T)^{-1}\Big) = \delta\Big(\frac{S_T}{\sigma T}\Big) = \frac{1}{\sigma T}\Big(S_T\int_0^T dW_t - \int_0^T D_t S_T dt\Big) = \\ = \frac{1}{\sigma T}\Big(S_T W_T - \sigma T S_T\Big)\end{aligned} \tag{20}$$

Replacing the result in the integration by parts formula we obtain

$$\begin{aligned}\frac{e^{-rT}}{S_0^2}\mathbb{E}_Q[\Phi''(S_T)S_T^2] = \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\Big[\Phi'(S_T)\frac{1}{\sigma T}\Big(S_T W_T - \sigma T S_T\Big)\Big] = \\ = \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\Big[\Phi'(S_T)S_T\Big(\frac{W_T}{\sigma T} - 1\Big)\Big]\end{aligned} \tag{21}$$

Now we apply again the integration by parts formula for $F = S_T$, $G = S_T\Big(\frac{W_T}{\sigma T} - 1\Big)$ and $u = 1$

$$\begin{aligned}\delta\Big(S_T\Big(\frac{W_T}{\sigma T} - 1\Big)(\sigma T S_T)^{-1}\Big) = \frac{1}{\sigma T}\delta\Big(\frac{W_T}{\sigma T} - 1\Big) = \\ = \frac{1}{\sigma T}\Big(\frac{1}{\sigma T}\Big(W_T\int_0^T dW_t - \int_0^T D_t W_T dt\Big) - \int_0^T dW_t\Big) = \\ = \frac{1}{\sigma T}\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\Big)\end{aligned} \tag{22}$$

Thus, replacing once more in the integration by parts formula

$$\frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\Big[\Phi'(S_T)S_T\Big(\frac{W_T}{\sigma T} - 1\Big)\Big] = \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\Big[\Phi(S_T)\frac{1}{\sigma T}\Big(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\Big)\Big] \tag{23}$$

Finally, we obtain the Gamma

$$\Gamma = \frac{e^{-rT}}{S_0^2 \sigma T} \mathbb{E}_Q \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right] \tag{24}$$

The Vega is the partial derivative of the option value with respect to the volatility of the underlying asset, and measures the rate change of the option value with respect to changes of the volatility of the underlying asset

$$\nu = \frac{\partial V_0}{\partial \sigma} = \mathbb{E}_Q \left[e^{-rT} \Phi'(S_T) \frac{\partial S_T}{\partial \sigma} \right] = e^{-rT} \mathbb{E}_Q [\Phi'(S_T) S_T (W_T - \sigma T)] \tag{25}$$

Hence, we have for $F = S_T$, $G = S_T(W_T - \sigma T)$ and $u = 1$

$$\begin{aligned} \delta\Big(S_T(W_T - \sigma T)(\sigma T S_T)^{-1} \Big) &= \delta\left(\frac{W_T}{\sigma T} - 1 \right) = \\ = \frac{1}{\sigma T} \left(W_T \int_0^T dW_t - \int_0^T D_t W_T dt \right) - \int_0^T dW_t &= \frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \end{aligned} \tag{26}$$

Replacing the result in the integration by parts formula we obtain for the Vega

$$\nu = e^{-rT} \mathbb{E}_Q \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right] \tag{27}$$

Comparing this expression for the Vega with the previous one for the Gamma, we can relate both of them as follows

$$\nu = (S_0^2 \sigma T) \Gamma \tag{28}$$

The Theta is the partial derivative of the option value with respect to the time, and measures the rate change of the option value with respect to changes of the time to maturity

$$\begin{aligned} \theta = -\frac{\partial V_0}{\partial T} &= \mathbb{E}_Q \left[r e^{-rT} \Phi(S_T) - e^{-rT} \Phi'(S_T) \frac{\partial S_T}{\partial T} \right] = \\ &= \mathbb{E}_Q \left[r e^{-rT} \Phi(S_T) - e^{-rT} \Phi'(S_T) S_T \left(\left(r - \frac{\sigma^2}{2} \right) + \frac{\sigma W_T}{2T} \right) \right] = \\ &= r e^{-rT} \mathbb{E}_Q [\Phi(S_T)] - e^{-rT} \mathbb{E}_Q \left[\Phi'(S_T) S_T \left(\left(r - \frac{\sigma^2}{2} \right) + \frac{\sigma W_T}{2T} \right) \right] \end{aligned} \tag{29}$$

We focus on the second term for applying the integration by parts formula. We have for $F = S_T$, $G = S_T\left(\left(r - \frac{\sigma^2}{2} \right) + \frac{\sigma W_T}{2T} \right)$ and $u = 1$

$$\begin{aligned} \delta\left(S_T \left(\left(r - \frac{\sigma^2}{2} \right) + \frac{\sigma W_T}{2T} \right) (\sigma T S_T)^{-1} \right) &= \frac{1}{\sigma T} \delta\left(\left(r - \frac{\sigma^2}{2} \right) + \frac{\sigma W_T}{2T} \right) = \\ = \frac{1}{\sigma T} \left(\left(r - \frac{\sigma^2}{2} \right) \int_0^T dW_t + \frac{\sigma}{2T} \left(W_T \int_0^T dW_t - \int_0^T D_t W_T dt \right) \right) &= \\ &= \frac{W_T^2}{2T^2} + \frac{1}{\sigma T} \left(r - \frac{\sigma^2}{2} \right) W_T - \frac{1}{2T} \end{aligned} \tag{30}$$

Therefore, we have for the Theta

$$
\begin{aligned}
\theta &= re^{-rT}\mathbb{E}_Q[\Phi(S_T)] - e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\left(\frac{W_T^2}{2T^2} + \frac{1}{\sigma T}\left(r - \frac{\sigma^2}{2}\right)W_T - \frac{1}{2T}\right)\right] = \\
&= e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\left(r - \frac{1}{2T}\left(\frac{W_T^2}{T} + \frac{2}{\sigma}\left(r - \frac{\sigma^2}{2}\right)W_T - 1\right)\right)\right] \qquad (31)
\end{aligned}
$$

The Rho is the partial derivative of the option value with respect to the rate of interest, and measures the rate change of the option value with respect to changes of the rate interest

$$
\begin{aligned}
\rho = \frac{\partial V_0}{\partial r} &= \mathbb{E}_Q\left[-Te^{-rT}\Phi(S_T) + e^{-rT}\Phi'(S_T)\frac{\partial S_T}{\partial r}\right] = \\
&= \mathbb{E}_Q[-Te^{-rT}\Phi(S_T) + e^{-rT}\Phi'(S_T)TS_T] = \\
&= -Te^{-rT}\mathbb{E}_Q[\Phi(S_T)] + e^{-rT}\mathbb{E}_Q[\Phi'(S_T)TS_T] \qquad (32)
\end{aligned}
$$

As in the previous Greek, we focus on the second term for applying the integration by parts formula. We have for $F = S_T$, $G = TS_T$ and $u = 1$

$$
\delta(TS_T(\sigma TS_T)^{-1}) = \delta\left(\frac{1}{\sigma}\right) = \frac{1}{\sigma}\int_0^T dW_t = \frac{W_T}{\sigma} \qquad (33)
$$

Hence, we have for the Rho

$$
\begin{aligned}
\rho &= -Te^{-rT}\mathbb{E}_Q[\Phi(S_T)] + e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\frac{W_T}{\sigma}\right] = \\
&= e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\left(\frac{W_T}{\sigma} - T\right)\right] \qquad (34)
\end{aligned}
$$

The Elasticity or Lambda is the rate of percentage change of option value with respect to the percentage change of the underlying asset.

$$
\lambda = \frac{S_0}{V_0}\frac{\partial V_0}{\partial S_0} = \frac{S_0}{V_0}\Delta \qquad (35)
$$

Hence, we have for the Lambda

$$
\lambda = \frac{e^{-rT}}{V_0\sigma T}\mathbb{E}_Q[\Phi(S_T)W_T] \qquad (36)
$$

4 Localization technique for variance reduction

Now we present the localization technique developed by Founié et al. [2] for variance reduction in the Monte Carlo simulation for Greeks computation in the case of Malliavin calculus. The idea is to localize the computation of the integration by parts formula around the singularity of the payoff. If the weights are large then the expected quantities increase and have also a large variance. So in order to reduce the variance we focus the calculations around the singularities. Our goal is to define the specific localization function for each Greek in the following European options with continuous and discontinuous payoff functionals.

4.1 European call vanilla option

For an European call vanilla option we have the payoff $\Phi(S_T) = (S_T - K)^+$ which has no discontinuity but a singularity at the point $S_T = K$.

We introduce the localization technique employing the following function

$$H_\delta(S_T) = \begin{cases} 0 & , if \quad S_T - K < -\delta \\ \frac{(S_T-K)+\delta}{2\delta} & , if \quad -\delta \leq S_T - K \leq \delta \\ 1 & , if \quad S_T - K > -\delta \end{cases} \tag{37}$$

where $\delta > 0$ is the localization parameter. Moreover, we also define the functions $G_\delta(S_T) = \int_{-\infty}^{S_T} H_\delta(s)ds$ and $F_\delta(S_T) = (S_T - K)^+ - G_\delta(S_T)$.

Hence, for the Delta we have

$$\begin{aligned} \Delta &= \frac{\partial}{\partial S_0}\mathbb{E}_Q[e^{-rT}(S_T - K)^+] = \\ &= \frac{\partial}{\partial S_0}\mathbb{E}_Q[e^{-rT}G_\delta(S_T)] + \frac{\partial}{\partial S_0}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\ &= \mathbb{E}_Q\left[e^{-rT}H_\delta(S_T)\frac{S_T}{S_0}\right] + \mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\frac{W_T}{S_0\sigma T}\right] = \\ &= \frac{e^{-rT}}{S_0}\mathbb{E}_Q[H_\delta(S_T)S_T] + \frac{e^{-rT}}{S_0\sigma T}\mathbb{E}_Q[F_\delta(S_T)W_T] \end{aligned} \tag{38}$$

We get the expression for the variance reduction of the Delta

$$\Delta = e^{-rT}\mathbb{E}_Q\left[H_\delta(S_T)\frac{S_T}{S_0} + F_\delta(S_T)\frac{W_T}{S_0\sigma T}\right] \tag{39}$$

Now, it is necessary to define a new localization function for the Gamma. So we define $I_\delta(S_T) = \frac{1}{2\delta}1_{\{|S_T-K|<\delta\}}$ and $J_\delta(S_T) = (S_T-K)^+ - \int_0^{S_T}\int_0^s I_\delta(u)duds$. With these localization functions we have for the Gamma

$$\begin{aligned} \Gamma &= \frac{\partial^2}{\partial S_0^2}\mathbb{E}_Q[e^{-rT}(S_T - K)^+] = \\ &= \mathbb{E}_Q\left[e^{-rT}I_\delta(S_T)\left(\frac{S_T}{S_0}\right)^2\right] + \mathbb{E}_Q\left[e^{-rT}J_\delta(S_T)\frac{1}{S_0^2\sigma T}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right] \end{aligned} \tag{40}$$

Hence, we get for the Gamma the variance reduction formula

$$\Gamma = e^{-rT}\mathbb{E}_Q\left[I_\delta(S_T)\left(\frac{S_T}{S_0}\right)^2 + J_\delta(S_T)\frac{1}{S_0^2\sigma T}\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right] \tag{41}$$

For the Vega we have

$$
\begin{aligned}
\nu = \frac{\partial}{\partial\sigma}\mathbb{E}_Q[e^{-rT}(S_T-K)^+] = \\
= \frac{\partial}{\partial\sigma}\mathbb{E}_Q[e^{-rT}G_\delta(S_T)] + \frac{\partial}{\partial\sigma}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\
= \mathbb{E}_Q[e^{-rT}H_\delta(S_T)S_T(W_T-\sigma T)] + \mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right]
\end{aligned}
\tag{42}
$$

We get the expression for the variance reduction of the Vega

$$
\nu = e^{-rT}\mathbb{E}_Q\left[H_\delta(S_T)S_T(W_T-\sigma T) + F_\delta(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right] \tag{43}
$$

For the Theta we have

$$
\begin{aligned}
\theta = -\frac{\partial}{\partial T}\mathbb{E}_Q[e^{-rT}(S_T-K)^+] = -\frac{\partial}{\partial T}\mathbb{E}_Q[e^{-rT}G_\delta(S_T)] - \frac{\partial}{\partial T}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\
= -\mathbb{E}_Q\left[-re^{-rT}G_\delta(S_T) + e^{-rT}H_\delta(S_T)\frac{\partial S_T}{\partial T}\right] - \mathbb{E}_Q\left[-re^{-rT}F_\delta(S_T) + e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial T}\right] = \\
= \mathbb{E}_Q[re^{-rT}(G_\delta(S_T)+F_\delta(S_T))] - \mathbb{E}_Q\left[e^{-rT}H_\delta(S_T)S_T\left(\left(r-\frac{\sigma^2}{2}\right) + \frac{\sigma W_T}{2T}\right)\right] - \\
-\mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\frac{1}{\sigma T}\left(\left(r-\frac{\sigma^2}{2}\right)W_T + \frac{\sigma}{2T}(W_T^2-T)\right)\right]
\end{aligned}
\tag{44}
$$

We get the expression for the variance reduction of the Theta

$$
\begin{aligned}
\theta = e^{-rT}\mathbb{E}_Q\left[r(S_T-K)^+ - H_\delta(S_T)S_T\left(\left(r-\frac{\sigma^2}{2}\right) + \frac{\sigma W_T}{2T}\right) - \right. \\
\left. -F_\delta(S_T)\frac{1}{\sigma T}\left(\left(r-\frac{\sigma^2}{2}\right)W_T + \frac{\sigma}{2T}(W_T^2-T)\right)\right]
\end{aligned}
\tag{45}
$$

For the Rho we have

$$
\begin{aligned}
\rho = \frac{\partial}{\partial r}\mathbb{E}_Q[e^{-rT}(S_T-K)^+] = \frac{\partial}{\partial r}\mathbb{E}_Q[e^{-rT}G_\delta(S_T)] + \frac{\partial}{\partial r}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\
= \mathbb{E}_Q\left[-Te^{-rT}G_\delta(S_T) + e^{-rT}H_\delta(S_T)\frac{\partial S_T}{\partial r}\right] + \mathbb{E}_Q\left[-Te^{-rT}F_\delta(S_T) + e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial r}\right] = \\
= \mathbb{E}_Q[-Te^{-rT}(G_\delta(S_T)+F_\delta(S_T))] + \mathbb{E}_Q[e^{-rT}H_\delta(S_T)TS_T] + \mathbb{E}_Q\left[e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial r}\right] = \\
= \mathbb{E}_Q[-Te^{-rT}(S_T-K)^+] + \mathbb{E}_Q[e^{-rT}H_\delta(S_T)TS_T] + \mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\frac{W_T}{\sigma}\right]
\end{aligned}
\tag{46}
$$

We get the expression for the variance reduction of the Rho

$$
\rho = e^{-rT}\mathbb{E}_Q\left[T(H_\delta(S_T)S_T - (S_T-K)^+) + F_\delta(S_T)\frac{W_T}{\sigma}\right] \tag{47}
$$

For the Lambda we have

$$\lambda = \frac{S_0}{V_0}\frac{\partial V_0}{\partial S_0} = \frac{S_0}{V_0}\Delta \tag{48}$$

Hence, we have for the Lambda with variance reduction

$$\lambda = e^{-rT}\mathbb{E}_Q\left[H_\delta(S_T)\frac{S_T}{V_0} + F_\delta(S_T)\frac{W_T}{V_0\sigma T}\right] \tag{49}$$

4.2 European call digital option

Now for an European call digital option we consider the payoff $\Phi(S_T) = A\theta(S_T - K)$ which has a discontinuity at the point $S_T = K$.

We introduce the localization technique employing the following function

$$H_\delta(S_T) = \begin{cases} 0 & ,if \quad S_T - K < -\delta \\ A\frac{(S_T-K)+\delta}{2\delta} & ,if \quad -\delta \le S_T - K \le \delta \\ A & ,if \quad S_T - K > -\delta \end{cases} \tag{50}$$

with $\delta > 0$ the localization parameter. In this case, we define the functions $I_\delta(S_T) = \frac{A}{2\delta}1_{\{|S_T-K|<\delta\}}$ and $F_\delta(S_T) = A\theta(S_T - K) - H_\delta(S_T)$.

Hence, for the Delta we have

$$\begin{aligned}\Delta = \frac{\partial}{\partial S_0}\mathbb{E}_Q[e^{-rT}A\theta(S_T - K)] = \frac{\partial}{\partial S_0}\mathbb{E}_Q[e^{-rT}H_\delta(S_T)] + \frac{\partial}{\partial S_0}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\ = \mathbb{E}_Q\left[e^{-rT}I_\delta(S_T)\frac{\partial S_T}{\partial S_0}\right] + \mathbb{E}_Q\left[e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial S_0}\right] = \\ = e^{-rT}\mathbb{E}_Q\left[I_\delta(S_T)\frac{S_T}{S_0}\right] + e^{-rT}\mathbb{E}_Q\left[F_\delta(S_T)\frac{W_T}{S_0\sigma T}\right]\end{aligned} \tag{51}$$

We get the expression for the variance reduction of the Delta

$$\Delta = e^{-rT}\mathbb{E}_Q\left[I_\delta(S_T)\frac{S_T}{S_0} + F_\delta(S_T)\frac{W_T}{S_0\sigma T}\right] \tag{52}$$

In this case, there is no possibility to define a variance reduction formula for Gamma, because there is no derivative for a function as the Dirac delta ($I_\delta(S_T) \to \delta_D(S_T)$ as $\delta \to 0$).

For the Vega we have

$$\begin{aligned}\nu = \frac{\partial}{\partial \sigma}\mathbb{E}_Q[e^{-rT}A\theta(S_T - K)] = \frac{\partial}{\partial \sigma}\mathbb{E}_Q[e^{-rT}H_\delta(S_T)] + \frac{\partial}{\partial \sigma}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\ = \mathbb{E}_Q\left[e^{-rT}I_\delta(S_T)\frac{\partial S_T}{\partial \sigma}\right] + \mathbb{E}_Q\left[e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial \sigma}\right] = \\ = \mathbb{E}_Q[e^{-rT}I_\delta(S_T)S_T(W_T - \sigma T)] + \mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right]\end{aligned} \tag{53}$$

We get the expression for the variance reduction of the Vega

$$\nu = e^{-rT}\mathbb{E}_Q\left[I_\delta(S_T)S_T(W_T-\sigma T)+F_\delta(S_T)\left(\frac{W_T^2}{\sigma T}-\frac{1}{\sigma}-W_T\right)\right] \tag{54}$$

For the Theta we have

$$\begin{aligned}\theta &= -\frac{\partial}{\partial T}\mathbb{E}_Q[e^{-rT}A\theta(S_T-K)] = -\frac{\partial}{\partial T}\mathbb{E}_Q[e^{-rT}H_\delta(S_T)] - \frac{\partial}{\partial T}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\ &= -\mathbb{E}_Q[-re^{-rT}H_\delta(S_T)+e^{-rT}I_\delta(S_T)\frac{\partial S_T}{\partial T}] - \mathbb{E}_Q[-re^{-rT}F_\delta(S_T)+e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial T}] = \\ &= \mathbb{E}_Q[re^{-rT}(H_\delta(S_T)+F_\delta(S_T))] - \mathbb{E}_Q\left[e^{-rT}I_\delta(S_T)S_T\left(\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma W_T}{2T}\right)\right] - \\ &\quad -\mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\frac{1}{\sigma T}\left(\left(r-\frac{\sigma^2}{2}\right)W_T+\frac{\sigma}{2T}(W_T^2-T)\right)\right]\end{aligned} \tag{55}$$

We get the expression for the variance reduction of the Theta

$$\begin{aligned}\theta = e^{-rT}\mathbb{E}_Q\Bigg[&rA\theta(S_T-K) - I_\delta(S_T)S_T\left(\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma W_T}{2T}\right) - \\ &-F_\delta(S_T)\frac{1}{\sigma T}\left(\left(r-\frac{\sigma^2}{2}\right)W_T+\frac{\sigma}{2T}(W_T^2-T)\right)\Bigg]\end{aligned} \tag{56}$$

For the Rho we have

$$\begin{aligned}\rho &= \frac{\partial}{\partial r}\mathbb{E}_Q[e^{-rT}A\theta(S_T-K)] = \frac{\partial}{\partial r}\mathbb{E}_Q[e^{-rT}H_\delta(S_T)] + \frac{\partial}{\partial r}\mathbb{E}_Q[e^{-rT}F_\delta(S_T)] = \\ &= \mathbb{E}_Q[-Te^{-rT}H_\delta(S_T)+e^{-rT}I_\delta(S_T)\frac{\partial S_T}{\partial r}] + \mathbb{E}_Q[-Te^{-rT}F_\delta(S_T)+e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial r}] = \\ &= \mathbb{E}_Q[-Te^{-rT}(H_\delta(S_T)+F_\delta(S_T))] + \mathbb{E}_Q[e^{-rT}I_\delta(S_T)TS_T] + \mathbb{E}_Q\left[e^{-rT}F'_\delta(S_T)\frac{\partial S_T}{\partial r}\right] = \\ &= \mathbb{E}_Q[-Te^{-rT}A\theta(S_T-K)] + \mathbb{E}_Q[e^{-rT}I_\delta(S_T)TS_T] + \mathbb{E}_Q\left[e^{-rT}F_\delta(S_T)\frac{W_T}{\sigma}\right]\end{aligned} \tag{57}$$

We get the expression for the variance reduction of the Rho

$$\rho = e^{-rT}\mathbb{E}_Q\left[T(I_\delta(S_T)S_T - A\theta(S_T-K)) + F_\delta(S_T)\frac{W_T}{\sigma}\right] \tag{58}$$

For the Lambda we have

$$\lambda = \frac{S_0}{V_0}\frac{\partial V_0}{\partial S_0} = \frac{S_0}{V_0}\Delta \tag{59}$$

Hence, we have for the Lambda with variance reduction

$$\lambda = e^{-rT}\mathbb{E}_Q\left[I_\delta(S_T)\frac{S_T}{V_0} + F_\delta(S_T)\frac{W_T}{V_0\sigma T}\right] \tag{60}$$

5 Numerical results

In this section we examine numerical results, mainly for an European call digital option for the model parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$ and $A = 10$. First, we check the results of Delta for an European call vanilla, extending the comparison to all methods.

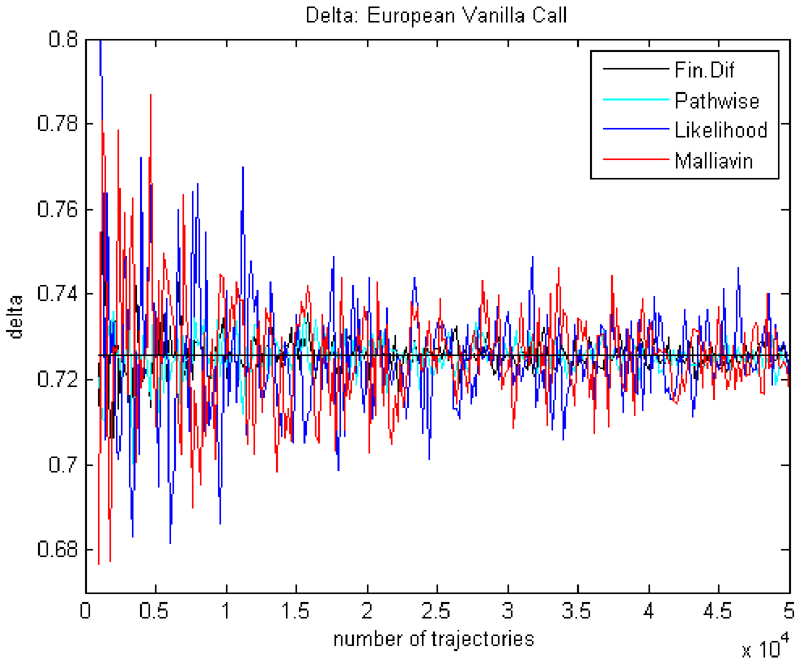

Figure 1: Delta for vanilla option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$.

The behavior observed is general for all Greeks, except the Gamma which is a second derivative. The finite-difference and pathwise methods have the least variance while the likelihood and Malliavin methods have large variance. Given that the most general method, until now, is the finite-difference method, from now on we focus on the comparison between this one and the Malliavin calculus, together with the variance reduction technique.

In the case of Gamma, we observe the largest variance for the finite-difference method because we estimate a second derivative finding even worse computational problems than in the case of first derivative.

Nonetheless, we also check the effectiveness of the variance reduction technique in reducing the variance of the Malliavin estimators and increasing the convergence rate to the exact value given by the Black-Scholes model formulae. We present this results for Delta and Gamma, which can be extended to the rest of Greeks. At the end of the section we include different tables with numerical values for different simulations to check this facts.

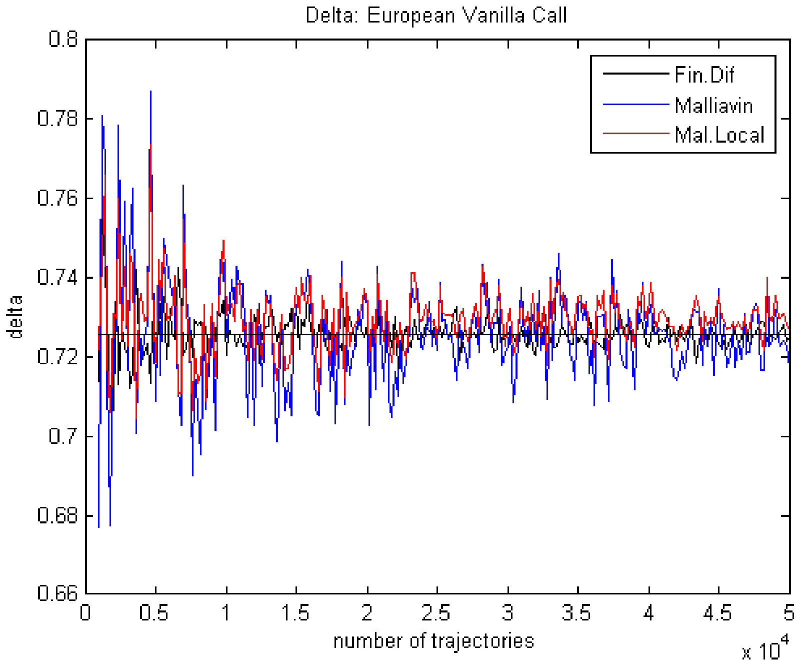

Figure 2: Delta for vanilla option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$.

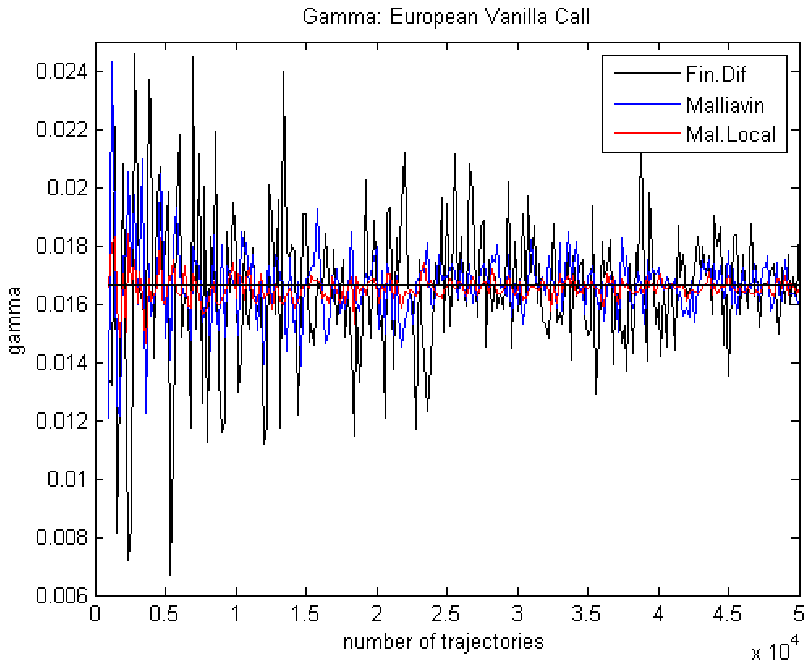

Figure 3: Gamma for vanilla option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$.

Now, we present the results for the case of an European call digital option. Here, the pathwise method is not applicable and, as we saw, the likelihood ratio method provides large variance and is not practical, so we will not make use of this two methods.

We see that in the case of European option with discontinuous payoff the Malliavin calculus is much more competitive than any other method even without applying the reduction variance technique, specially for the second derivative computation.

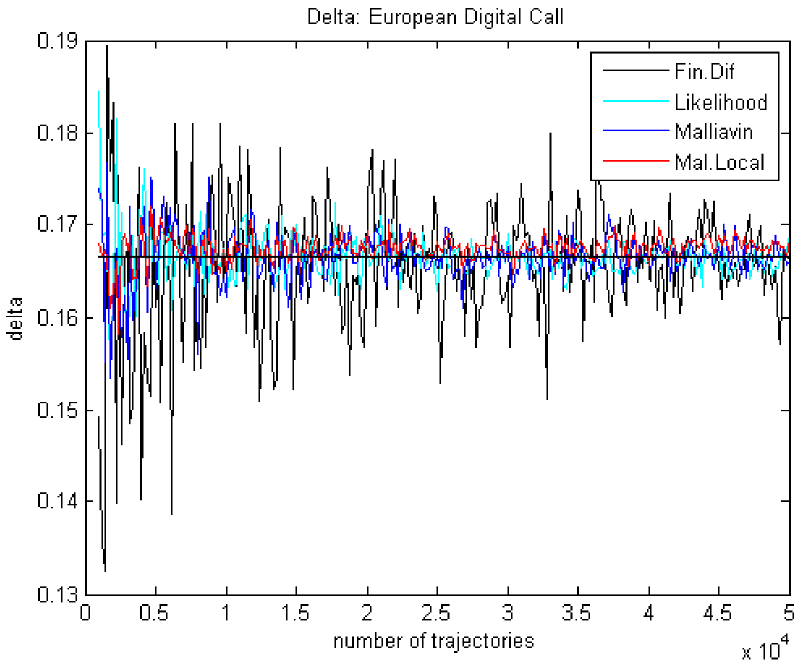

Figure 4: Delta for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

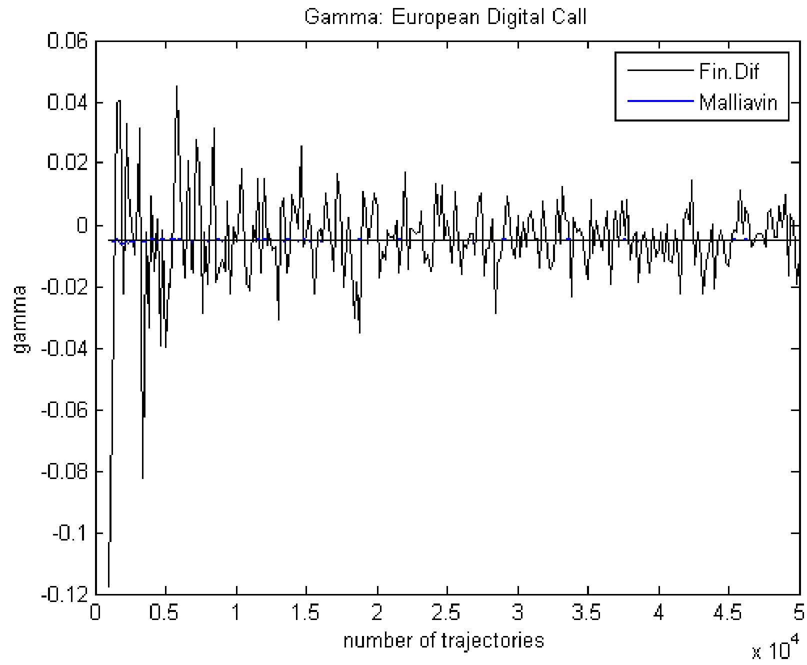

Figure 5: Gamma for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

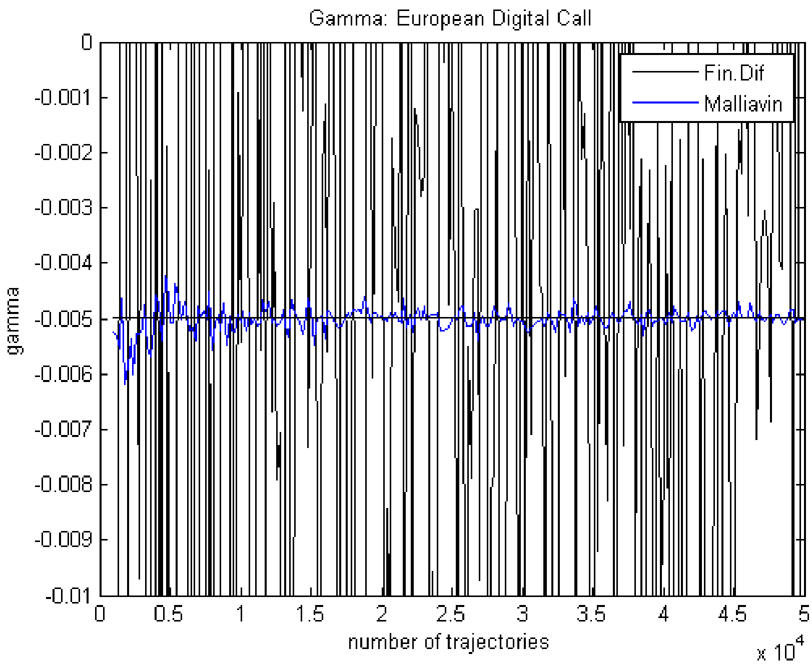

Figure 6: Zoom of Gamma for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

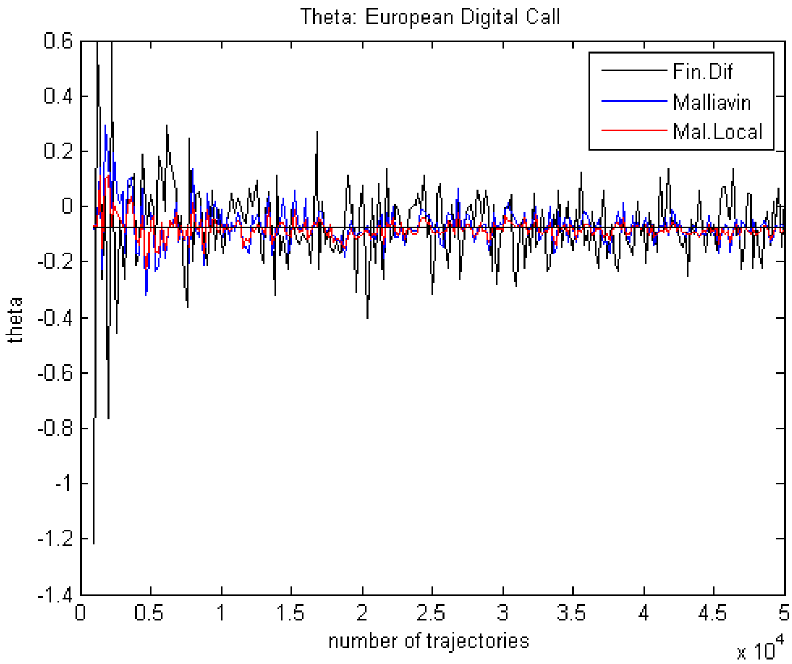

Figure 7: Theta for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

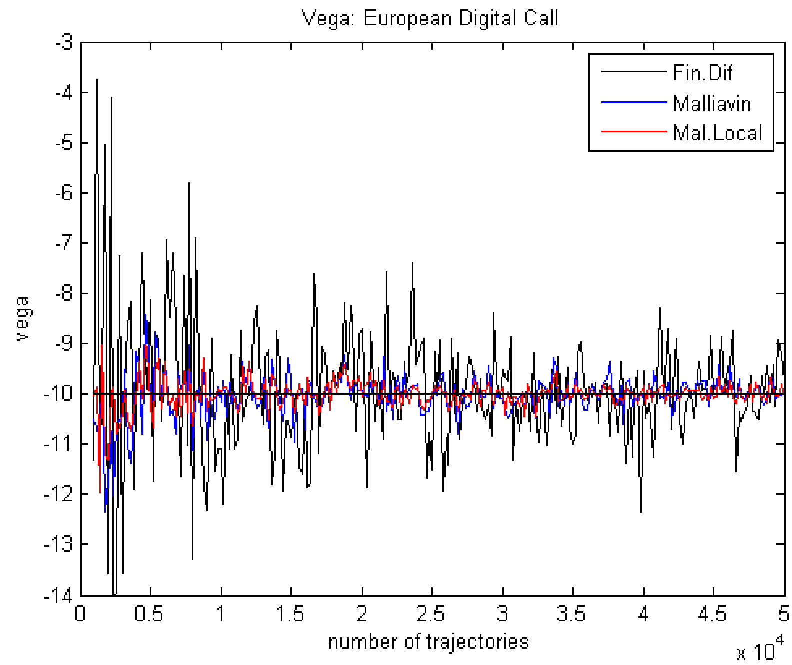

Figure 8: Vega for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

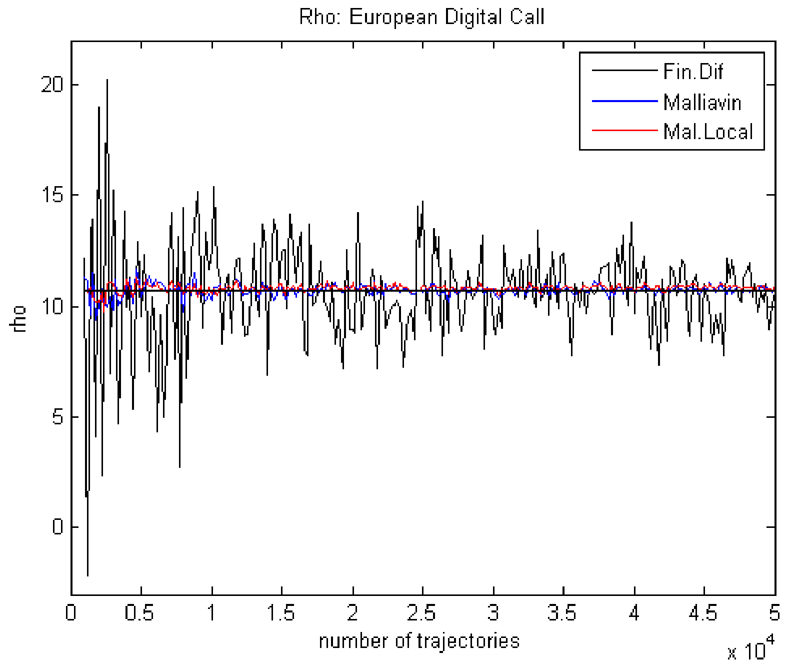

Figure 9: Rho for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

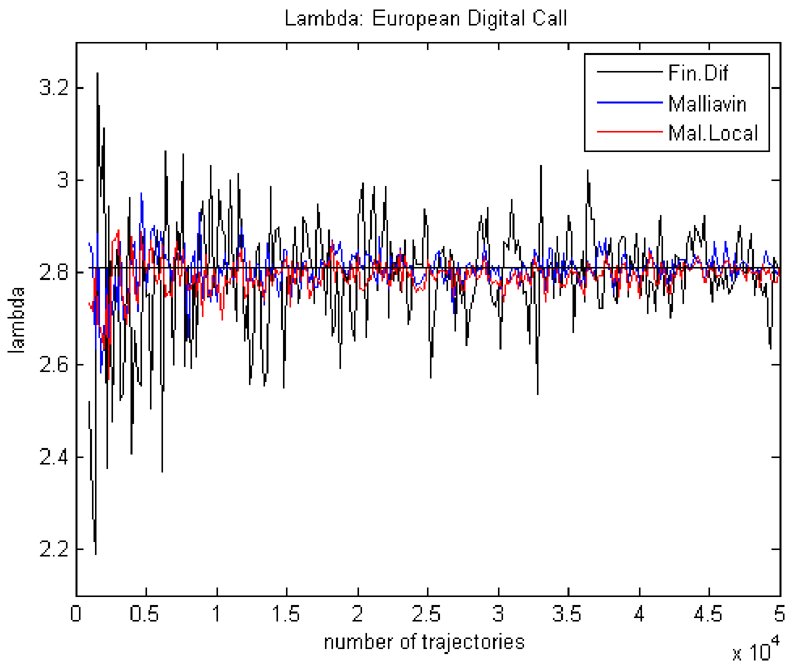

Figure 10: Lambda for digital option with parameters $S_0 = 100$, $K = 100$, $r = 0.1$, $T = 1$, $\sigma = 0.2$, $A = 10$.

We complete this section with some tables showing the numerical results for Greeks computation employing different number of simulations for each method and option. We do not include the likelihood ratio method but its results have the same magnitude as the obtained for Malliavin calculus.

We have the exact values for the Black-Scholes model in the case of an European call vanilla option: $Delta = 0.7257$, $Gamma = 0.0167$, $Theta = -9.2627$, $Vega = 33.3225$, $Rho = 59.3050$ and $Lambda = 5.4692$. The numerical results obtained are the following.

Greeks	**Fin.Difer.**	**Pathwise**	**Malliavin**	**Mal.Local**
Delta	0.7342(0.0054)	0.7319(0.0055)	0.7439(0.0160)	0.7495(0.0098)
Gamma	0.0195(0.0035)	-	0.0167(0.0014)	0.0174(0.0005)
Theta	-9.3730(0.1017)	-9.4577(0.1050)	-9.4231(0.4212)	-9.5924(0.1956)
Vega	33.7336(0.7987)	34.9503(0.8328)	33.4468(2.8589)	33.1885(1.3964)
Rho	59.9963(0.4320)	59.6269(0.4333)	60.7844(1.4570)	61.3534(0.8334)
Lambda	5.4680(0.0778)	5.3955(0.0780)	5.4692(0.0976)	5.1394(0.0736)

Table 1: Greeks of a call vanilla option calculated with 10000 simulated trajectories.

Greeks	**Fin.Difer.**	**Pathwise**	**Malliavin**	**Mal.Local**
Delta	0.7255(0.0031)	0.7282(0.0031)	0.7214(0.0092)	0.7229(0.0057)
Gamma	0.0182(0.0019)	-	0.0167(0.0009)	0.0163(0.0003)
Theta	-9.2089(0.0580)	-9.2574(0.0580)	-9.2387(0.2483)	-9.3290(0.1119)
Vega	32.7070(0.4543)	33.0068(0.4539)	33.4241(1.7077)	32.9596(0.6068)
Rho	59.3822(0.2490)	59.5675(0.2486)	58.9629(0.8385)	59.1198(0.3844)
Lambda	5.5085(0.0455)	5.4956(0.0452)	5.4757(0.0581)	5.4077(0.0446)

Table 2: Greeks of a call vanilla option calculated with 30000 simulated trajectories.

Greeks	**Fin.Difer.**	**Pathwise**	**Malliavin**	**Mal.Local**
Delta	0.7248(0.0024)	0.7252(0.0024)	0.7170(0.0070)	0.7270(0.0043)
Gamma	0.0187(0.0015)	-	0.0159(0.0006)	0.0165(0.0002)
Theta	-9.3131(0.0457)	-9.2543(0.0449)	-9.0378(0.1887)	-9.2847(0.0855)
Vega	34.0088(0.3598)	33.2850(0.3510)	31.8634(1.2985)	32.9532(0.6156)
Rho	59.1221(0.1927)	59.2578(0.1924)	58.5147(0.6388)	59.5188(0.3685)
Lambda	5.4272(0.0348)	5.4690(0.0348)	5.4392(0.0442)	5.4145(0.0340)

Table 3: Greeks of a call vanilla option calculated with 50000 simulated trajectories.

In the case of an European call digital option we have the exact values: *Delta* $= 0.1666$, *Gamma* $= -0.0050$, *Theta* $= -0.0734$, *Vega* $= -9.9967$, *Rho* $= 10.7307$ and *Lambda* $= 2.8094$. For different number of simulated trajectories for the underlying asset we get the following results.

Greeks	**Fin.Difer.**	**Malliavin**	**Mal.Local**
Delta	0.1639(0.0085)	0.1642(0.0028)	0.1673(0.0015)
Gamma	-0.0009(0.0174)	-0.0049(0.0002)	-
Theta	-0.0084(0.1664)	-0.0665(0.0669)	-0.0956(0.0369)
Vega	-11.0796(1.5954)	-9.8688(0.4462)	-9.8839(0.3081)
Rho	11.6253(2.8407)	10.5335(0.2632)	10.8402(0.1740)
Lambda	2.7695(0.1457)	2.7893(0.0517)	2.8095(0.0331)

Table 4: Greeks of a call digital option calculated with 10000 simulated trajectories.

Greeks	**Fin.Difer.**	**Malliavin**	**Mal.Local**
Delta	0.1623(0.0049)	0.1660(0.0016)	0.1669(0.0009)
Gamma	-0.0106(0.0099)	-0.0050(0.0001)	-
Theta	-0.1034(0.1030)	-0.0573(0.0380)	-0.0806(0.0210)
Vega	-10.09599(0.8735)	-10.0897(0.2532)	-9.9516(0.1753)
Rho	10.7503(1.5903)	10.6627(0.1506)	10.7581(0.0944)
Lambda	2.7277(0.0827)	2.7965(0.0294)	2.7809(0.0187)

Table 5: Greeks of a call digital option calculated with 30000 simulated trajectories.

Greeks	**Fin.Difer.**	**Malliavin**	**Mal.Local**
Delta	0.1663(0.0038)	0.1660(0.0012)	0.1685(0.0007)
Gamma	-0.0105(0.0078)	-0.0050(0.0001)	-
Theta	-0.0931(0.0721)	-0.0785(0.0296)	-0.0749(0.0163)
Vega	-9.7722(0.6635)	-9.9169(0.1977)	-10.0979(0.1377)
Rho	10.7078(1.2257)	10.7020(0.1162)	10.9467(0.0770)
Lambda	2.7993(0.0647)	2.8138(0.0228)	2.8134(0.0147)

Table 6: Greeks of a call digital option calculated with 50000 simulated trajectories.

6 Conclusions

In this study, we have seen that the Malliavin calculus and, in particular, the integration by parts formula can be extended to the case of discontinuous payoffs for European options with better results, in general, than any other method avaliable for Greeks computation. This fact is of particular interest for financial derivatives with not smooth payoff functionals, because in these cases the only practical alternative is the finite-difference method which exhibits large variance and possible biases. As we see, the Malliavin calculus provides unbiased estimators for Greeks but we find the handicap of large variances. Nevertheless, this can be solved suitably employing the variance reduction technique for the appropriate localization functions.

There are many possibilities for further research to this work. The first one is to develop a systematic way to estimate optimally the localization parameter δ, for the variance reduction technique. A second one is to extend this study to more complicated discontinuous payoffs and check the results with the rest of the methods. The last one is to extend this study to the multidimensional case and to more sophisticated and realistic models for the underlying asset.

7 Appendix

7.1 Finite-difference method

This method relies on the definition of the derivative. Considering the price of an option V_0, its sensitivity with respect to a parameter α of the problem is

$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \lim_{h \to 0} \frac{V_0(\alpha + h) - V_0(\alpha)}{h}$$

Then, the finite-difference method requires to evaluate or simulate at more than one value of the parameter of derivation. Employing the Monte Carlo methodology we have the following estimator

$$\frac{\partial V_0(\alpha)}{\partial \alpha} \simeq \frac{\Delta V_0(\alpha)}{\Delta \alpha} = \frac{\hat{V_0}(\alpha + h) - \hat{V_0}(\alpha)}{h}$$

$$\mathbb{E}\left[\frac{\Delta V_0(\alpha)}{\Delta \alpha}\right] = \frac{V_0(\alpha + h) - V_0(\alpha)}{h}$$

Nevertheless, this method produces biased estimators. Considering a Taylor series expansion

$$V_0(\alpha + h) = V_0(\alpha) + \frac{dV_0(\alpha)}{d\alpha} h + \frac{1}{2}\frac{d^2 V_0(\alpha)}{d\alpha^2} h^2 + O(h^3)$$

$$\frac{V_0(\alpha + h) - V_0(\alpha)}{h} = \frac{dV_0(\alpha)}{d\alpha} + \frac{1}{2}\frac{d^2 V_0(\alpha)}{d\alpha^2} h + O(h^2) = \frac{dV_0(\alpha)}{d\alpha} + O(h)$$

Then, the expectation of the estimator has a bias of order h

$$\mathbb{E}\left[\frac{\Delta V_0(\alpha)}{\Delta \alpha}\right] = \frac{V_0(\alpha + h) - V_0(\alpha)}{h} = \frac{\partial V_0(\alpha)}{\partial \alpha} + O(h)$$

In order to reduce the bias of the sensitivity estimator we consider an alternative definition for the derivative given by

$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \lim_{h \to 0} \frac{V_0(\alpha + h) - V_0(\alpha - h)}{2h}$$

Then, in this case the error of the estimator will be lower, given that $h < 1$:

$$\frac{\partial V_0(\alpha)}{\partial \alpha} \simeq \frac{\Delta V_0(\alpha)}{\Delta \alpha} = \frac{\hat{V}_0(\alpha + h) - \hat{V}_0(\alpha - h)}{2h}$$

$$\mathbb{E}\left[\frac{\Delta V_0(\alpha)}{\Delta \alpha}\right] = \frac{V_0(\alpha + h) - V_0(\alpha - h)}{2h} = \frac{\partial V_0(\alpha)}{\partial \alpha} + O(h^2)$$

The variance of the estimator is

$$\mathbb{V}ar\left[\frac{\Delta V_0(\alpha)}{\Delta \alpha}\right] = \frac{1}{(2h)^2}[\hat{V}_0(\alpha + h) - \hat{V}_0(\alpha - h)]$$

For the second derivatives we have the approximation

$$\frac{\partial^2 V_0(\alpha)}{\partial \alpha^2} \simeq \frac{V_0(\alpha + h) + 2V_0(\alpha) - V_0(\alpha - h)}{h^2}$$

The implementation of the second derivatives by means of finite differences has numerical problems worse than the case of first derivatives.

7.2 Pathwise derivative method

We remember that the price of an European option and the sensitivities at time $t = 0$ are given by

$$V_0(\alpha) = \mathbb{E}_Q[e^{-rT}\Phi(S_T; \alpha)]$$
$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\mathbb{E}_Q[e^{-rT}\Phi(S_T; \alpha)]$$

When it is possible, if we interchange the expectative and derivative operators we can calculate the sensitivity of the option with respect to a parameter α as follows

$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\mathbb{E}_Q[e^{-rT}\Phi(S_T; \alpha)] = e^{-rT}\mathbb{E}_Q\left[\frac{\partial}{\partial \alpha}\Phi(S_T; \alpha)\right]$$

The ambit of this method is limited by the required condition of continuity of the payoff, which is the function to derivate. This is an unbiased estimator of the corresponding Greek. Again, for error estimations we have to calculate the variance of the estimators.

In the case of European call vanilla option we have for the Greeks the following estimators

Greeks	European call vanilla option
Delta	$e^{-rT}\mathbb{E}_Q\left[\frac{S_T}{S_0}\theta(S_T-K)\right]$
Gamma	$e^{-rT}\mathbb{E}_Q\left[\left(\frac{S_T}{S_0}\right)^2\delta(S_T-K)\right]$
Vega	$e^{-rT}\mathbb{E}_Q\left[\sigma T S_T\left(\frac{W_T}{\sigma T}-1\right)\theta(S_T-K)\right]$
Theta	$e^{-rT}\mathbb{E}_Q\left[r(S_T-K)^+ - S_T\left(\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma W_T}{2T}\right)\theta(S_T-K)\right]$
Rho	$e^{-rT}\mathbb{E}_Q\left[T\left(S_T\theta(S_T-K)-(S_T-K)^+\right)\right]$
Lambda	$\frac{S_0}{V_0}e^{-rT}\mathbb{E}_Q\left[\frac{S_T}{S_0}\theta(S_T-K)\right]$

On the other hand, for an European call digital option

Greeks	European call digital option
Delta	$e^{-rT}\mathbb{E}_Q\left[\frac{S_T}{S_0}A\delta(S_T-K)\right]$
Vega	$e^{-rT}\mathbb{E}_Q\left[\sigma T S_T\left(\frac{W_T}{\sigma T}-1\right)A\delta(S_T-K)\right]$
Theta	$e^{-rT}\mathbb{E}_Q\left[rA\theta(S_T-K) - S_T\left(\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma W_T}{2T}\right)A\delta(S_T-K)\right]$
Rho	$e^{-rT}\mathbb{E}_Q\left[T\left(S_TA\delta(S_T-K)-A\theta(S_T-K)\right)\right]$
Lambda	$\frac{S_0}{V_0}e^{-rT}\mathbb{E}_Q\left[\frac{S_T}{S_0}A\delta(S_T-K)\right]$

We see that in the case of the digital option this method is not applicable in practice due to the discontinuity of the payoff, because the derivative of the heaviside step function is the Dirac delta function which is hard to implement computationally.

7.3 Likelihood ratio method

In the previous method we derivate the payoff function. Then, the pathwise derivative method fails when we have discontinuities in the payoff function. One possible alternative is to derivate, not the payoff itself, but the density function of the payoff. In this sense, we can see this method as the complementary of the pathwise method. Of course, this method is the less used of all because we are limited by the knowledge of the payoff density function.

For an European option the price at time $t = 0$ is given by the expectation

$$V_0(\alpha) = \mathbb{E}_Q[e^{-rT}\Phi(S_T)] = \int e^{-rT}\Phi(S_T)f(S_T;\alpha)dS_T$$

Then, the sensitivity with respect to the α parameter is given by

$$\frac{\partial V_0(\alpha)}{\partial \alpha} = \int e^{-rT}\Phi(S_T)\frac{\partial f(S_T;\alpha)}{\partial \alpha}dS_T =$$
$$= \int e^{-rT}\Phi(S_T)f(S_T;\alpha)\frac{\partial log(f(S_T;\alpha))}{\partial \alpha}dS_T =$$
$$= \mathbb{E}_Q\left[e^{-rT}\Phi(S_T)\frac{\partial log(f(S_T;\alpha))}{\partial \alpha}\right]$$

Considering the Black-Scholes model we have the density function

$$f(S_T;\alpha) = \frac{1}{\sigma\sqrt{T}S_T}\frac{1}{\sqrt{2\pi}}exp\left[-\frac{\zeta(S_T)^2}{2}\right]; \quad \zeta(S_T) = \frac{log\left(\frac{S_T}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

In the case of vanilla and digital European call options we have for the Greeks the same estimators, for the corresponding payoffs $\Phi(S_T)$ in each of both cases, given by

Greeks	**European call vanilla/digital option**
Delta	$e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\frac{1}{S_0\sigma\sqrt{T}}\zeta(S_T)\right]$
Gamma	$e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\frac{1}{S_0^2\sigma^2 T}\left(\zeta(S_T)^2 - 1 - \sigma\sqrt{T}\zeta(S_T)\right)\right]$
Vega	$e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\left(\frac{1}{\sigma}(\zeta(S_T)^2 - 1) - \sqrt{T}\zeta(S_T)\right)\right]$
Theta	$-e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\left(\frac{1}{2T}(\zeta(S_T)^2 - 1) + \frac{r-\frac{\sigma^2}{2}}{\sigma\sqrt{T}}\zeta(S_T)\right)\right]$
Rho	$e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\frac{\sqrt{T}}{\sigma}\zeta(S_T)\right]$
Lambda	$\frac{S_0}{V_0}e^{-rT}\mathbb{E}_Q\left[\Phi(S_T)\frac{1}{S_0\sigma\sqrt{T}}\zeta(S_T)\right]$

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