ASSET ALLOCATION IN HEDGE FUND STRATEGIES UNDER HIGHER MOMENTS: NON PARAMETRIC AND PARAMETRIC MODELS

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Abstract

This paper investigates empirically the asset allocation in a three hedge fund strategy portfolio under higher moments, first four moments (4M), and comparing the results with those obtained with the traditional mean-variance (MV) analysis. We begin showing the investor's expected utility maximization problem, we consider a CRRA utility function, using a Taylor series expansion until the fourth moment to approximate the expected utility function. Then, we focus in present the co-skewness and co-kurtosis matrix tensors M_3 and M_4, estimating the elements in those matrices by three methods: sample estimators, single factor estimators (Martellini and Ziemman, 2010), and a multivariate Variance Gamma distribution (Hitaj and Mercuri, 2011). We observe that the optimal portfolios obtained under the three methods, seem to be very similar to changes in the risk aversion degree, being the parametric Variance Gamma model the furthest in the results. We find that optimal portfolios vary greatly from MV to 4M under all the approaches. We perform an out-of-sample analysis comparing the different optimal portfolios, finding 4M models perform better than MV in this sample.

Keywords: Asset allocation; utility function; non normality; higher moments; improved estimators; Variance gamma distribution.

1 Introduction

Markowitz's (1952) portfolio theory has been the theoretical framework for academics and fund managers, and it represents the foundation of the current portfolio's choice. Despite its easy interpretation and friendly implementation, the Mean-Variance strategy become imprecise when (i) we are dealing with non-Gaussian distribution of the asset returns, (ii) the investor's utility function has a higher order than the quadratic and (iii) the first and the second centered moments do not determine the distribution, therefore is necessary to consider higher moments.

Since is widely accepted that financial returns are not normally distributed showed for strong empirical evidence as long as the fact that expected utility function might be approximated by higher order moments (Samuelson, 1970; Scott and Hovarth, 1980), mean-variance approach would lead to welfare losses for investors.

To overcome the inadequacy of mean-variance strategy in presence of the situations above referred; there have been attempts to include higher order moments in portfolio selection. In the existing literature that references the problem, has been considered up to the fourth moment, which makes skewness and kurtosis to be incorporated in the portfolio selection theory. However, several approaches have been developed with the porpoise of including the third and the forth moment in the analysis. One of them extends the literature by introducing higher dimensional 'efficient frontiers'. It coincides with the standard efficient frontier approach that for a variety of utility functions the target of selecting an optimal portfolio becomes the task of selecting a point on the high dimensional 'efficient frontiers', Athayde and Flores (2003) investigate in this sense. Further studies regarding to this approach have been done by Malvergne and Sornette (2005) or Prakash et al. (2003). However, all of them do not consider the theoretical motivation of the expected utility maximization. Another way of including higher moments consists in assuming certain kind of distribution for asset returns in pursuit to obtain closed-form portfolio solutions starting from the expected utility maximization problem, e.g. Mencia and Santana (2008), Hitaj and Mercuri (2011).

Alternatively, there has been introduced another method to consider higher moments in the asset allocation, it consists of doing an approximation to the expected utility function by using a Taylor expansion series of a certain order. This approach has been recently implemented by Harvey et al (2002), Guidolin and Timmermann (2006), Jondeau and Rockinger (2006), and Wongwachara (2009). The first study develops a Bayesian decision theoretic framework which assumes a skew normal distribution for modeling multivariate returns. The second study use Taylor expansion series in a Markov-switching framework with conditional normal innovations for returns. Jondeau and Rockinger measure in their paper the advantage of using higher moments in the optimization strategy for approximating the expected utility.

Nonetheless, the previous approach has some critics due to the arbitrariness of truncation of the Taylor series since there is no a particular rule at the moment of selecting the order of such a truncation. Further critics come from the fact that adding a new moment does not imply necessarily an improvement of the approximation. Furthermore, the convergence to the expected utility function using Taylor series expansion is assured under restrictive conditions, which makes the convergence is not guaranteed for all kind of utility functions.

When a Taylor series expansion is used to approximate an expected utility function, is especially important the estimation of the moments and the co-moments for each of them. Frequently sample estimators are the most used for estimating moments and co-moments, but it leads to high estimation error when the sample is small. Taking this into consideration, two methodologies have been introduced, (i) improved estimators, (ii) assuming a joint distribution for returns.

Improved estimators have been lately developed by Martellini and Ziemann (2010) and Hitaj et. al (2010), where they extend to co-skewness and co-kurtosis tensors the constant correlation approach (Elton and Gruber, 1973) and the single-factor approach (Sharpe, 1963), originally established for covariance matrix. Furthermore, they extend the concept of optimal shrinkage intensities to the presence of higher moments. They found that improved estimators diminish the estimator error, being particularly significant for the co-skewness matrix in accordance with the intuition that estimators for odd moments have more noise than estimators for even moments.

In the second methodology, is supposed a particular joint distribution for returns, e.g. Hitaj and Mercuri (2011) assume a Variance gamma distribution; it was introduced by Madan and Seneta (1990), where univariate log-returns are said to be variance gamma distributed. Hitaj and Mercuri's paper starts considering the expected utility maximization problem, then they suggest a parametric model constructed assuming a Multivariate Variance Gamma joint distribution using three different models: The symmetric case (Madan and Seneta, 1990), Semeraro model (Semeraro, 2008) and Wang model (Wang, 2000).

In this study, we focus in investigating empirically the effect that non-normality returns bring to allocation problem of wealth for an investor with a Constant Relative Risk Aversion utility function in a static single-period context, based on three hedge fund strategies sample. We use the Taylor series expansion up to the fourth moment to approximate the utility function following the Jondeu and Rockinger (2006) study, and comparing the allocation obtained with mean-variance and higher moments analysis. For estimating moments and co-moments, we use sample estimators as well as single factor estimators (improved estimators), in line with the study of Martellini and Ziemman (2010). Finally, we will focus in applying the multivariate variance gamma joint distribution model in the symmetric case, following Hitaj and Mercuri (2011) paper.

This document is organized as follows. In section 2 we present the investors problem and introduce the higher moment theory. Section 3 is dedicated to present the generalized hyperbolic distribution family and introduced variance gamma distribution as a special case, showing the estimation procedure and the asset allocation process using a common gamma mixing density. In Section 4 data is described. Section 5 is dedicated to explain the Empirical Analysis and the results, and finally section 6 is reserved to conclusions.

2 Portfolio Selection under Higher Moments

In this section we present the investor's problem, then we continue with the approximation of the expected utility function by using a Taylor series expansion, and we explain how higher moments' tensors can be computed from asset returns series.

2.1 Investor's problem

With the objective of evaluating the impact of higher-order moment on portfolio choice, we start by considering an expected utility function maximization problem, finding the optimal portfolio weights that assure the maxim expected utility to the investor. The expected utility function is then expressed as follows.

$$E[U(W)] = \int U(w)f(w)dw$$

Where $f(w)$ is the density distribution function of the final period's wealth, $W = W_0(1 + r_p)$ represents the investor's final wealth. W_0 is the initial wealth whereas $r_p = \alpha' R$ is the portfolio returns, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)'$ is the vector of weights assigned to the risky assets R.

Working with a infinitely differentiable function U, we approximate utility function by doing a Taylor expansion series, centered in the mean of the terminal wealth, $\overline{W} = W_0(1 + E(r_p))$:

$$U(W) = \sum_{k=0}^{\infty} \left[\frac{U^{(k)}E(W)}{k!} \left(W - E(W)\right)^k \right]$$

Where the expected wealth is $E(W) = \overline{W} = 1 + \alpha'\mu$, being $\mu = E(R)$ the expected return vector[1]. Then the utility can be represented as follows

$$U(W) = \sum_{k=0}^{\infty} \left[\frac{U^{(k)}(\overline{W})}{k!} \, (W - \overline{W})^k \right]$$

Due to the complex interpretation of moments higher than the forth, we assume that the previous utility function might be represented by the first four centered moments of the distribution of the portfolio return, as well as the derivatives of the utility function. The expression below shows such approximation.

$$E[U(W)] \approx U(\overline{W}) + \mathrm{U}^{(1)}(\overline{W})E[W - \overline{W}] + \frac{1}{2!}\mathrm{U}^{(2)}(\overline{W})E[(W - \overline{W})^2] + \frac{1}{3!}\mathrm{U}^{(3)}(\overline{W})E[(W - \overline{W})^3] + \frac{1}{4!}\mathrm{U}^{(4)}(\overline{W})E[(W - \overline{W})^4]$$

Following the notation in Jondeau and Rockinger (2006) the four centered moments above represented: the expected return, variance, skewness and kurtosis of the end-period are defined as:

$$E[(W - \overline{W})] = E\left[\left(r_p - \mu_p\right)\right] = \mu_p = 0$$

$$E[(W - \overline{W})^2] = E\left[\left(r_p - \mu_p\right)^2\right] = \sigma_p^2$$

$$E[(W - \overline{W})^3] = E\left[\left(r_p - \mu_p\right)^3\right] = s_p^3$$

$$E[(W - \overline{W})^4] = E\left[\left(r_p - \mu_p\right)^4\right] = k_p^4$$

By replacing the previous relations we obtain the following expression, which is the approximation of the expected utility function:

[1] W_0 is set equal to one with the objective to make the problem free-scale of the wealth. It means that relative risk aversion does not depend on wealth. This is a reason to prefer the power or logarithm functions in this kind of problems, since they are scale-independent utility functions.

$$E[U(W)] \approx U(\bar{W}) + \frac{1}{2!}\mathrm{U}^{(2)}(\bar{W})\sigma_p^2 + \frac{1}{3!}\mathrm{U}^{(3)}(\bar{W})s_p^3 + \frac{1}{4!}\mathrm{U}^{(4)}(\bar{W})k_p^4 \qquad (1)$$

2.2 The Case of CRRA function

In this document we consider an investor or asset manager in a static single-period context (short-term investor). Thus, the investor is said to do a myopic portfolio choice since she ignores what could happen in the next periods[2]. This investor has some preferences that are represented by a power utility function also known as the Constant Relative Risk Aversion function (CRRA).

$$U(W) = \begin{cases} W^{1-\lambda}/1-\lambda & ; \lambda \neq 1 \\ \log(W) & ; \lambda = 1 \end{cases} \qquad (2)$$

Where λ measures the coefficient of relative risk aversion. CRRA function shows a decreasing absolute risk aversion. This utility function has been widely studied in the literature.

In this case, the approximation using the expression (1) for the CRRA function is[3]:

$$E[U(W)] \approx \frac{\bar{W}^{1-\lambda}}{1-\lambda} - \frac{1}{2!}\lambda\bar{\mathrm{W}}^{-(\lambda+1)}\sigma_p^2 + \frac{1}{3!}(\lambda^2+\lambda)\bar{\mathrm{W}}^{-(\lambda+2)}s_p^3 - \frac{1}{4!}(\lambda^3+3\lambda^2+2\lambda)\bar{\mathrm{W}}^{-(\lambda+3)}k_p^4 \qquad (3)$$

We get the first order conditions (FOC), and after doing some simplifications we obtain the next condition:

$$\frac{\partial E[U(W)]}{\partial \alpha} = \bar{W}^{-\lambda}\frac{\partial \mu_p}{\partial \alpha} - \frac{1}{2!}\lambda(\bar{W})^{-(\lambda+1)}\frac{\partial \sigma_p^2}{\partial \alpha} + (\lambda^2+\lambda)(\bar{W})^{-(\lambda+2)}\left[\frac{1}{2!}\frac{\partial \mu_p}{\partial \alpha}\sigma_p^2 + \frac{1}{3!}\frac{\partial s_p^3}{\partial \alpha}\right] - (\lambda^3+3\lambda^2+ 2\lambda)(\bar{W})^{-(\lambda+3)}\left[\frac{1}{3!}\frac{\partial \mu_p}{\partial \alpha}s_p^3 + \frac{1}{4!}\frac{\partial k_p^4}{\partial \alpha}\right]\frac{1}{4!}(\lambda^4+6\lambda^3+11\lambda^2+6\lambda)(\bar{W})^{-(\lambda+4)}\frac{\partial \mu_p}{\partial \alpha}k_p^4 \qquad (4)$$

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[2] There exist some special cases in which long-term investors should make the same decisions that shor-term investors, becoming the investment horizon irrelevant. See Campbell and Viceira, 2002.

[3] Notice that an indispensable requirement in mean-variance analysis is that utility function must be quadratic. Equivalently, for the sake of incorporating skewness and kurtosis the utility function must be quartic.

2.3 Higher order tensors

Now, we express the moments of a portfolio in a simple way that will help us to solve the asset allocation problem, introducing the higher order tensors. The M_2 is the second moment tensor and consist in the variance co-variance matrix. Equivalently, tensor M_3 is the skewness co-skewness matrix and M_4 is the kurtosis co-kurtosis matrix.

The M_3 and M_4 matrices are defined by the next relations:

$$M_3 = E[(R-\mu)(R-\mu)'\otimes(R-\mu)'] = \{s_{ijk}\}$$

$$M_4 = E[(R-\mu)(R-\mu)'\otimes(R-\mu)'\otimes(R-\mu)'] = \{k_{ijkl}\} \qquad (5)$$

Elements s_{ijk} and k_{ijkl} are defined as:

$$s_{ijk} = E\left[(R_i-\mu_i)\left(R_j-\mu_j\right)'\otimes(R_k-\mu_k)'\right] \qquad i,j,k = 1,\dots,n$$

$$k_{ijkl} = E\left[(R_i-\mu_i)\left(R_j-\mu_j\right)'\otimes(R_k-\mu_k)'\otimes(R_l-\mu_l)'\right] \qquad i,j,k,l = 1,\dots,n$$

Last notation is found in Harvey et al (2002), Athayde and Flores (2004) and Jondeau and Rockinger (2006).

As an example, let suppose we are dealing with tree different asset (n=3), then we would obtain a (3,9) co-skewness matrix and a (3,27) co-kurtosis matrix.

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right] = [S_{1jk} \;\; S_{2jk} \;\; S_{3jk}]$$

S_{1jk} is a short notation for the (n,n) matrix $\{s_{1jk}\}_{j,k=1,2,3}$. With this notation we present the (3,27) M_4 matrix.

$$M_4 = [K_{11kl} \;\; K_{12kl} \;\; K_{13kl} | K_{21kl} \;\; K_{22kl} \;\; K_{23kl} | K_{31kl} \;\; K_{32kl} \;\; K_{33kl}]$$

where K_{11kp} denotes the (n,n) matrix $\{k_{11kl}\}_{j,k=1,2,3}$.

As we can see there is no need to compute all the elements in M_3 and M_4 due to symmetries, for instance, element s_{112} is the same than elements s_{121} and s_{211}, it conduces to compute only $n(n+1)(n+2)/6$ and $n(n+1)(n+2)(n+3)/24$ elements in M_3 and M_4 matrices, respectively.

Moments of the portfolio can be computed given a vector of weights α as:

$$\mu_p = \alpha'\mu$$

$$\sigma_p^2 = \alpha' M_2 \alpha$$

$$s_p^3 = \alpha' M_3(\alpha \otimes \alpha) \tag{6}$$

$$k_p^4 = \alpha' M_4(\alpha \otimes \alpha \otimes \alpha)$$

Where the derivatives with respect to weights α are:

$$\frac{\partial \mu_p}{\partial \alpha} = \mu$$

$$\frac{\partial \sigma_p^2}{\partial \alpha} = 2M_2\alpha$$

$$\frac{\partial s_p^3}{\partial \alpha} = 3M_3(\alpha \otimes \alpha) \tag{7}$$

$$\frac{\partial k_p^4}{\partial \alpha} = 4M_4(\alpha \otimes \alpha \otimes \alpha)$$

Now we have obtained all the elements needed to solve the asset allocation problem. We replace values (6) and (7) in the equation (4) and recalling that $\bar{W} = (1+\alpha'\mu)$ we get the expression required to do the optimization when $\lambda \neq 1$.

$$\frac{\partial E[U(W)]}{\partial \alpha} = (1+\alpha'\mu)^{-\lambda}\mu - \lambda(1+\alpha'\mu)^{-(\lambda+1)}M_2\alpha + \frac{1}{2}(\lambda^2+\lambda)(1+\alpha'\mu)^{-(\lambda+2)}[\mu\alpha' M_2\alpha + M_3(\alpha \otimes \alpha)] - \frac{1}{6}(\lambda^3+3\lambda^2+2\lambda)(1+\alpha'\mu)^{-(\lambda+3)}[\mu\alpha' M_3(\alpha \otimes \alpha) + M_4(\alpha \otimes \alpha \otimes \alpha)] + \frac{1}{24}(\lambda^4+6\lambda^3+11\lambda^2+6\lambda)(1+\alpha'\mu)^{-(\lambda+4)}\mu\,\alpha' M_4(\alpha \otimes \alpha \otimes \alpha) \tag{8}$$

So far, we have showed the asset allocation problem starting from the investor's expected utility maximization problem, considering a CRRA utility function. After that we have approximated CRRA by using a Taylor series expansion until the fourth moment, later we got the FOC needed to find the optimal weights. Then, we have focused in present the co-skewness and co-kurtosis matrix tensors M_3 and M_4 in very suitable way. The elements in those matrices can be estimated using diverse methods; as we have mentioned earlier in this document, we use three methods; sample estimators, single factor estimators (Martellini and Ziemman, 2010), and establishing a joint density distribution for asset returns. In the coming section we explore the last approach, specifying a multivariate variance gamma distribution developed by Hitaj and Mercuri (2011).

3 The Variance Gamma multivariate model with a common gamma density

We move to present a parametric model which assumes a joint variance gamma distribution for asset returns. First we make a brief review of generalized hyperbolic distributions family where the variance gamma distribution belongs as a special case. Then, we present the estimated closed formulas used to obtain the moments and the co-moments needed to compute M_2, M_3 and M_4 and then the estimation procedure assuming a common mixing density.

3.1 The Generalized Hyperbolic Distribution

The principal features of the family of generalized hyperbolic distribution (GH), are mentioned in McNeil et al (2005). This family is built by using a mean variance mixture and a conditional mean specification. Thus, a random vector X has a GH distribution if it can be represented as:

$$X = \mu + \theta W + \sqrt{W}\Sigma^{1/2} Z \tag{9}$$

where,

(i) $Z \sim N_k(0, I_k)$

(ii) $\mu, \theta \in \mathbb{R}^d$

(iii) $\Sigma^{1/2} \in \mathbb{R}^{dxk}$ is a matrix

(iv) $W \geq 0$, is a scalar valued random variable which is independent Z and has a Generalized Inverse Gaussian Distribution, $GIG(\lambda, \chi, \psi)$.

Consequently, X depends on six parameters, $X \sim GH_d(\lambda, \chi, \psi, \mu, \Sigma, \theta)$. Parameters λ, χ and ψ give the shape of the distribution, while μ is the location parameter, θ is the skewness parameter, and dispersion matrix Σ is a dxd positive semidefinite matrix given by the relation $\Sigma = \Sigma^{1/2}(\Sigma^{1/2})'$.

The joint density distribution in case Σ had rank d is provided by mixing $X|W$ with respect to W.

$$f(x) = \int_0^\infty f_{X|W}(x|w) f_W(w) \mathrm{d}w$$

$$f(x) = \int_0^\infty \frac{e^{(x-\mu)'\Sigma^{-1}\gamma}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}w^{\frac{d}{2}}} \exp\left\{-\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{2w} - \frac{\gamma'\Sigma^{-1}\gamma}{\frac{2}{w}}\right\} f_W(w)\mathrm{d}w$$

By evaluating the integral above, we obtain the generalized hyperbolic density function.

$$f(x) = \frac{\left(\sqrt{\chi\psi}\right)^{-\lambda}\psi^{\lambda}(\psi + \gamma'\Sigma^{-1}\gamma)^{\left(\frac{d}{2}\right)-\lambda}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}K_\lambda\left(\sqrt{\chi\psi}\right)} \frac{K_{\lambda-\frac{d}{2}}\sqrt{\left(\chi + (x-\mu)'^{\Sigma^{-1}}(x-\mu)\right)(\psi + \gamma\Sigma^{-1}\gamma)}\, e^{(x-\mu)'\Sigma^{-1}\gamma}}{\left(\chi + (x-\mu)'\Sigma^{-1}(x-\mu)\right)(\psi + \gamma'\Sigma^{-1}\gamma\,))^{\left(\frac{d}{2}\right)-\lambda}} \tag{10}$$

where K_λ is a modified Bessel function of the third kind.

3.1.1 Special Cases

The family of Generalized Hyperbolic distribution is composed by several special cases according to the values of the parameters.

- If $\lambda = \frac{d+1}{2}$, we call this case the d-dimensional hyperbolic distribution, nevertheless its margins are not hyperbolic .
- If $\lambda = 1$ we get the multivariate generalized hyperbolic distribution and its margins are one dimensional hyperbolic distribution.
- If $\lambda = -\frac{1}{2}$ we refer to this case as the Normal Inverse Gaussian NIG.
- If $\lambda > 0$ and $\chi = 0$, we get a limiting case known as the Variance Gamma distribution. Generalized Laplace or Bessel function.
- If $\lambda = -\frac{\nu}{2}, \chi = \nu$ and $\psi = 0$, we obtain a limiting case called the skewed-t distribution, containing the usual Student t-distribution by setting the skewness parameter $\theta = 0$.

So far we have introduced the special cases of the Generalized hyperbolic distribution, next we will focus in the Variance Gamma distribution following the Hitaj & Mercuri (2011) approach for asset allocation with higher order moments.

3.2 Multivariate Variance Gamma with a common gamma mixing density

The Variance Gamma distribution was first introduced in finance by Madan and Seneta (1990), formulating a three parameter model for market returns, being possible to control skewness and kurtosis of the return distribution.

Let's begin with the general representation of the GH distribution seen in equation (9).

$$X = \mu + \theta W + \sqrt{W}\Sigma^{1/2}Z$$

As we already told, the Variance Gamma (VG) is obtained by setting in the GH distribution parameters to be $\chi = 0$ and $\lambda > 0$, hence $VG \sim (\lambda, 0, \psi, \mu, \Sigma, \theta)$. Therefore, W that in the GH case is a Generalized Inverse Gaussian random variable $GIG(\lambda, \chi, \psi)$ becomes Gamma distributed $\Gamma(\lambda, \psi)$ in the Variance Gamma model (See appendix A). The parameters λ and ψ are identified as the shape parameter and the scale parameter respectively, while the $\Sigma^{1/2}$ is a lower triangular matrix:

$$\Sigma^{1/2} = \begin{cases} a_{ij} & si\ i \geq j \\ 0 & \text{otherwise} \end{cases}$$

The i^{th} component of the vector X is said to be a univariate variance gamma model.

$$X_i = \mu_i + \theta_i W + \sqrt{W} \sum_{h=1}^{i} a_{ih} Z_h \tag{11}$$

3.2.1 Computing moments and co-moments

We can proceed to compute the moments and co-moments[4] in order to get M_2, M_3 and M_4 tensors, for the expected utility maximization problem.

The mean of each asset is given:

$$E(X_i) = \mu_i + \theta_i \lambda$$

The covariance matrix is calculated:

$$Cov(X_i, X_j) = \lambda(\theta_i\theta_j + \sigma_{ij}) \qquad \text{for } \ i \neq j$$

$$Var(X_i) = \lambda(\theta^2 + \sigma_i^2) \qquad \text{for } \ i \neq j$$

where covariance is computed as $\sigma_{ij} = \sum_{h=1}^{\min(i,j)} a_{ih} a_{jh}$

The co-skewness matrix is calculated:

$$\mathrm{s}_{iii} = \lambda\theta_\mathrm{i}(3\sigma_\mathrm{i}^2 + 2\theta_\mathrm{i}^2) \qquad \text{for } i = j = k$$

$$\mathrm{s}_{\mathrm{ii}k} = \lambda(\theta_k\sigma_i^2 + 2\theta_i\sigma_{ik} + 2\theta_k\theta_\mathrm{i}^2) \qquad \text{for } i = j \neq k$$

$$s_{ijk} = \lambda(\theta_k\sigma_{ij} + \theta_i\sigma_{jk} + \theta_j\sigma_{ik} + 2\theta_k\theta_j\theta_i) \qquad \text{for } i \neq j \neq k \tag{12}$$

The co-kurtosis matrix is calculated:

[4] The moments and the co-moments in this VG model are obtained through the moment generating function.

$$k_{iiii} = 3(\lambda^2 + \lambda)\sum_{h=1}^{i} a_{ih}^4 + (\alpha^2 + 2\alpha)(6\theta_i^2\sigma_i^2 + 3\theta_i^4)$$

$$k_{iiij} = 3(\lambda^2 + \lambda)\sum_{h=1}^{\min(i,j)} a_{ih}^3 a_{jh} + (\lambda^2 + 2\lambda)(3\theta_i\theta_j\sigma_i^2 + 3\theta_i^2\sigma_{ij} + 3\theta_i^3\theta_j)$$

$$k_{iijj} = 3(\lambda^2 + \lambda)\sum_{h=1}^{\min(i,j)} a_{ih}^2 a_{jh}^2 + (\lambda^2 + 2\lambda)\left(\theta_j^2\sigma_i^2 + 4\theta_i\theta_j\sigma_{ij} + \theta_i^2\sigma_j^2\right) + (3\lambda^2 + 6\lambda)\theta_i^2\theta_j^2$$

$$k_{iijk} = 3(\lambda^2 + \lambda)\sum_{h=1}^{\min(i,j,k)} a_{ih}^2 a_{jh} a_{kh} + (\lambda^2 + 2\lambda)\left(\theta_j\theta_k\sigma_i^2 + 2\theta_i\theta_k\sigma_{ij} + 2\theta_i\theta_j\sigma_{ik} + \theta_i^2\sigma_{jk}\right) + (3\lambda^2 + 6\lambda)\theta_i^2\theta_j\theta_k$$

3.2.2 Parametrical estimation

Following Hitaj & Mercuri (2011), we will assume parameter $\psi = 1$. Therefore, we get that $W \sim \Gamma(\lambda, 1)$. Then, we prepare to estimate the multivariate VG model with a common mixing random variable. However, as we have seen, the joint density distribution in this kind of family distributions has a difficult form. It conduces that there exist some problems with the estimation based in a joint likelihood distribution. Taking this into account, it is implemented an univariate estimation and later we try to advance to the multivariate model. According with the Hitaj & Mercuri (2011) the estimation procedure to do so is the following:

i). Estimate the parameters $\mu_i, \theta_i, \sigma^2$and λ_i for each financial time series by the Maximum Likelihood method.

ii). After estimating we take a common shape parameter λ by computing $\hat{\lambda} = \frac{1}{N}\sum_{i=1}^{N}\lambda_i$.

iii). We re-estimate $\mu_i, \theta_i, \sigma_i^2$ for one financial time series by one by Maximum Likelihood Method with $\hat{\lambda}$ fixed.

iv). To estimate the elements in $\Sigma^{1/2}$, is proposed the minimization of the Frobenius norm between covariance matrix and theoretical under the constrains $\sigma_i^2 = \sum_{h=1}^{i} a_{ih}^2$.

4 Data

4.1 Data source and descriptions

The focus is to construct a portfolio on the basis of tree hedge fund strategies: Equity Hedge (EH), Event Driven (ED) and Macro. The time series used in this study are the Hedge Fund Research (HFRX) indices prices extracted from Datastream[5]. The indices are measured in US dollars, from January 2009 to December 2011. Despite hedge fund strategies are not properly tradable assets, the Hedge fund series can replicate the risk borne of taking positions according to any of those individual strategies (See appendix B).

The reasons behind the choice to select hedge fund as assets susceptible to be traded are fundamentally three: i) the main motivation of the asset allocation problem is to build optimal portfolios with a large amount of assets. Nonetheless, as we have seen in section 2 the task of constructing matrices M_3 and M_4 growths exponentially as we include a new single asset which means a great computationally cost at the optimization process if we are dealing with a large based equity portfolio. Considering this, it is not ideal to have a portfolio based on merely three equities. ii) Given the previous, using hedge fund strategies the asset manager or investor can develop a wide range of buys and sells in any kind of security, to execute the strategy and she would not be restricted to a barely three assets. iii) Hedge fund series of returns are said not to be normally distributed (Stefanini, 2006)

[5] Other possible hedge fund index providers are: C/S Tremont, S&P Hedge index, MSCI/Lyxor, Hennessee, Van Hedge, Eurekahedge, Altvest/Investorforce and FRM. It is important to mention that the any hedge fund index databases have two biases: Self selection bias and Survivorship bias. The first is due to the fact that only best funds tend to report data, while the second is caused by the exclusion from the databases of funds that have disappeared over timer because their negative performance. These elements make databases from diverse providers to be incomparable among them.

which make them a priori ideal for our porpoise to observe the investor's welfare change of taking into account higher moments instead of the mean-variance analysis.

The data set is described in Table 1, the study uses daily returns computed as:

$$R_{it} = \ln(P_t) - \ln(P_{t-1})$$

Where R_{it} denote the log return of the asset i at date t.

Table 1. Description of data set

Name of Variable	Period	Frequency	No. Observations
Equity Hedge strategy index (EH) Event Driven strategy index (ED) Macro strategy index (MACRO)	From 01/01/2009 to 31/12/2011	Daily	758

The HFRX indices are constructed by using quantitative techniques and analysis, multi-level screening, cluster analysis. Monte-Carlo simulation and optimization techniques ensure that each Index is a representation of its corresponding investment focus, reflecting the associated risk implicit in each strategy[6].

In Table 2, we report some descriptive statistic features for each time series of log-returns on the whole period to considerate. We start estimating its first four moments and testing the null hypothesis of normality in the univariate case, being the last especially important in this study, we compute the Jarque-Bera test and the Kolmogorov Smirnov tests.

We can observe that the three time series are bearish, in line with the markets in the proper period. The three hedge fund strategies returns are negatively skewed and shows great excess of kurtosis, which is consistent with the general characteristics of hedge fund strategies returns.

[6] Source: Datastream.

The correlation coefficient estimation for the relation EH-ED strategies, in comparison with the others relations is high, which means the two series of returns are highly linear correlated. The relations EH-Macro and ED-Macro exhibit low linear correlation.

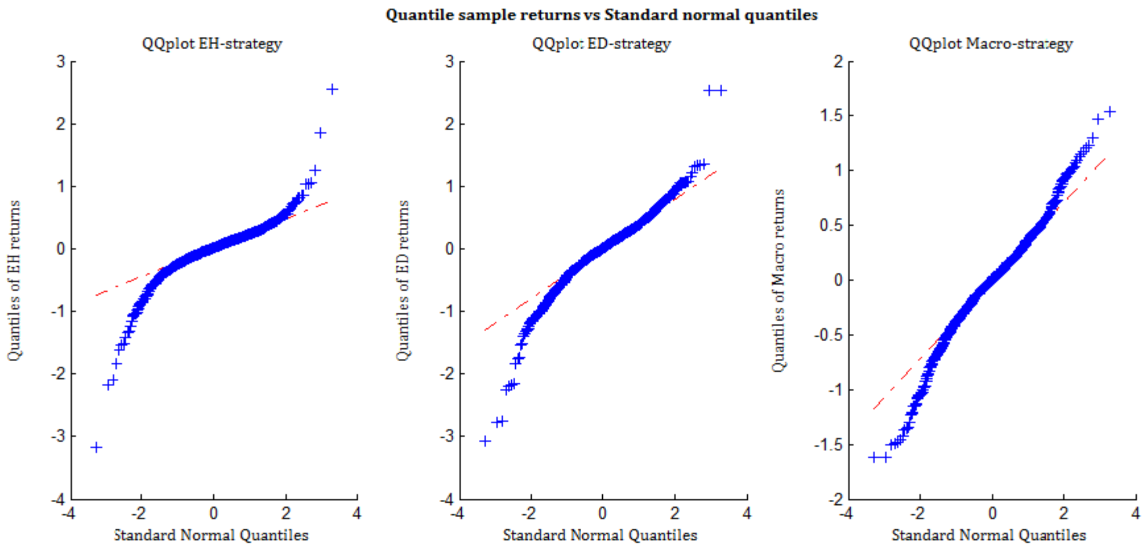
Looking at the Jarque-Bera and Kolgomorov-Smirnov tests we reject the null hypothesis of Normality of any single time series returns, we can assert they are clearly non-Gaussian. This result is important since we look to work with non-normal returns. In order to illustrate this graphically, figure 1 shows the QQplots of the sample returns for all strategies.

Table 2. Descriptive statistics

Return(%)	EH	ED	Macro
Mean	-1.2599	-2.9385	-1.0381
Std. Dev	0.3566	0.5159	0.4301
Skewness	-1.3013	-0.7685	-0.3230
Kurtosis	16.5605	7.6766	4.4375
Normality			
Jarque-Bera	8015.66	1018.803	104.423
p-val	0.001	0.001	0.001
Kol – Smirnov	0.2903	0.2081	0.2231
p-val	0.000	0.000	0.000
Correlation			
EH	1.000		
ED	0.7356	1.000	
Macro	-0.0133	0.0435	1.000

Notes: Jarque-Bera and Kolmogorov-Smirnov test are computed using Matlab specific function, as well as p-values.

Figure 1. Returns Nomatility



5 Empirical Analysis

In this section we report the empirical analysis based on a portfolio composed of three different hedge fund strategies: Equity Hedge (EH), Event Driven (ED), and Macro.

In order to minimize the F.O.C (equation 8), we need to estimate μ, M_2, M_3, and M_4 elements. Therefore, we need to estimate the moments (mean, variance, skewness and kurtosis) and the co-moments (covariance, co-skewness and co-kurtosis) necessaries to build such matrices.

5.1 Sample Approach

The easiest way and the most intuitive, consists in using the sample estimators, which are constructed as follows.

The sample mean of hedge strategy i is given by:

$$\widehat{\mu_i} = \frac{1}{T}\sum_{t=1}^{T} X_{i,t}$$

The sample covariance between strategy i and strategy j is given by:

$$\widehat{\sigma_{ij}^2} = \frac{1}{(T-1)} \sum_{t=1}^{T} (R_{it} - \widehat{\mu_i})(R_{jt} - \widehat{\mu_j})$$

The sample co-skewness between strategies i, j, k is given by:

$$\widehat{s_{ijk}} = \frac{T}{(T-1)(T-2)} \sum_{t=1}^{T} (R_{it} - \widehat{\mu_i})(R_{jt} - \widehat{\mu_j})(R_{kt} - \widehat{\mu_k}) \tag{13}$$

The sample co-kurtosis between strategies i, j, k, l is given by:

$$\widehat{k_{ijkl}} = \frac{T}{(T-2)(T-3)} \sum_{t=1}^{T} (R_{it} - \widehat{\mu_i})(R_{jt} - \widehat{\mu_j})(R_{kt} - \widehat{\mu_k})(R_{lt} - \widehat{\mu_l})$$

where R_{it} is the log-return of hedge fund strategy i in time t, while T is the length of the sample. The table below report the moments and co-moments resulting from using this approach in our sample.

Table 3. Multivariate Statistics. Sample Approach.

Covariance matrix	stat.	stat.	stat.	Co-skewness matrix	stat.	stat.	stat.	stat.
	x_1	x_2	x_3		x_1^2	x_2^2	x_3^2	x_1x_2
x_1	0.057	0.066	0.004		-0.012	-0.035	-0.004	
x_2	0.066	0.192	0.024		-0.021	-0.049	-0.008	
x_3	0.004	0.024	0.130		-0.002	-0.010	-0.014	-0.00007
Co-kurtosis matrix								
	x_1^3	x_2^3	x_3^3	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$	
x_1	0.025	0.032	0.004	0.00002	0.00006			
x_2	0.095	0.222	0.027		0.00003	0.00002		
x_3	0.003	0.012	0.076	0.00002			0.0001	

Note: x_1 represents EH strategy, x_2 represents ED strategy, $x_3$represents Macro strategy.

5.2 Single Factor Approach

For the improved estimators approach we use the Sharpe's single–factor methodology introduced for higher moments by Martellini and Ziemman (2009). It establishes a single-factor for any asset return:

$$R_{it} = c + \beta_i F_t + \varepsilon_{\mathrm{it}} \tag{14}$$

We have a lineal model, where F_t is the value of the market factor represented by a stock market index in time t, ε_{it} captures the idiosyncratic error term of each asset, while β_i is the regression coefficient. We assume the regression residuals are homoscedastic and cross-sectionally uncorrelated, then $\varepsilon \sim (0, \Psi)$[7]. The last two assumptions lead to be zero all off-diagonal elements in the matrix Ψ, while the diagonal elements represent the idiosyncratic risk of the assets. We take as single factor the S&P500 index.

The variance-covariance matrix with size (3,3) for the three strategies is given by (following the Martellini and Ziemman notation):

$$M_2 = (\beta\beta')\mu_0^{(2)} + \Psi$$

β is extracted from the previous regression, whereas $\mu_0^{(2)}$ is the second centered moment (variance) of the single-factor used.

Replacing equation (14) in equations (5) we get

$$M_3 = E[(\beta\bar{F} + \varepsilon)(\beta\bar{F} + \varepsilon)' \otimes (\beta\bar{F} + \varepsilon)'] = \{s_{ijk}\}$$

$$M_4 = E[(\beta\bar{F} + \varepsilon)(\beta\bar{F} + \varepsilon)' \otimes (\beta\bar{F} + \varepsilon)' \otimes (\beta\bar{F} + \varepsilon)'] = \{k_{ijkl}\} \tag{15}$$

Term $\bar{F}$ is the centered market returns $\bar{F} = F - \mu_0$.

From equation (15) result that M_3 and M_4 tensors are given by:

[7] This assumption is stronger than the no-correlation assumption.

$$M_3 = (\beta\beta' \otimes \beta')\mu_0^{(3)} + \phi$$

$$M_4 = (\beta\beta' \otimes \beta' \otimes \beta')\mu_0^{(4)} + \Upsilon \tag{16}$$

Estimating regression model with last-squares technique we obtain a factor return process independent of the residual return process, in addition according with the argument of factor model approach, we suppose all cross-sectional residuals ε_i and ε_j $(i \neq j)$ are independent.

The composition of the (3,3) covariance matrix of residual returns Ψ, will be defined by

$\psi_{ii} = E(\varepsilon_i^2)$, estimated as $\frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_i^2$

$\psi_{ii} = 0$

The composition of the (3,9) co-skewness matrix of residual returns ϕ, will be defined by

$\phi_{iii} = E(\varepsilon_i^3)$, estimated as $\frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_i^3$

$\phi_{iij} = 0,$

$\phi_{ijk} = 0, \quad \forall i \neq j \neq k$

In the superdiagonals of this matrix, can be found the idiosyncratic skewnesses proxied by the third order moment of the residuals, while all off-superdiagonal elements are zero due to the assumption of independence.

The composition of the (3,27) co-kurtosis matrix of residual returns Υ will be defined by

$\upsilon_{iiii} = E(\varepsilon_i^4)$, estimated as $\frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_i^4$

$\upsilon_{iiij} = 3\beta_i\beta_j\mu_0^{(2)}\psi_{ii},$

$\upsilon_{iijj} = \beta_i^2\mu_0^{(2)}\psi_{jj} + \beta_j^2\mu_0^{(2)}\psi_{ii} + \psi_{ii}\psi_{jj},$

$$\upsilon_{iijk} = \beta_j \beta_k \mu_0^{(2)} \psi_{ii},$$

$$\upsilon_{ijkl} = 0 \qquad \forall i \neq j \neq k \neq l$$

In the superdiagonals of this matrix can be found the idiosyncratic kurtosis proxied by the forth order moment of the residuals, elements in all off-superdiagonal come from the equation (15). For the sake of illustration the case υ_{iiij} is obtained as follows

$$\begin{aligned} M_4^{iiij} &= E\left[(\beta_i \bar{F} + \varepsilon_i)^3 \left(\beta_j \bar{F} + \varepsilon_j\right)\right] \\ &= E\left[(\beta_i^3 \bar{F}^3 + 3\beta_i^2 \bar{F}^2 \varepsilon_i + 3\beta_i \bar{F} \varepsilon_i^2 + \varepsilon_i^3)\left(\beta_j \bar{F} + \varepsilon_j\right)\right] \\ &= \beta_i^3 \beta_j \mu_0^{(4)} + 3\beta_i \beta_j \mu_0^{(2)} \psi_{ii} \end{aligned}$$

Where the first term $\beta_i^3 \beta_j \mu_0^{(4)}$ is an element of the $(\beta\beta' \otimes \beta' \otimes \beta')\mu_0^{(4)}$ matrix in equation (16), this is why it does not make part in matrix ϒ.

The following table report the moments and co-moments resulting from using the single-factor approach in our three hedge fund strategies sample.

Table 4. Multivariate Statistics. Single Factor Approach.

Covariance matrix	stat.	stat.	stat.	Co-skewness matrix	stat.	stat.	stat.	stat.
	x_1	x_2	x_3		x_1^2	x_2^2	x_3^2	x_1x_2
x_1	0.057	0.042	-0.004		-0.00283	-0.00099	-0.00001	
x_2	0.042	0.192	-0.007		-0.00048	-0.00481	-0.00001	
x_3	-0.004	-0.007	0.130		0.00004	0.00017	-0.0152	0.00008
Co-kurtosis matrix								
	x_1^3	x_2^3	x_3^3	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$	
x_1	0.011	-0.002	-0.001	0.0182	0.0074			
x_2	0.009	-0.006	-0.003		0.0027	0.0227		
x_3	-0.001	0.023	0.074	-0.0020			-0.001	

Note: x_1 represents EH strategy, x_2 represents ED strategy, $x_3$represents Macro strategy

Finally, to estimate the parameters for the multivariate VG model needed to construct tensors M_2, M_3 and M_4 we follow Hitaj and Mercuri (2011) procedure described in section 3.2.2. We deviate from their steps in the estimation method since they use Maximum Likelihood and we use Method of Moments[8]. Then, the estimation procedure goes like:

1). we estimate the parameters $\mu_i, \theta_i, \sigma^2$and λ_i for each financial time series by the Method of moments.

As is usually known, the idea of this method is to match theoretical moments with the sample moments. The theoretical moments in this model are (Loregian et al, 2011):

$$E(x) = \mu + \theta\lambda$$

$$Var(x) = \lambda(\theta^2 + \sigma^2)$$

$$Skew(x) = \frac{(2\theta^2 + 3\sigma^2)\theta}{\sqrt{\lambda}\sqrt{(\theta^2 + \sigma^2)^3}} \tag{17}$$

$$k(x) = \frac{(2\theta^2 + 3\sigma^2)\theta}{\sqrt{\lambda}\sqrt{(\theta^2 + \sigma^2)^3}}$$

The next table shows the result of the previous step using fsolve in Matlab to find the zeros of the functions.

Table 5. Estimated parameters for VG model

Strategy	$\hat{\mu}$	$\hat{\theta}$	$\hat{\sigma}^2$	$\hat{\lambda}$
EH	0.0753	-0.0801	0.3174	0.7035
ED	0.0856	-0.0877	0.4195	1.0886
Macro	0.0542	-0.0362	0.2450	2.1314

[8] We have also estimated parameters μ, θ, σ^2 and λ for each time series using Maximum likelihood. However, the results vary greatly as we modify the initial conditions of the parameters; the previous is consistent with the findings of Loregian *et al* (2011). Contrary, the results with the Method of Moments converge independent to the initial conditions. With both of them ML and MoM is observed a high number of iterations to stop the algorithm, which makes these procedures computationally expensive working with a large quantity of assets.

2). After estimating, we take a common shape parameter λ by computing $\hat{\lambda} = \frac{1}{N}\sum_{i=1}^{N}\lambda_i$.

$$\hat{\lambda} = \frac{1}{3}\sum_{i=1}^{N}\lambda_{EH} + \lambda_{ED} + \lambda_{Macro} = 1.308$$

3). We re-estimate $\mu_i, \theta_i, \sigma_i^2$ for each hedge funds strategies returns series by Method of Moments with $\hat{\lambda}$ fixed.

Having a unique parameter $\hat{\lambda}$ for the three series returns, means we already have a common gamma mixing density. Then, with equations (17) we re-estimate the remaining parameters $\mu_i, \theta_i, \sigma_i^2$ by finding the zeros of each function. Using fsolve in Matlab we obtain the following parameters.

Table 6. Estimated parameters for VG model

Strategy	$\hat{\mu}$	$\hat{\theta}$	$\hat{\sigma}^2$
EH	0.1234	-0.0743	0.2268
ED	0.1110	-0.0859	0.3848
Macro	0.0256	-0.0369	0.3150

4). To estimate the elements in $\Sigma^{1/2}$, we use the minimization of the Frobenius norm between sample covariance matrix and theoretical variance under the constrains $\hat{\sigma}_i^2 = \sum_{h=1}^{i} a_{ih}^2$.

Following the first three steps we get the theoretical variance for each asset return, now we require to estimate the elements of matrix $\Sigma^{1/2}$ to assure we have a multivariate variance gamma model. In order to obtain $\Sigma^{1/2}$ we use Cholesky inferior decomposition of sample covariance matrix and theoretical variances $\hat{\sigma}_i^2$, and then we minimize the norm of frobenius constrained to:

$$\hat{\sigma}_1^2 = a_{11}^2$$

$$\hat{\sigma}_2^2 = a_{21}^2 + a_{22}^2$$

$$\hat{\sigma}_3^2 = a_{31}^2 + a_{32}^2 + a_{33}^2$$

The function fmincon in Matlab allows us to minimize this norm using no linear constraints. The following table shows the results of this step for matrix $\hat{\Sigma}^{1/2}$.

Table 7. Cholesky inferior matrix

Cholesky	a_1	a_2	a_3
a_1	0.476191	0.000000	0.000000
a_2	0.000042	0.620359	0.000000
a_3	0.000044	0.000044	0.561269

We have already obtained the estimators $\hat{\mu}_i, \hat{\theta}_i, \hat{\lambda}$ and the matrix $\hat{\Sigma}^{1/2}$. We can proceed to compute the moment and co-moments in order to get M_2, M_3 and M_4 tensors for the expected utility maximization problem. For this, we take equations (12) to compute moments and co-moments for any tensor. The table below shows such relations.

Table 8. Multivariate Statistics. Variance Gamma Model (VGM)

Covariance matrix	stat.	stat.	stat.	Co-skewness matrix	stat.	stat.	stat.	stat.
	x_1	x_2	x_3		x_1^2	x_2^2	x_3^2	x_1x_2
x_1	0.305	0.009	0.004		-0.067	-0.039	-0.031	
x_2	0.009	0.513	0.004		-0.027	-0.131	-0.036	
x_3	0.004	0.004	0.414		-0.011	-0.019	-0.046	-0.0006
Co-kurtosis matrix								
	x_1^3	x_2^3	x_3^3	$x_1x_2^2$	$x_1x_3^2$	$x_2x_3^2$	$x_1^2x_2$	
x_1	0.499	0.036	0.012	0.019	0.010			
x_2	0.021	1.419	0.013		0.003	0.013		
x_3	0.009	0.016	0.909	0.005			0.010	

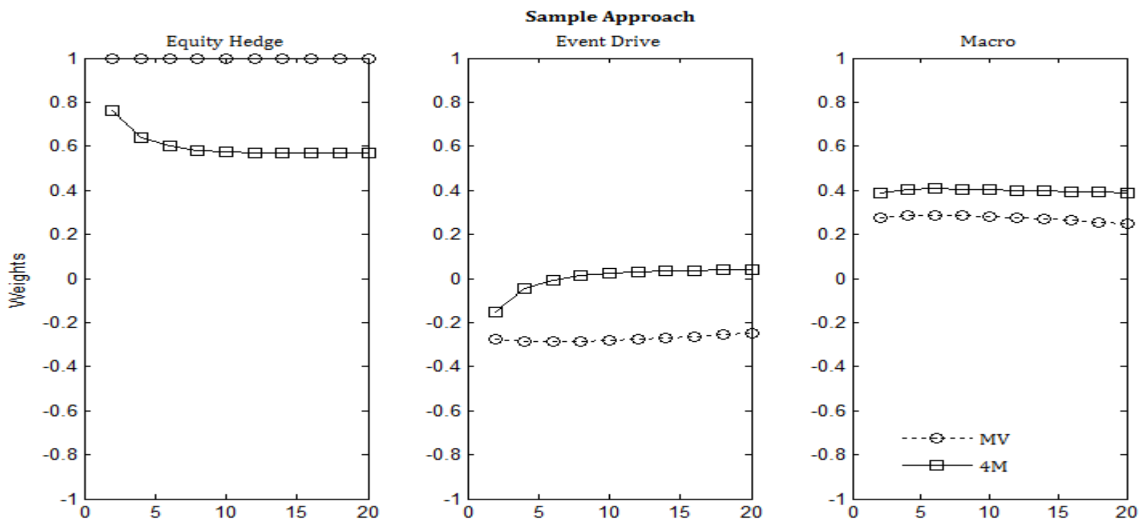
Note: x_1 represents EH strategy. x_2 represents ED strategy. $x_3$represents Macro strategy.

Having the moments and co-moments estimated from non parametric and parametric approaches, we can solve the equation (8) through the function fmincon in Matlab. Since Hedge funds do not have restrictions to short selling, the bounds for the weights in each asset will be between -1 and 1, meaning that short selling are not restricted in our models.

We will construct optimal portfolios for different levels of risk aversion in the CRRA investor's utility function, varying $\lambda = [1, 2, \ldots, 20]$. In addition we get optimal weights using two moments (mean-variance MV) and four moments (4M).

The next figure shows the optimal weights for the three Hedge fund strategies portfolio using the complete sample, daily log-returns from 01/01/2009 to 31/12/2011 and computing tensors M_2, M_3 and M_4 with the sample approach.

Figure 2. Optimal Weights. Sample Approach with MV and 4M.



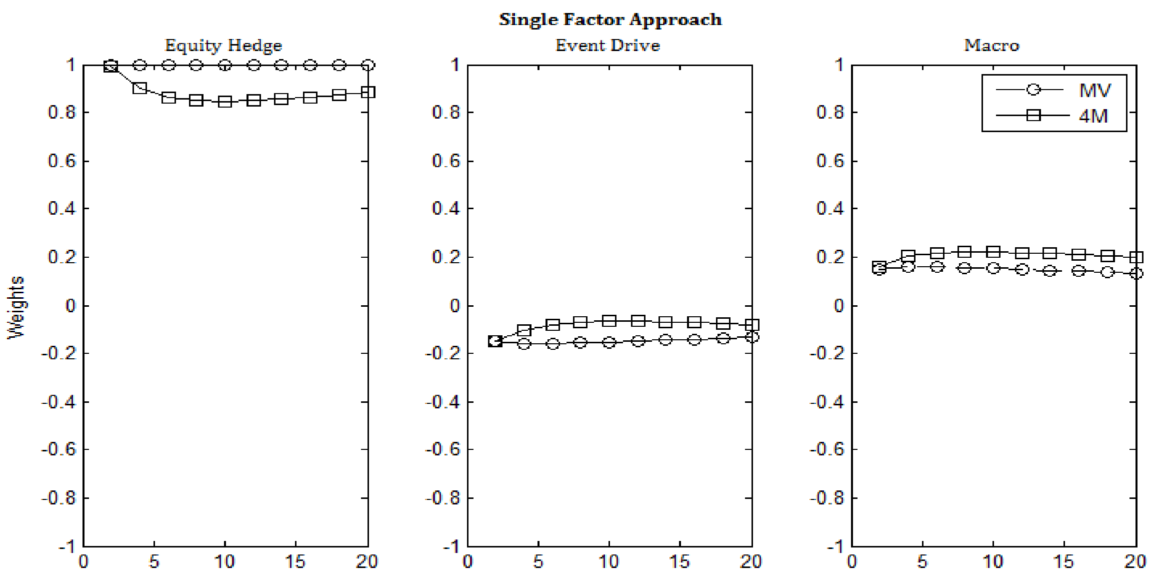
This figure shows optimal weights for the Equity Hedge- Event driven- Macro hedge funds strategies portfolio with different degrees of risk aversion. Here MV refers for mean-variance model and 4M for four moment optimal weights. Short sales are permitted.

In the sample approach we can see extreme values for EH in mean-variance (MV), the difference in the allocation for diverse levels of risk λ is subtle in Macro strategy in both MV and four moments (4M), unlike EH weights in 4M where is reduced as λ rises, the contrary happens for ED. The disparity of weights in MV and 4M is noticeable for any level of risk aversion being greater in EH and ED. The EH appears to have an important role in diversification and dominates the EH-ED-Macro portfolio in MV, however, the influence of

EH in 4M portfolios is diminished due to its returns being more negatively skewed and leptokurtic than ED and Macro. The graph depicts ED strategy would be ‘shorted’ in both MV and 4M which is of central importance in this case due to the predominantly ‘bearish’ period taken and it.

The optimal weights for the three Hedge fund strategies portfolio using the complete sample, daily log-returns from 01/01/2009 to 31/12/2011 and computing tensors M_2, M_3 and M_4 with the single-factor approach is showed in the following figure.

Figure 3. Optimal Weights. Single Factor Approach with MV and 4M.

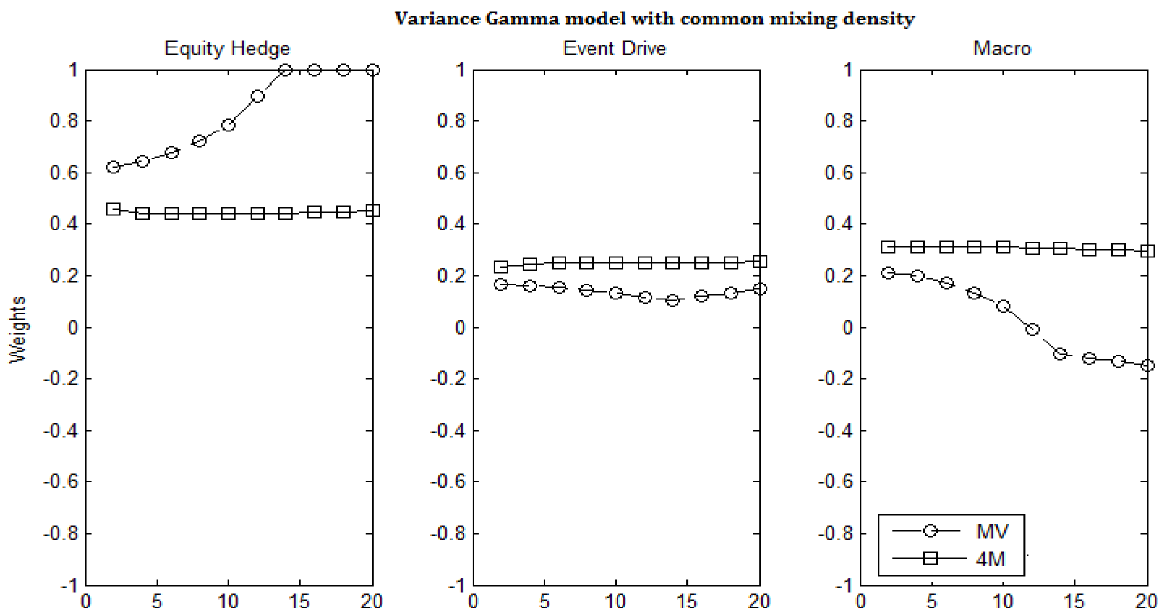


This figure shows optimal weights for the Equity hedge - Event driven - Macro hedge fund strategies portfolio with different degrees of risk aversion. Here MV refers to mean-variance and 4M for four moment optimal weights. Short sales are permitted

In the graphic above we see some similar results with the sample approach. In general, the differences in the portfolios’ weights between MV and 4M are not as large as the previous case. Nonetheless, 4M portfolios appear to response more as the level of risk aversion moves. Alike the previous case the EH appears to have an important role in diversification and dominates the EH-ED-Macro portfolio in MV, as well as 4M. Again the role of ED is crucial in the diversification since it would be necessary to take short positions in this asset. Interaction between ED and Macro is visible in single-approach, almost the same weights are allocated but in contrary position, this is due to the extreme weights allocated in EH.

The optimal weights for the three hedge fund strategies portfolio using the complete sample, daily log-returns from 01/01/2009 to 31/12/2011 and computing tensors M_2, M_3 and M_4 with the multivariate Variance Gamma model approach is showed in the next figure.

Figure 4. Optimal Weights. Variance Gamma Model with MV and 4M.



This figure shows optimal weights for the Equity hedge- Event driven - Macro hedge fund strategies portfolio with different degrees of risk aversion. Here MV refers to mean-variance and 4M for four moment optimal weights. Short sales are permitted.

The VG approach shows dissimilar results in comparison with the two cases analyzed before. In this case no assets would be 'shorted' in 4M, and none of them dominates the portfolio. The 4M portfolios do not exhibit change with the level of risk aversion λ whereas MV portfolios are more sensible to the level of risk aversion. The interaction in VG optimal weights between EH and Macro is visible in MV due to the few relevance ED takes, being near zero the weight placed in this asset.

Here, we perform an out-of-sample analysis in order to examine the possible gain that this investor would have considering the first four moments (4M) instead of just the first two

(MV). We implement a buy and hold strategy, in which, we use the information of one month to determine the optimal portfolio. Then we hold the optimal weights during the next month computing at the end of the month the out-of sample returns for each approach (sample approach, single factor approach, MVG model approach) in both cases 4M and MV.

Table 9 shows the out-of-sample statistics results of each approach considering MV and 4M with different degrees of aversion λ of the utility function.

Table 9. Out-of-sample statistics

lambda	lambda = 1				lambda = 10			
	annual mean %	annual std %	skewness	kurtosis	annual mean %	annual std %	skewness	kurtosis
sample MV	6.1533	21.8225	-0.0579	3.6117	4.9460	20.2195	-0.2212	3.9803
single f MV	6.9390	21.1034	-0.5021	5.6169	4.9911	20.4861	-0.4523	6.0806
VGM MV	4.4561	19.4643	-0.5946	3.3053	4.4860	19.9837	-0.4072	4.4213
sample 4M	7.5889	23.8772	-0.4358	4.5459	6.8874	19.1256	-0.2772	3.6380
single f 4M	7.8237	24.7027	-0.3379	5.2172	6.7124	20.8691	-0.0882	4.9321
VGM 4M	5.0924	17.9549	-0.5788	3.0924	3.7550	17.8428	-0.5653	2.8059
lambda	lambda = 15				lambda = 20			
	annual mean %	annual std %	skewness	kurtosis	annual mean %	annual std %	skewness	kurtosis
sample MV	2.7801	18.9880	-0.0254	4.4651	4.4651	21.8612	-0.0250	3.9224
Single f MV	4.3692	18.9836	-0.3777	5.0496	4.1442	17.5959	-0.1479	4.5556
VGM MV	4.2445	18.6962	-0.3540	4.6442	4.1893	19.5782	-0.0155	5.9907
sample 4M	6.7599	19.5033	-0.4386	4.2939	5.8837	19.0885	-0.3717	3.8110
single f 4M	6.9382	19.7685	-0.0820	5.9308	6.2128	19.1929	-0.0686	5.6470
VGM 4M	3.2200	17.2976	-0.5604	2.0463	3.1026	23.4849	-0.1118	2.2926

As is noticed above, the investor obtains higher returns with 4M than MV in the sample and single approach, while in the VGM approach the results are mixed, so we can not assert the same. In general, the annual returns fall, the variance becomes shorter and the skewness gets less negative as lambda increases. For the kurtosis results, there is no a clear tendency in any model. These results are consistent with the theory, where more risk averse investors prefer to get shorter returns with lower variance, in addition the skewness is expected to be less negative.

6 Conclusions

In this study, we have showed the asset allocation problem starting from the investor's expected utility maximization problem, considering a CRRA utility function, using a Taylor series expansion until the fourth moment to approximate the expected utility function. Then, we have focused in present the co-skewness and co-kurtosis matrix tensors M_3 and M_4, estimating the elements in those matrices by three methods: sample estimators, single factor estimators (Martellini and Ziemman, 2010), and VG distribution by Hitaj and Mercuri (2011). Using three hedge fund strategies we find the optimal weights under the methods mentioned in MV and 4M analysis.

We have found that optimal portfolios vary greatly from MV to 4M under all the approaches, noticed that the EH strategy is preferred in the MV analysis, but considering higher moments (4M) this strategy is less demanded by the investor, this can be explained by its larger kurtosis and more negative skewness in comparison with the other strategies.

The optimal portfolios obtained under the three methodologies, seem to be very similar to changes in the risk aversion degree, being the parametric VG model the furthest in the results.

In this paper we have used a VG model with a common mixed density parameter, implementing an univariate estimation and then we move to the multivariate model. A possible extension to this paper, is to use directly the multivariate generalized hyperbolic density function (10) estimating by maximum likelihood method with the algorithms of the EM (Expectation – Maximization) type[9].

The estimation procedure for VG model is computationally more expensive than the non-parametric models, which means that allocating a large number of assets will require more hardware capacity.

[9] See McNeil et al (2005). Quantitative Risk Management: Concepts, techniques and tools. Princenton University Press.

As the out-of-sample results show, the methodologies perform similar for the four moments and have parallel tendencies to changes in lambda, making complicated to choose a model that performs better than the others.

Appendix

A. Generalized Inverse Gaussian

The random variable X is a generalized inverse Gaussian (GIG) distributed, $X \sim N^{-}(\lambda, \chi, \psi)$, if its density is

$$f(x) = \frac{\chi^{-\lambda}\left(\sqrt{\chi\psi}\right)^{\lambda}}{2K_{\lambda}(\sqrt{\chi\psi})} x^{\lambda-1} \exp\left(-\frac{1}{2}(\chi x^{-1} + \psi x)\right), \qquad x > 0,$$

K_{λ} is a modified Bessel function of the third kind with index λ. The parameters satisfy $\chi > 0, \psi \geq 0$ if $\lambda < 0$; $\chi > 0, \psi > 0$ if $\lambda = 0$; $\chi \geq 0, \psi > 0$ if $\lambda > 0$.

The GIG contains the gamma and inverse gamma densities as limiting cases, corresponding to $\chi = 0$ and $\psi = 0$, respectively. In this cases the density must be interpreted and a limit, which can be evaluated using the asymptotic relations $K_{\lambda}(x) \sim \Gamma(\lambda)2^{\lambda-1}x^{-\lambda}$ as $x \to 0+$ for $\lambda > 0$ and $K_{\lambda}(x) \sim \Gamma(-\lambda)2^{-\lambda-1}x^{\lambda}$ as $x \to 0+$ for $\lambda < 0$.

B. Hedge Fund Strategies

Next we make a brief review of the three strategies, equity hedge, event driven and global macro, mention their main features, the management style, time horizon, the assets they trade, etc.

Equity Hedge strategy

Also known as long/short equity, equity hedge strategy finds its root in the original idea of hedge fund. It consists mainly in combining long and short positions in equities, as a result are obtained portfolios that have reduced market risk. Managers look for shares that they think have undervaluing prices, and shares they believe are being overvalued, taking a long position in the first and shorting the second. The profit would come when long positions go up and short positions go down, if the contrary happens they would suffer losses. The goal of any equity hedge strategy is to minimize exposure to the market in general and taking advantage of the spread between two stocks, this is why equity hedge is referred as 'double alpha' strategy. The short portfolio serves for both provide a negative exposure to securities which are supposed to be overvalued and reducing the market exposure by hedging the systematic risk.

There exist a wide variety of equity hedge strategies, including market neutral where the core idea in this strategy is that long position outperforms the short position on a relative basis, if both stocks fall in price the portfolio is still profitable since long position declines less than short position, in addition managers can maintain directional strategies generally positive market exposure called long bias strategy, which seeks to make profits of generally bullish periods.

Equity hedge manager may display further approaches in the investment process, they can used a bottom-up or top down analysis, also can be distinguished by the geographical market, the sector or their investment style. Most equity hedge managers apply fundamental techniques or quantitative analysis, employed by traditional equity managers with the difference that equity hedge managers can make profits even in declining markets. Furthermore, they may follow a management style with an intense short-term oriented trading or a more long-term oriented investment horizon, also can be characterized according to the market capitalization they invest (large cap or small cap), the kind of approach to the reference market's prevailing trend in followers (momentum) or forerunners of trend reversal (counter trend).

Event Driven strategy

Event driven or also called "special situations" is an investment strategy that seeks to exploit pricing inefficiencies that are likely to occur before or after wide variety of corporate events including but not limited to merger, acquisition, bankruptcy, spinoffs, restructurings, liquidations, financial distress, tender offers, shareholder buy-backs, recapitalizations, hostile takeover-bias or other capital structure adjustments. Depending on the opportunities available on the market, managers allocate their capital across the different sub-strategies, with this porpoise they make researches on the operating and financial profiles of companies. Because of its concern with micro events this strategy follows a bottom up analysis. Nevertheless, anticipating events is not the core of this investment strategy but the ability of manage them, the success will be in trying to predict the outcome of a given deal along with indentify the best moment to allocating capital in the investment. Given the nature of the strategy the performance does not depend on market direction, but the bearish equity markets may make fail the deal or at least be redefined, and bullish markets represents investment opportunities for even driven strategy.

Even driven includes also other strategies such as activist, distressed securities, merger arbitrage, credit arbitrage, regulation D among others. It is usually used by Distressed securities managers, since distressed securities are often corporate bonds, bank debt and trade claims of companies in some kind of distress, which makes both strategies complementaries. Event driven is likely to work best when the state of the economy is performing well while distressed restructuring works best if the contrary happens.

Risk exposure of event driven strategy results from a combination of sensitivities to equity markets, credit markets and idiosyncratic company specific factors.

Global Macro Strategy

The Global Macro strategy is the broadest implemented strategy among managers, being able of allocating capital on almost any market, region or sector using any financial instrument. The macro term comes from managers' to employ macroeconomic principles to distinguish distortions in asset prices. This strategy has the larger size in terms of assets under management in comparison with any other hedge fund strategy.

Global macro managers can execute a variety of strategies in which the investment process and its choices are taken considering principally a macro-economic analysis, identifying possible movements in economic variables and the impact that these have on the investment which makes Global macro a top-down strategy mainly. Some managers design trades based on their subjective opinion of the market conditions, this is called the discretionary approach, while others follow the systematic approach where are used quantitative tools and econometric models for forecasting those economic variables in order to find discrepancies between information provided by models and macro-economic variables like GDP, public deficit, public debt, interest rates, exchange rates, inflation, equity market returns, net exports, etc. Then these views are established in global markets through appropriate positions -long or short- in equities, bonds, commodities and currencies.

Global macro managers try to anticipate price changes i.e. on capital markets and execute directional positions, for this they have to identify whole the factors may affect those quotes like political events, macro-economic announcements and external factors, the trade is usually done in liquidity markets like commodity, currency or treasury markets, facilitating quick changes in positioning as new opportunities are identified. Besides, the use of liquid financial contracts, for instance index options, forwards and futures guarantees that minimal expenditure is afforded in the execution of the strategy. As outcome, the performance of this strategy would deeply depend on the quality and timing of the prediction.

Global macro trading strategies can be split in two categories, directional and relative. In the first they implement a long or short position as a stand alone strategy and try to take benefit from the direction of the movements on financial markets or assets prices, establishing directional positions that reflect their predictions. Directional strategies deviate from the original hedge fund philosophy since they do not provide hedge but they keep the original idea of the non existence of constrains as well as the non limitations of mutual funds. In the second category, they structure by pairing a long and a short position in similar assets to take advantage of a relative mispricing while maintaining neutral exposure to the broader asset class, for instance take a short position in corn and a long position in wheat. Nonetheless, Global macro strategy differs from Relative value strategy as the idea behind the first is the prediction of future movements whereas in Relative value is the discrepancy of valuation between securities.

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