EXTENDING THE NOWAK-ROMANIUK APPROACH FOR PRICING CATASTROPHE BONDS.

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Abstract

Classic insurance mechanisms have proven to be inadequate to deal with the losses caused by natural catastrophes, such as hurricanes, earthquakes, floods and so on. Therefore, new financial and insurance instruments are needed. In this framework, catastrophe bond arise as a useful tool to deal with catastrophe losses. The aim of this paper is to extend the Nowak & Romaniuk approach for cat bonds pricing to new models for the risk-free spot interest rate, as well as to implement different distribution for the losses and two types of payoff. First we prove a general pricing formula and then we apply it to price some particular cases combining the three factors mentioned above. Some studies of sensitivities are carried out. After that, we study which factors have a bigger influence on the price.

Acknowledgements

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Contents

1 Introduction

Nowadays, insurance companies are not well prepared to face off losses caused by natural catastrophes, such as hurricanes or earthquakes, losing huge amounts of money when they try to use classic insurance mechanisms with such events. As some examples, losses from Hurricane Katrina in 2005 are bigger than \$40 billions, and more than 60 insurance companies broke after Hurricane Andrew.

One of the main reasons that explains why classic insurance mechanisms are not able to cover natural catastrophes lies on the fact that these phenomena present a high dependence on location and period. The probability of occurrence of some natural catastrophe is not the same in every region and in every period of time. Additionally, we must take into account the fact that, classically, insurance companies are used to face off losses that are small compared to the whole insurance portfolio.

One possible way to deal with the huge losses caused by catastrophe events consists of converting the losses into tradable assets, also known as catastrophe derivatives, since financial markets all over the world move even bigger quantities of money every day.

In this framework, catastrophe bonds (*cat bonds* from now on) have arisen as the most confident instrument to deal with losses from natural catastrophe events and they are starting to be used more and more, since they work well and their mechanism is very simple to understand.

When someone acquires a cat bond, it comes along with some predetermined triggering point, such as the occurrence of certain natural catastrophe in some region and time predefined. If no event crossing that triggering point happens, the bondholder is paid the face value, but, in case that such an event occurs and the triggering point is reached, the bondholder would be paid the face value minus a certain quantity, quantity that would be used by the insurance company to face off the losses of his clients due to the natural catastrophe.

Being an instrument quite recent, techniques for cat bond pricing are still needing to be developed and improved, because the methods implemented are very simplified ones. Only few approaches developed include stochastic processes with continuous time, and amongst them is the paper of *Vaugirard (2003)* [8], which is the benchmark of the work done by *Nowak* & *Romaniuk, (2013)* [7], work that we extend on this paper.

Vaugirard was one of the first authors to apply non-arbitrage conditions and solved the problem of market incompleteness using the Merton method. Nowak & Romaniuk used this method to price cat bonds through 3 different models for the risk-free spot interest rate, *Hull-White (1993)*, *Vasicek* and *CIR*, simulating losses following two different distributions, *lognormal* and *Weibull* and also taking into account 2 different types of payoff: *stepwise* and *piecewise linear* payoffs.

As we mentioned before, in this paper we will extend *Nowak* & *Romaniuk* contributions, computing prices of cat bonds, using the combinations of factors that they did not use in their paper, and also adding some other models for the risk-free spot interest rate, such as *Black-Derman-Toy (1990),* and *Mercurio-Moraleda (2000)*, and considering different distributions for the losses, like *Gamma* and *GPD* distributions.

We will assume the following set of hypothesis:

1. Arbitrage is not possible.
2. Market behavior is completely independent from the occurrence of catastrophe events.
3. Interest rate changes can be replicated by other financial instruments.

The paper is organized as follows. Section 2 introduces a general formula to price cat bonds. Section 3 explains the different cases considered for the risk-free spot interest rate and the two types of payoff implemented. In Section 4, we simulate the price of the cat bonds using the formulas calculated in Section 3 from each model, and in the corresponding subsections, we study the sensitivity of each of the parameters involved and which of the choices for the possible factors has a bigger impact on the price. Section 5 presents the conclusions obtained from the work previously done.

2 General Formula to Price a Cat Bond

In this Section we introduce a general pricing formula for catastrophe bonds, following the approach used by Nowak & Romaniuk.

As mentioned in the previous section, our paper follows the method described by Vaugirard, so we need to apply stochastic models with continuous time, between [0,T']. According to that, some stochastic processes and random variables describing the dynamics of the spot interest rate and aggregated losses need to be defined, as we see below.

One of the most important issues we have to deal with is the choice of the distribution for the losses. Let $(U_i)_{i=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables, where U_i represents the losses caused by the i-th catastrophe event. Since the losses are always a positive quantity, it would be pointless to choose any distribution with both positive and negative values, and we must only take into account those distributions bounded for positive values. Nowak & Romaniuk considered Lognormal and Weibull distributions. Gamma and Generalized Pareto Distribution are added to our results.

We will be able then to simulate one single loss according to the chosen distribution, but we still have a problem to face off: the number of catastrophe events that will take place between [0,T'] remains unknown. To deal with that, we also need to introduce $\widetilde{N}_t = \sum_{i=1}^{N_t} U_i$, $t \in [0, T']$, where N_t is an homogeneous Poisson process (HPP) with $\lambda > 0$ that represents the number of catastrophe events occurred until time t. Let us remember now the main properties of an HPP:

- $N_0 = 0$
- For s $\leq$ t, $N_s \leq N_t$
- $E^P[N_t] = \lambda t$
- $P(N_t - N_s = k) = \mathrm{e}^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$

That means that, between t=0 and t = t, we expect to have λ catastrophe events and the distribution of the occurrence of those events follows a Poisson law. $\widetilde{N}_t$ is then a compound Poisson process and it shows the aggregated losses until time t. It is important to mention that N_t is totally independent from U_i.

It is also needed to define the banking account, $(B_t)_{t\in[0,T']}$, and its dynamics, that follows the equation

$$dB_t = r_t B_t dt, B_0 = 1$$

where $r = (r_t)_{t\in[0,T']}$ is the risk-free spot interest rate.

In addition to the non-arbitrage condition and the possibility to replicate the changes in r, we must add another assumption: investors are neutral towards the nature jump risk. This will have important implications that will be seen later.

Not only catastrophe bonds can be found in this market, zero coupon bonds are also traded in it. B(t,T) is the price of a zero coupon bond at time t, with a maturity date T $\leq$T', following a geometric Brownian motion:

$$\frac{dB(t,T)}{B(t,T)} = \mu_t^T dt + \sigma_t^T dW_t$$

where W_t is a Brownian motion, $\mu^T = (\mu_t^T)_{t\in[0,T]}$ and $\sigma^T = (\sigma_t^T)_{t\in[0,T]}$ are the drift and the volatility of the bond price process, respectively. As we know, investors will need to receive a prime over the risk-free interest rate to compensate them for taking a riskier position. This prime, called market price of risk, is defined as follows:

$$\bar{\lambda}_t \equiv \frac{\mu_t^T - r_t}{\sigma_t^T}$$

This prime must be equal for all bonds, no matter their maturity time, if we are in a no-arbitrage market and we assume that it satisfies the Novikov condition:

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \bar{\lambda}_t^2 dt\right)\right] < \infty$$

This is a sufficient condition that guarantees that $\bar{\lambda}_t$ is a martingale and that enables us to apply the Girsanov's theorem to change from the real probability measure P to the neutral probability measure Q, defined by the Radon-Nikodym derivative as follows:

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \bar{\lambda}_t dW_t - \frac{1}{2}\int_0^T \bar{\lambda}_t^2 dt\right)$$

Once we have found the neutral probability measure Q, pricing is as simple as taking the conditional expectation under Q of the payoff multiplied by the discount factor. That is, for the zero coupon bond:

$$B(t,T) = E^Q \left[\exp \left(- \int_t^T r_u du \right) |F_t \right], t \in [0,T]$$

Since the payoff at time T is B(T,T) = 1, it is not included in the equation. And, for a catastrophe bond IB(t) with payoff $\nu_{IB_{cat}(T,F_v)}$, the equation is:

$$IB(t) = E^Q \left[\exp \left(- \int_t^T r_u du \right) \nu_{IB_{cat}(T,F_v)} |F_t \right], t \in [0,T]$$

Since we are assuming that the investors are neutral towards the risk, $\lambda = 0$ and both payoff and risk-free spot interest rate are independent so we can write

$$IB(0) = E^Q \left[\exp \left(- \int_0^T r_u du \right) \right] E^Q \left[\nu_{IB_{cat}(T,F_v)} |F_t \right], t \in [0,T]$$

This is our general pricing equation for any cat bond. Next section will focus in some particular cases, in which we consider the previously mentioned models for the risk-free spot interest rate, distribution of losses and types of payoff.

3 Particular Cases

3.1 Payoff

Once we have found a general formula to price catastrophe bonds, let us define the two different models considered for the payoff, the stepwise model and the piecewise linear model.

3.1.1 Stepwise payoff

First of all, let us define a sequence of constants:

$$0 < K_1 < K_2 < \cdots < K_n, \; n > 1$$

that will act as the different triggering points of our cat bond. Using those triggering points, we can define a sequence of stopping times τ_i as follows

$$\tau_i(\omega) = inf_{t \in [0,T']} \left\{ \widetilde{N}(t)(\omega) > K_i \right\} \wedge T', \; 1 \leq i \leq n$$

where $\omega_1 < \omega_2 < ... < \omega_n$ is a sequence of nonnegative constants acting as weight parameters, so they must be such that $\sum_{i=1}^n \omega_i \leq 1$.

So we have a sequence of stopping times, each one corresponding to the moment in which the aggregated losses surpass a certain threshold. These stopping times sequence has a cumulative distribution function that follows the next lemma

Lemma 3.1. *The value of the cdf Φ_i at the moment T is defined by*

$$\Phi_i(T) = 1 - \sum_{j=0}^{\infty} \frac{(\kappa T)^j}{j!} \mathrm{e}^{(-\kappa T)} \Phi_{\widetilde{U}_j}(K_i)$$

where $\widetilde{U}_j$ are the aggregated losses from 0 to j (different from $\widetilde{N}_t$, aggregated losses from 1 to N_t).

We can see how the factor that multiplies $\Phi_{\widetilde{U}_j}(K_i)$ is, actually, a Poisson distribution, due to the HPP defined previously.

As mentioned in the previous Section, the choice of the distribution of the losses is really important, cause its cdf might or not be analytically tractable.

Once we have our sequence of stopping times perfectly characterized, we can define the payoff of our bond, knowing that:

- If no catastrophe event occurs during the interval [0,T], the bondholder receives the face value :

$$\tau_1 > T \Rightarrow \nu_{IB_s(T,F_v)} = F_v$$

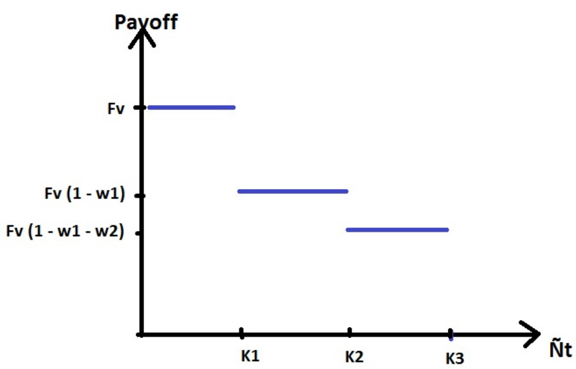
.

- If one (or many) catastrophe events occur between 0 and T, the bondholder will receive the face value minus a function of the cdf of the stopping times inside the interval [0,T], multiplied by the weight assigned to each stopping time, ω_i:

$$\tau_n > T \Rightarrow \nu_{IB_s(T,F_v)} = F_v \left[1 - \sum_{i=1}^{n} \omega_i \Phi_i\right]$$

.

So if catastrophe events are present, as the cumulative losses grow and surpass the thresholds K_i defined previously, new stopping times need to be considered and then the payoff starts to be reduced following the above formula. This can be easily seen in the following graph, that synthesizes the behavior of the payoff:



3.1.2 Piecewise linear payoff

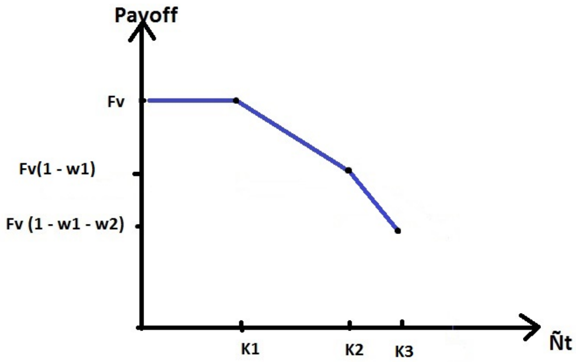
We now consider a bond with a piecewise linear payoff. The mechanism is very similar to the one applied above, but in this case, the quantity multiplying ω_i is not 0 or 1, but a changing value bounded between them, so that allows our payoff to be a continuous function, with no jumps.

For this payoff we also define some thresholds K_i, but starting from K_0 this time. We will need some weight factors as well: $0 \leq K_0 < K_1 < K_2 < \cdots < K_n$ and $\omega_1 < \omega_2 < \cdots < \omega_n$ with $\sum_{i=1}^{n} \omega_i \leq 1$.

In this case, the payoff follows the next equation:

$$\nu_{IB_p(T,F_v)} = F_v \left[1 - \sum_{j=0}^{n-1} \frac{\widetilde{N}_T \wedge K_{j+1} - \widetilde{N}_T \wedge K_j}{K_{j+1} - K_j} \omega_{j+1} \right]$$

The behavior of the payoff is showed in the next graph:



So the payoff is here a linear payoff, but the slope does not have to be necessarily constant, different slopes are allowed between different triggering points

This following lemma allows us to compute numerically the expected value under Q of the payoff we have just defined:

Lemma 3.2. *Let*

$$\begin{aligned} \varphi_m &= P(\widetilde{N}_T \leq K_m), m = 0, 1, 2, ...n \\ e_m &= E\left[\widetilde{N}_T 1_{K_m < \widetilde{N}_T \leq K_{m+1}}\right], m = 0, 1, 2, ..., n-1 \end{aligned}$$

Then we can write our payoff as

$$E^Q \nu_{IB_p(T,F_v)} = F_v \Bigg\{ 1 - (1 - \varphi_n) \sum_{j=1}^{n} \omega_j$$

$$-\sum_{m=0}^{n-1}\left[(\varphi_{m+1}-\varphi_m)\sum_{0\le j<m}\omega_{j+1}+\frac{e_m-(\varphi_{m+1}-\varphi_m)K_m}{K_{m+1}-K_m}\omega_{m+1}\right]\Bigg\}$$

3.2 Risk-free spot interest rate

Once we have the payoff models defined, let us focus on modeling the risk-free interest rate, following each one of the models mentioned above.

3.2.1 Vasicek

Let us assume that the risk-free interest rate behaves according to Vasicek's model, that is,

$$dr_t = a(b-r_t)dt+\sigma dW_t$$

for positive constants a,b and σ. We also assume $\lambda = 0$, since investors are assumed to be neutral towards the risk.

Applying Itô's lemma and going through some calculation (Appendix A) we obtain the expressions to price a zero coupon bond. Once we have done this, pricing both a cat bond with stepwise and piecewise payoff is straightforward:

$$\begin{aligned}
IB_s(0) &= F_v \mathrm{e}^{-A(t,T)r_t+D(t,T)}[1-\Phi(T)]\\
IB_p(0) &= \mathrm{e}^{-A(t,T)r_t+D(t,T)}E^Q\left[\nu_{IB_p(T,F_v)}\right]\\
A(t,T) &= \frac{1-\mathrm{e}^{-a(T-t)}}{a}\\
D(t,T) &= \left(b-\frac{\sigma^2}{2a^2}\right)[A(t,T)-(T-t)]-\frac{\sigma^2 A(t,T)^2}{4a}
\end{aligned}$$

where $E^Q\left[\nu_{IB_p(T,F_v)}\right]$ can be calculated using lemma 3.2.

3.2.2 Hull-White

Now we are assuming r follows the Hull-White model. Once again, we will have to price first a zero coupon bond and then add the payoff of our catastrophe, just the same way we did in the Vasicek case (see Appendix B).

Let us denote by $f^M(t,T)$ the market instantaneous forward rate at time t for maturity T. We know that $f^M(0,T)$ and the zero coupon bond are related through the formula

$$f^M(0,T) = -\frac{\partial ln P^M(0,T)}{\partial T}$$

and the dynamic of the HW model follows

$$dr_t = (\vartheta(t)-ar_t)dt+\sigma dW_t$$

for constants a, $\sigma > 0$ and function ϑ, whose expression is computed in Appendix B.

After all the calculation, the obtained pricing formulas are:

$$\begin{aligned} IB_s(0) &= P^M(0,T)\mathrm{e}^{B(0,T)f^M(0,0)}\mathrm{e}^{-B(0,T)r_0}F_v[1-\Phi(T)] \\ IB_p(0) &= P^M(0,T)\mathrm{e}^{B(0,T)f^M(0,0)}\mathrm{e}^{-B(0,T)r_0}E^Q\left[\nu_{IB_p(T,F_v)}\right] \end{aligned}$$

with

$$B(t,T) = \frac{1}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)$$

3.2.3 CIR

Let us now study the CIR model, whose dynamics is described by

$$dr(t) = a(b-r_t)dt + \sigma\sqrt{r_t}dW_t$$

for constants a, b, $\sigma > 0$, with $2ab > \sigma^2$.

This model is an extension of Vasicek's, with a volatility proportional to $\sqrt{r_t}$ that avoids the possibility of negative interest rates for all positive values of a and b, the main problem of Vasicek's model.

After all the corresponding computation, the obtained pricing formulas are

$$\begin{aligned} IB_s(0) &= P(r,0)F_v[1-\Phi(T)] \\ IB_p(0) &= P(r,0)E^Q\left[\nu_{IB_p(T,F_v)}\right] \end{aligned}$$

where the expression of P(r,0) is given in Appendix C.

3.2.4 Black-Derman-Toy (BDT)

This is a one factor model with no closed solution, even if we know that it follows this stochastic differential equation:

$$d\ln(r) = \left[\theta_t + \frac{\sigma_t'}{\sigma_t}\ln(r)\right]dt + \sigma_t dW_t$$

where

- θ_t is the value of the underlying asset at option expiry.
- σ_t is the instant short rate volatility.
- W_t is a standard Brownian motion.

That means we have to use a binomial tree to calibrate the model parameters to fit both the current term structure of interest rates (yield curve), and the volatility structure for interest rate caps.

To calibrate the model, we have to take into account the following assumptions (see *Black-Derman-Toy*[1]):

- Both probabilities of going up or down on a one-step move are 1/2, so that means that today's price S can be calculated as

$$S = \frac{\frac{1}{2} * (S_u + S_d)}{1 + r}$$

where r is the instant short rate valid today.

- For each input spot rate, we iteratively adjust the rate at the upper node at the current instant of time, i, then we find all other nodes in that time, using $0.5 * \ln(r_u/r_d) = \sigma_i * \sqrt{\Delta t}$.
- Once we have done this, we have to discount recursively through the tree, from the instant of time we are at to the first node in the tree, and repeat this process until the calculated spot-rate equals the assumed spot-rate.

3.2.5 Mercurio-Moraleda. Humped volatility model

Due to the mean reversion property, it is known that short-term rates are more volatile than long-term rates. Taking this into account, the mean-reverting effect has always been modeled considering that the volatility of interest rates is a strictly decreasing function of the maturity. Empirical studies, however, have shown that the volatility structure is actually humped. That is, the volatility has an slope that increases at first until a maximum, and then it decreases as the maturity grows, as an effect of the mean reversion property.

Mercurio & *Moraleda*[6] modeled the yield curve dynamics according to a humped form of the volatility according to the Heath et al. [4](HJM from now on) framework.

Let us denote by P(t,T) a zero coupon bond and by f(t,T) the instantaneous forward rate at time t for a maturity T. As we saw already, the following relation can be written:

$$f(t,T) = -\frac{\partial lnP(t,T)}{\partial T}$$

The following diffusion process models the instantaneous forward rate, assuming a fixed maturity T:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

with f(0,T) given and deterministic, $\sigma(t,T)$ is a deterministic function, integrable over closed interval, and $\alpha(t,T)$ is a deterministic function related with $\sigma(t,T)$ through the HJM no-arbitrage condition:

$$\alpha(t,T) = \sigma(t,T) \left(\int_t^T \sigma(t,s)ds - \theta(t) \right)$$

where $\theta(t) = 0$ under the Q measure of probability, which we are assuming throughout our work.

Application of Ito's Lemma gives the following dynamics of the bond price:

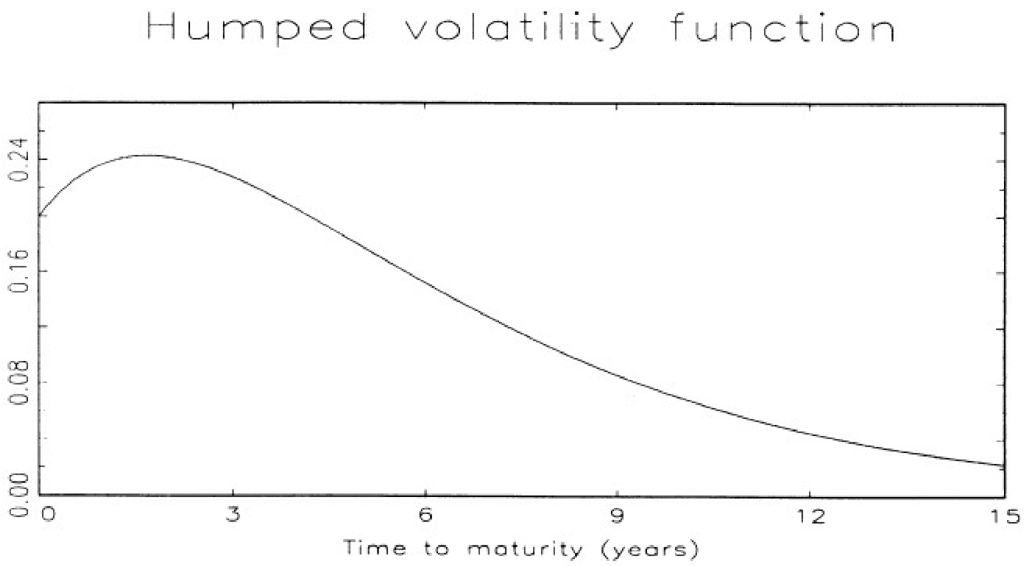
$$dP(t,T) = P(t,T)\left[\left(r(t) - \int_t^T \alpha(t,v)dv + \frac{1}{2}\left(\int_t^T \sigma(u,T)du\right)^2\right)dt - \left(\int_t^T \sigma(t,v)dv\right)dW(t)\right]$$

where r(t) is the instantaneous short-term interest rate at time t, that is $r(t) = f(t,t)$.

In their paper, Mercurio & Moraleda considered the following characterization of the volatility of the instantaneous forward rates:

$$\sigma(t,T) = \sigma\left[\gamma(T-t)+1\right]\exp(-\lambda/2(T-t))$$

where σ, γ and λ are non-negative constants. This expression defines a humped volatility with this form:



for $\sigma = 0.2, \lambda = 0.6$ and $\gamma = 0.6$.

With this humped volatility formula, the integrals on the expression of the zero bond coupon price have no closed form, and we must solve them using numerical methods to get the coupon price.

The initial term structure of interest rates is assumed to be $r_T = 0.08 - 0.05\exp(0.18T)$ as in Hull & White.

4 Numerical Experiments

Now that all the pricing formulas have been obtained, it is time to conduct some simulations to check if they can be appropriate to price catastrophe bonds.

Since we have 5 different models for the risk-free spot interest rate, 4 distributions for the losses and 2 kinds of payoff, 40 different models can be simulated. To avoid being repetitive, we will only present here some of the most relevant of those 40 models.

In the simulations, the face value of the cat bond is set to 1, and the trading horizon is 1 year. The parameters of the different models for the interest rate have been previously fitted in *Episcopos (2000)*[3] and *Hull & White (1993)*[5], as well as the adjusted parameters for all the losses distributions can be found in *Chernobai (2005)*[2].

For our first model, let us consider the Vasicek interest rate model, combined with the gamma distribution for the losses and the stepwise payoff. The parameters of the model can be found in the following table:

	Parameters
Vasicek model	a = 0.0235, b = 0.0055, σ= 0.0, r0 = 0.0614
Intensity of HPP	κ_{HPP} = 31.7143
Gamma distribution	α = 0.5531, β = 1.5437e-9
Triggering points	$K_1 = Q_{HPP-GAM}(0.75)$, $K_2 = Q_{HPP-GAM}(0.95)$
Losses coefficients	ω_1 = 0.2,ω_2 = 0.3

Table 1: Parameters for model 1.

Where, for the triggering points we have taken the 0.75 and 0.95 quantiles of the corresponding gamma distribution. With these parameters, the price obtained for the cat bond is 0.865546.

For the next model, we consider the CIR model, combined with a GPD distribution and a piecewise payoff, according to the following parameters:

	Parameters
CIR model	a = 0.0241, b = 0.0539419, σ= 0.0141421, r0 = 0.0614
Intensity of HPP	κ_{HPP} = 31.7143
GPD distribution	ξ = -0.8090, σ_{GPD} = 0.534e+8, θ = 0
Triggering points	$K_0 = Q_{HPP-GPD}(0.75)$, $K_1 = Q_{HPP-GPD}(0.85)$, $K_2 = Q_{HPP-GPD}(0.95)$
Losses coefficients	ω_1 = 0.2,ω_2 = 0.3

Table 2: Parameters for model 2.

And this time, the price obtained is 0.877098

Let us show now the result of combining the humped volatility model of Mercurio & Moraleda with a lognormal distribution and a stepwise payoff:

	Parameters
Humped volatility model	σ= 0.2, γ = 0.6, λ = 0.6, r0 = 0.03
Intensity of HPP	κ_{HPP} = 31.7143
LN distribution	μ_{LN} = 17.357, σ_{LN} = 1.7643
Triggering points	$K_1 = Q_{HPP-LN}(0.75)$, $K_2 = Q_{HPP-LN}(0.95)$
Losses coefficients	$\omega_1 = 0.2$,$\omega_2 = 0.3$

Table 3: Parameters for model 3.

In this case, the price obtained is 0.911380. As we will see later, the prices involving the humped volatility model are always considerably higher than those obtained from the other models.

To finish showing some model samples, we present here the model combining BDT with GPD distribution and stepwise payoff. The following table contains the yield curve and yield volatility used to compute the binomial tree needed to get the discount factor:

Maturity (years)	Yield (%)	Yield Volatility (%)
1	10	20
2	11	19
3	12	18
4	12.5	17
5	13	16

Table 4: Parameters for model 4.

In this model we compute the discount factor using different number of steps (1,2,4 and 12) in the binomial tree, simulating an annual, semiannual, quarterly and monthly tree, respectively. As we can see, the discount factor is slightly smaller as we implement more steps in our tree. This decreasing behavior is not constant with the number of steps, i.e the difference between considering one or two steps is much notorious that the one between 4 and 12 steps.

The prices computed in this model are:

1 step	2 steps	4 steps	12 steps
0.799823	0.798616	0.797640	0.798062

It is clear that the price decreases as the number of steps is higher. The fact that the price for 12 steps is a bit higher than the one for 4 steps is understandable if we take into account that we are working with simulations, so every time we run the simulation the output is slightly different.

As it is mentioned previously, 40 different models were implemented, but we only comment here some of the most relevant. In the next subsection, we perform some studies in order to characterize the sensitivity of the parameters involved.

4.1 Sensitivities of the parameters

In this subsection, our aim is to show the individual influence of each of the parameters considered in the final price. To show that, we present the prices obtained by simulation when an individual parameter changes its value a 5% (both positive and negative increments are considered).

This calculation is showed here for one model to avoid being redundant, but it could be applied to any other. The chosen model is the one with the combination CIR + Weibull + Piecewise, whose price is 0.878329. Let us show first, in table 5, the prices obtained when the parameters related with the interest rate are modified:

Parameter	Price (+5%)	Price (-5%)
a	0.879621 0.15%	0.879833 0.17%
b	0.878416 0.01 %	0.878476 0.02 %
σ	0.879043 0.08 %	0.879227 0.1%
r0	0.876154 -0.24 %	0.883095 0.55%

Table 5: Prices and relative changes when interest rate parameters are changed.

As we can see, changing one single parameter of the interest rate model does not really produce big changes in the price of the cat bond. If we now consider the parameters involving the payoff function (Table 6):

In this case, the weights ω do not have a big influence on the price, but the thresholds seem to be more important, specially as it gets closer to the quantile 1 of the distribution, i.e, changes on quantile 0.95 produce bigger diferences on the price that changes on quantile 0.75.

Table 7 presents the prices when the losses distribution parameters are changed:

In this case, it is clear that the changes in the parameters have a much bigger influence than in the previous tables, so we might conclude that the well fitting of the parameters of the losses distribution is a crucial point when it comes to price cat bonds.

Parameter	Price (+5%)	Price (-5%)
ω_1	0.878089 -0.02%	0.880836 0.28%
ω_2	0.879074 0.08 %	0.880050 0.19 %
K_0	0.882612 0.49 %	0.875681 0.30%
K_1	0.887855 1.08 %	0.871452 -0.79%
K_2	0.896837 2.11 %	0.870566 0.88%

Table 6: Prices and relative changes when payoff parameters are changed.

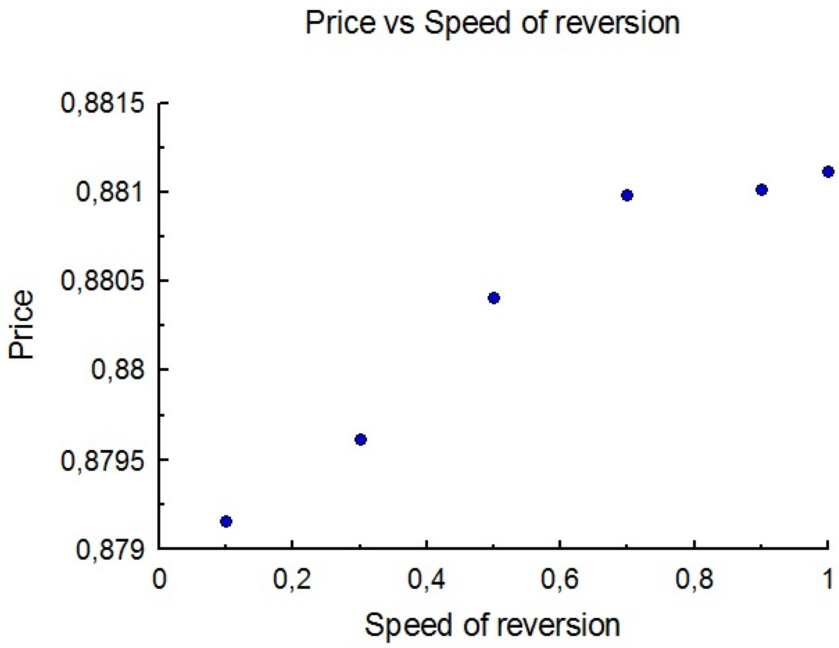
Parameter	Price (+5%)	Price (-5%)
β	0.895672 1.97%	0.856642 -2.47%
σ_{Wei}	0.933371 6.27 %	0.711184 -19.03 %

Table 7: Prices and relative changes when losses parameters are changed.

4.1.1 Dynamic of the prices when interest rate parameters are changed

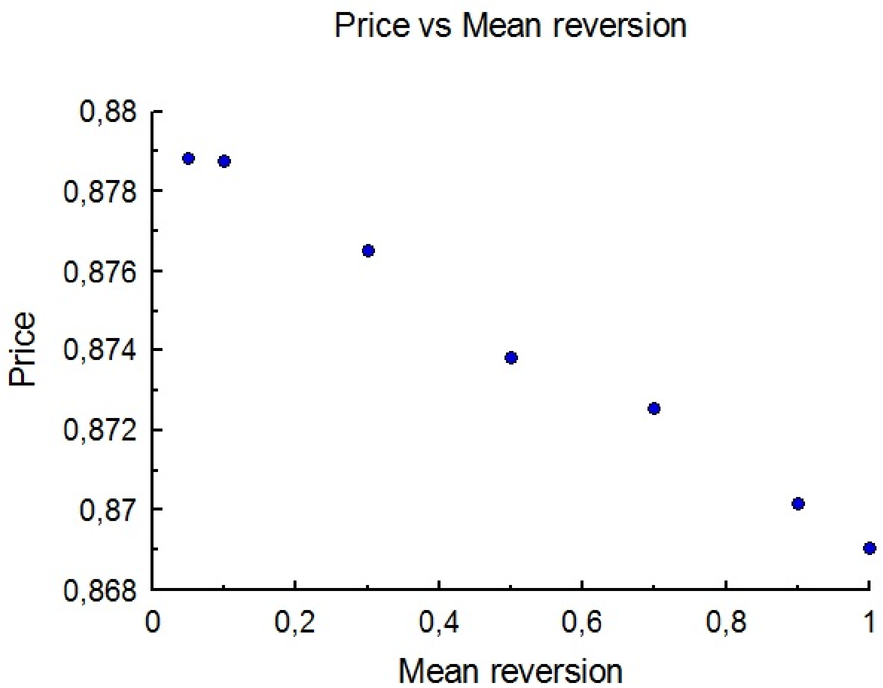
At this point, we want to see from a different point of view the study of variations in interest rate parameters we just carried out. Our aim here is to take a look at the evolution of prices when the parameters are changed and to check if the results agree what we could expect from the theory.

Let us consider first the parameter a of CIR model, which is nothing but the speed of reversion of the model. As we change its value, the prices obtained can be seen in the next graph:



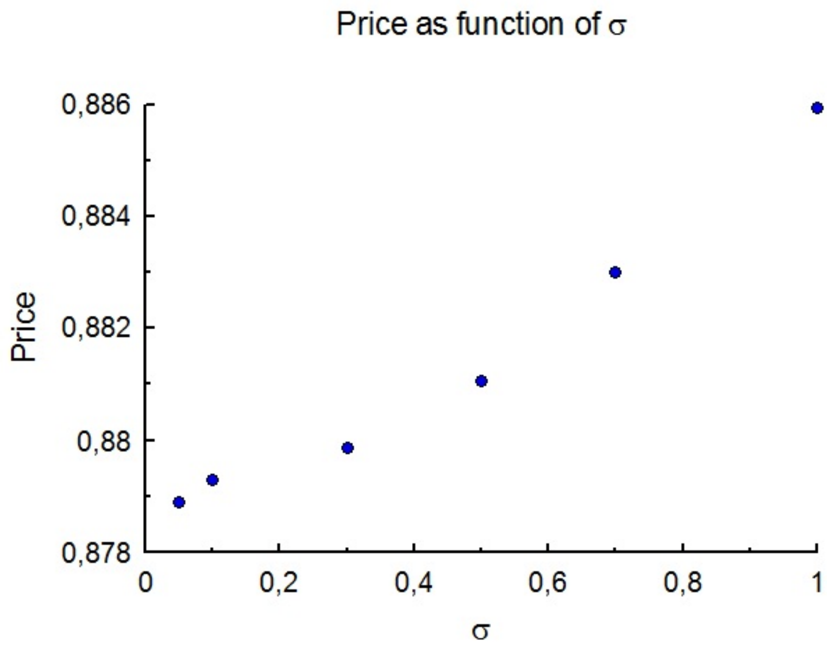
As the value for the speed of reversion grows, the price also becomes bigger. That is perfectly understandable if we take into account that the factor we are modifying tells us how fast the value of the risk-free spot interest rate reaches its final value, the mean reversion value. As the speed is higher, the risk-free spot interest rate becomes more stable, and then the price is higher.

If we now modify the value of b, that is, the mean reversion value, the price changes as follow:



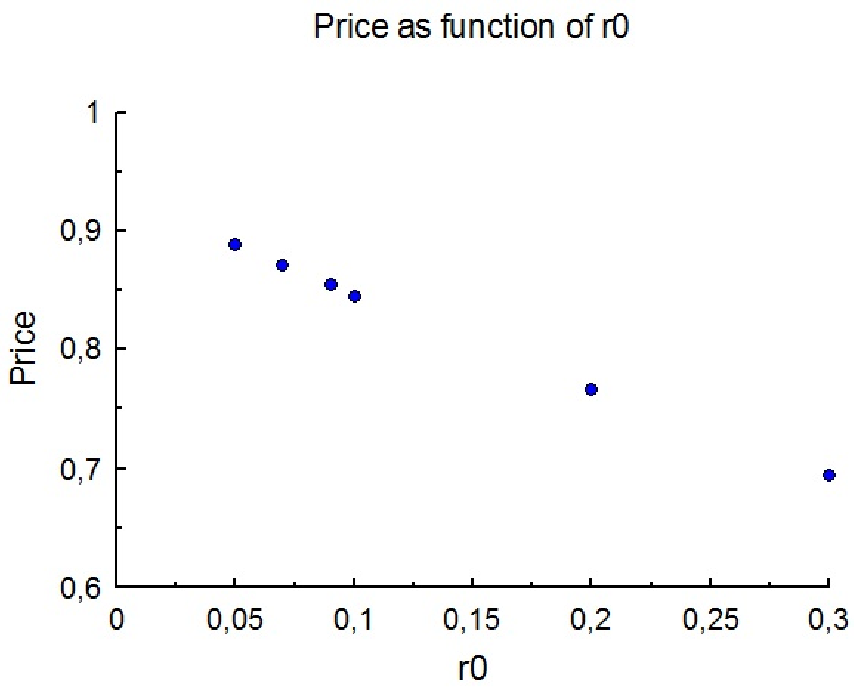
In this case, the price decreases as the mean reversion goes higher. This is also reasonable, since what it is being changed here is the final value that the risk-free spot interest rate will reach, and remembering that the discount factor goes like $exp(-r * T)$, it is straightforward that as r grows, the discount factor is smaller, so the price is too.

The following graph shows how the price changes when the volatility of the risk-free spot interest rate is changed:



In this case, we are not able, a priori, to expect that the price will be higher or lower when the value of the volatility grows. Looking at the graph, it is clear that the price is directly proportional to the value of the volatility and this may be a result of the fact that we are assuming the bondholders to be neutral towards the risk, so they are willing to pay more as the uncertainty grows, regarding more on their possible winnings than on their possible losses.

To conclude, we present the graph showing how changes on the value of r0 affect the price:

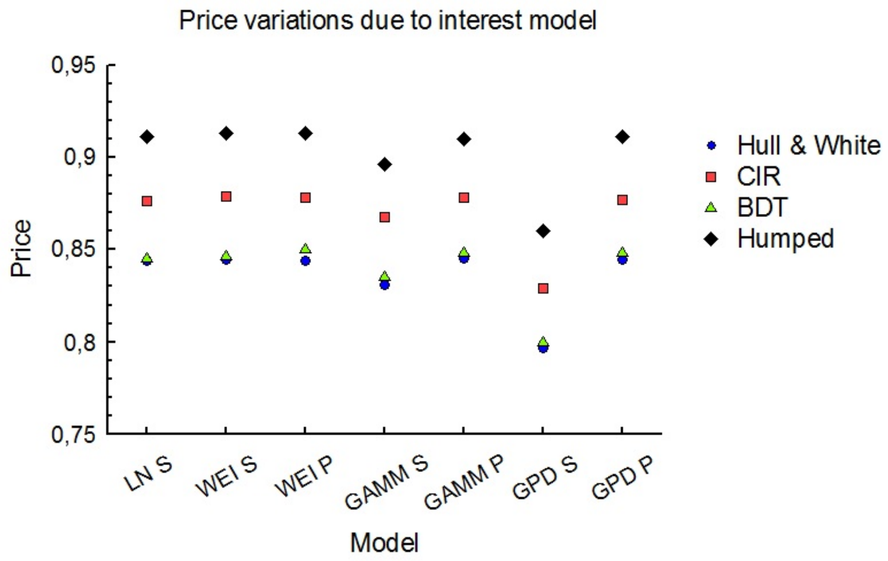


The parameter being changed here is the initial value of the risk-free spot interest rate. Let us remember that it will revert to a final value of b = 0.0539419, but, as we start the reversion from a higher value, the discount factor is initially smaller as r0 grows, and so the prices are.

4.2 Influence of the election of the factors

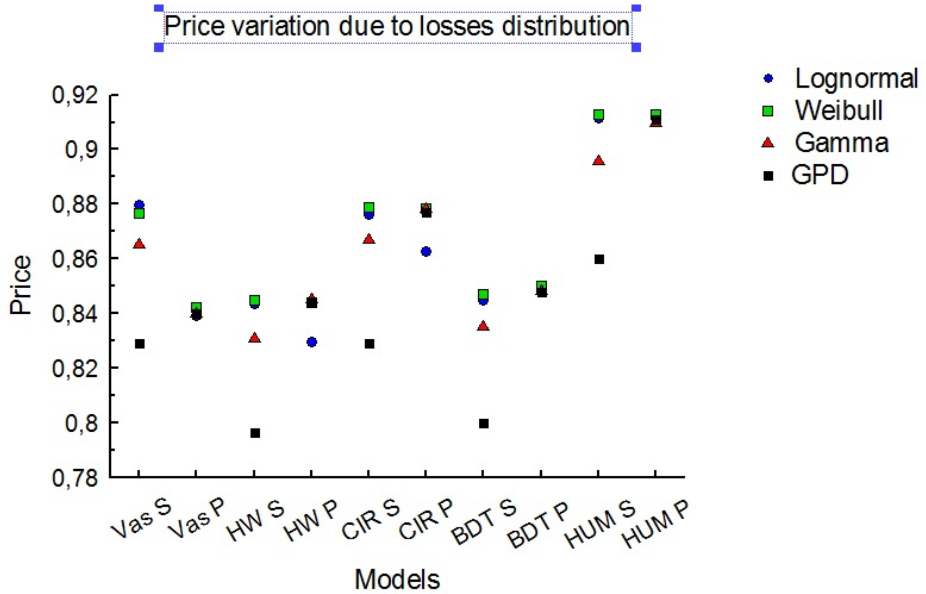
This subsection aims to answer the following question: Imagine that we want to price a cat bond; we have seen up to now that our cat bond can be modeled through choosing a combination of three factors: the payoff function, the distribution for the losses and the risk-free spot interest rate. Hence, a question arises: Which of the three decisions to make has a bigger influence on the price? To answer this, a very simple procedure is performed: We start by computing all the prices obtained that have 2 of the 3 factors in common, i.e all the models that have lognormal distribution and stepwise payoff, so the only factor changing is the interest rate. Then, we compute the dispersion of these results, as the maximum price minus the minimum, divided by the mean of the set of prices, and we get how much, in %, the price can vary due to the choice of one or another model for the interest rate. After that, the factor with bigger influence is found by repeating the same process with the payoff function and the losses distribution.

Once the procedure has been introduced, let us show in this graph the prices obtained for the different interest rate models:

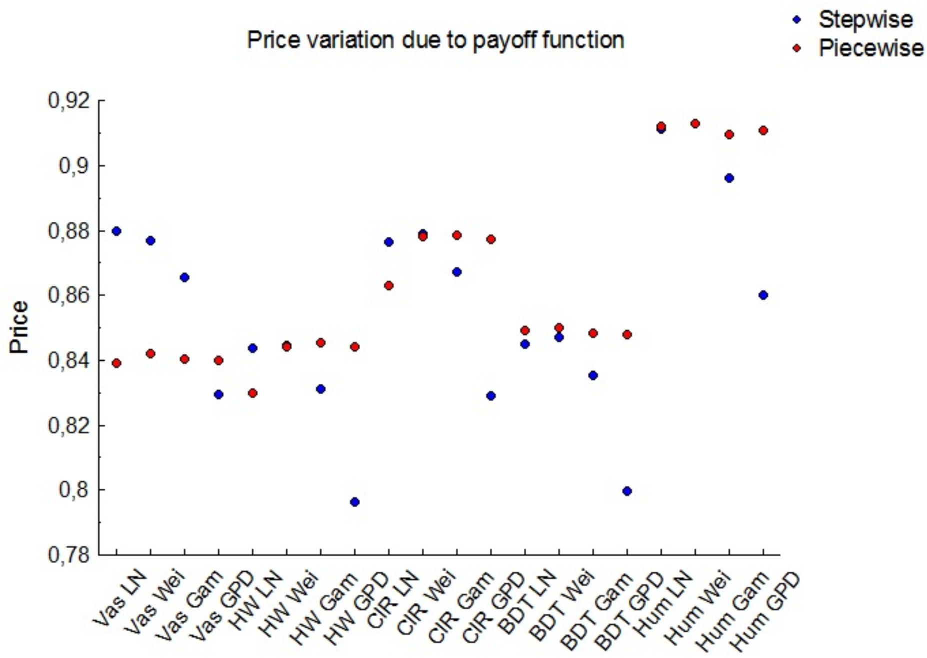


As we can see, the same pattern is repeated almost exactly in every model, the humped volatility model always on top, followed by CIR and then the others all very close amongst them. Vasicek model is not included in the graph because its results are very close to HW and BDT, but its taken into account when computing the dispersion.

Let us now present the equivalent graph for the losses distribution:



It is really easy to observe in the graph that there are two different patterns, one involving models with the stepwise payoff , in which lognormal and Weibull give always the higher prices, followed closely by gamma,with GPD always the lowest,and the other including all the models with the piecewise linear payoff, in which all the distribution give almost the same price.



Prices as a function of the payoff function implemented can be seen in the graph above. As we mentioned earlier, it is clear that the models with the piecewise linear function give almost the same price for all distributions for losses, while the stepwise payoff gives prices with much less correlation.

Next table computes the values obtained for the dispersion in each case:

Factor changing	Dispersion (%)
Interest rate	8.12
Losses distribution	3.36
Payoff function	2.24

Table 8: Dispersions in %.

Then, we can affirm that the factor whose election can affect more the price is the risk-free spot interest rate model, since the price can vary up to a 8 % depending on which model we choose.

5 Conclusions

Once it has been demonstrated that the classical insurance methods are not appropriate when it comes to natural catastrophe events, new approaches are needed, such as the conversion of losses into tradable assets called cat bonds.

In this paper we extend the Nowak & Romaniuk approach, introducing some new risk-free spot interest rates and distribution of losses that were not considered in their work. The pricing approach considers no possibility of arbitrage, independence of the catastrophe occurrence from the behavior of the market, and the possibility of replication of interest rate changes by other instruments. We use two types of payoff (stepwise and piecewise linear), four different losses distribution (Lognormal, Weibull, Gamma and GPD) and five models for the interest rate (Vasicek, Hull & White, CIR, Black-Derman-Toy and Mercurio & Moraleda).

On the first part of the paper, we show some of the prices obtained for certain combination of our models. If we take a look at the interest rate, the results always present the same pattern: Humped volatility model gives a price around 0.91, CIR model decreases to 0.87 and the other models give always prices around 0.84. This higher value obtained with the humped volatility model might be a consequence of the increasing slope present in the humped volatility expression for short maturities. If we extend our computation to, let us say, 10 years, we will expect a lower price with this model compared to the others.

After that, we carried out some studies to quantify the sensitivity of each of the parameters involving our models, modifying a 5% their values and computing the relative variation of the price. Results show that variations on the parameters of the loss distribution have a much bigger impact compared to the other parameters. In section we also studied the behavior of the price as a function on the parameters characterizing the risk-free spot interest rate dynamics, and the results obtained coincide with what we might expect from the theory: prices getting higher as we increase the speed of reversion and the volatility of the interest rate and decreasing when the mean reversion and the r0 grow. This studies were implemented for the combination CIR + Weibull + Piecewise linear payoff, and of course it is possible to extend them to any other combination treated on this paper.

And finally, we proposed the question of which of the three decisions to make (choices of the interest rate models, losses distributions and type of payoff) has a bigger impact on the price. This could be possibly the most important study on the paper, since it provides information about which factor should be chosen more carefully when pricing a certain cat bond. The results show that the impact of the interest rate model chosen can be up to more than twice the impact of the losses distribution and almost four times the influence of the type of payoff chosen.

6 Appendix

A Vasicek model

$$\begin{aligned} r_t &= \mathrm{e}^{at}\left[r_0 + \int_0^T ab\mathrm{e}^{au}du + \sigma\int_0^T \mathrm{e}^{au}dW_u\right] \\ r_t &= \mu_t + \sigma\int_0^T \mathrm{e}^{a(u-t)}dW_u \end{aligned}$$

where μ_t is the deterministic part of the equation, such that $E[r_t] = \mu_t$.

r_t is a gaussian random variable. This follows from the definition of the stochastic integral term, which is

$$\lim_{|\pi|\to 0}\sum_{i=0}^{n-1}\mathrm{e}^{a(u_i-t)}(W_{u_i+1} - W_{u_i})$$

where the increments are given by

$$(W_{u_i+1} - W_{u_i}) \sim N(0, u_{i+1} - u_i)$$

Consider we have a catastrophe bond, with the general pricing formula showed before,

$$IB(0) = E^Q\left[\exp\left(-\int_t^T r_u du\right)\right]E^Q\left[\nu_{IB_{cat}(T,F_v)}|F_t\right], t \in [0,T]$$

We need to compute the integral using Vasicek's model for r

$$E\left[\exp\left(-\int_0^T r_u du\right)|F_t\right]$$

.

To do that, we will use the following transformation

$$X(u) = r_u - b$$

where X(u) is the solution of the Ornstein-Uhlenbeck process

$$dX(t) = -aX(t) + \sigma dW_t$$

Applying Itô's lemma to this process, we obtain

$$X(u) = \mathrm{e}^{-au}\left(X(0) + \int_0^u \sigma\mathrm{e}^{as}dW_s\right)$$

.

X(u) is a Gaussian process with continuous path and then $\int_0^T X(u)du$ is also Gaussian, with expected value

$$E\left[\int_0^T X(u)du\right] = \int_0^T E[X(u)]du = \frac{X(0)}{a}(1 - \mathrm{e}^{-at})$$

Using that $X(u) = r_u - b$, it is possible to write

$$E\left[-\int_0^T r_u du\right] = -\frac{r_t - b}{a}(1 - \mathrm{e}^{-aT}) - bT$$

and the variance

$$Var\left[-\int_0^T r_u du\right] = Var\left[-\int_0^T X(u)du\right] = \frac{\sigma^2}{2a^3}\left(2aT - 3 + 4\mathrm{e}^{-aT} - \mathrm{e}^{-2aT}\right)$$

This computation can be easily done using the following relation

$$Var\left[\int_0^T r_u du\right] = Cov\left[\int_0^T r_u du, \int_0^T r_s ds\right]$$

Regarding the Itô representation of r_t, we can notice that r_t satisfies the Markov property, so we can write

$$E\left[\exp(-\int_0^T r_u du)|F_t\right] = E\left[\exp(-\int_0^T r_u du)|r_t\right] = E\left[\exp(-\int_0^T r_u(r_t)du)\right]$$

.

And, knowing that r_t is a Gaussian process, e^{r_t} is a lognormal process, and then

$$E\left[\exp(-\int_0^T r_u(r_t)du)\right] = \exp\left(E\left[-\int_0^T r_u(r_t)du\right] + \frac{1}{2}Var\left[-\int_0^T r_u(r_t)du\right]\right)$$

And we arrive at our final expression after some algebra

$$\begin{aligned} E\left[\exp(-\int_0^T r_u(r_t)du)\right] &= \mathrm{e}^{-A(t,T)r_t + D(t,T)} \\ A(t,T) &= \frac{1 - \mathrm{e}^{-a(T-t)}}{a} \\ D(t,T) &= \left(b - \frac{\sigma^2}{2a^2}\right)[A(t,T) - (T-t)] - \frac{\sigma^2 A(t,T)^2}{4a} \end{aligned}$$

This

is exactly the same formula that appears in Nowak & Romaniuk paper, if we take into account that

$$\begin{aligned} b - r &= \frac{\lambda\sigma}{a}, \text{ with } \lambda = \frac{\mu - r}{\sigma} \\ IB_s(0) &= F_v \mathrm{e}^{-A(t,T)r_t + D(t,T)}[1 - \Phi(T)] \\ IB_p(0) &= \mathrm{e}^{-A(t,T)r_t + D(t,T)} E^Q\left[\nu_{IB_p(T,F_v)}\right] \end{aligned}$$

where $E^Q\left[\nu_{IB_p(T,F_v)}\right]$ can be calculated using the lemma 3.2.

B Hull & White

The dynamics of the model are described by

$$dr_t = (\vartheta(t) - ar_t)dt + \sigma dW_t$$

where a and σ are positive constants, and $\vartheta(t)$ is a function perfectly fitted to the initial rate and volatility term structure.

The expression for this $\vartheta(t)$ can be deduced by the following procedure:

First we take the transformation $r(t) = x(t) + \alpha(t)$ where $dx(t) = -ax(t)dt + \sigma dW(t)$, $x(0) = 0$

Simple integration from 0 to t gives

$$x(t) = x(0)\mathrm{e}^{-at} + \sigma \int_0^t \mathrm{e}^{-a(t-s)} dW(s)$$

So, this integral can be computed:

$$\int_0^t x(u)du = x(0) \int_0^t \mathrm{e}^{-au} du + \sigma \int_0^t \int_0^u \mathrm{e}^{-a(u-s)dW(s)du}$$

with expected value and variance, obtained through simple algebra,

$$\begin{aligned} E\left[\int_0^t x(u)du\right] &= x(0)\int_0^t \mathrm{e}^{-au}du = 0 \\ Var\left[\int_0^t \mathrm{e}^{-au}du\right] &= \frac{\sigma^2}{a^2}\left(t + \frac{1-\mathrm{e}^{-2at}}{2a} - 2\frac{1-\mathrm{e}^{-at}}{a}\right) \end{aligned}$$

We now move to the pricing formula for a coupon zero bond,

$$P(t,T) = E^Q\left[\exp\left(\int_t^T r(u)du\right)|_t\right] = E^Q\left[\exp\left(\int_t^T x(u)du\right)|_t\right]\exp\left(-\int_t^T \alpha(u)du\right)$$

since α is deterministic.

As we know, x_t is a gaussian process, so we can write

$$\begin{aligned} P(0,T) &= \exp\left(\frac{1}{2}Var\left[-\int_0^T x(u)du|_0\right]\right)\exp\left(-\int_0^T \alpha(u)du\right) \\ &= \exp\left(\frac{\sigma^2}{2a^2}\left(t + \frac{1-\exp(-2at)}{2a} - 2\frac{1-\exp(-at)}{a}\right)\right)\exp\left(-\int_0^T \alpha(u)du\right) \end{aligned}$$

Let us now denote by $f^M(t,T)$ the market instantaneous forward rate at time t for maturity T, where

$$f^M(0,T) = -\frac{\partial lnP^M(0,T)}{\partial T}$$

Using this definition and rearranging the expression above, is not hard to arrive to the following expression

$$\alpha(t) = f(0,T) + \frac{\sigma^2}{2a^2}\left(1 + \mathrm{e}^{-2at} - 2\mathrm{e}^{-at}\right)$$

Now we only have to compare these two relations

$$\begin{aligned} dr(t) &= dx(t) + d\alpha(t)dt = [d\alpha(t) + a\alpha(t) - ar(t)]dt + \sigma dW(t) \\ dr_t &= (\vartheta(t) - ar_t)dt + \sigma dW_t \end{aligned}$$

And directly write

$$\vartheta(t) = d\alpha(t) + a\alpha(t) = f_t(0,T) + af(0,T) + \sigma^2 2a\left(1 - \mathrm{e}^{-2at}\right)$$

And we have finally reached the expression of $\vartheta(t)$.

Now the process to get the pricing formula is very similar to that used with Vasicek, we have to propose a solution for the differential equation of the form $A(t,T)\mathrm{e}^{-B(t,T)r_t}$, where in this case

$$\begin{aligned} B(t,T) &= \frac{1}{a}\left(1 - \mathrm{e}^{-a(T-t)}\right) \\ A(t,T) &= \frac{P^M(0,T)}{P^M(0,t)} \exp\left(B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - \exp(-2at))B(t,T)^2\right) \end{aligned}$$

And consequently the pricing formulas for the cat bonds are

$$\begin{aligned} IB_s(0) &= P^M(0,T)\mathrm{e}^{B(0,T)f^M(0,0)}\mathrm{e}^{-B(0,T)r_0}F_v[1 - \Phi(T)] \\ IB_p(0) &= P^M(0,T)\mathrm{e}^{B(0,T)f^M(0,0)}\mathrm{e}^{-B(0,T)r_0}E^Q\left[\nu_{IB_p(T,F_v)}\right] \end{aligned}$$

C CIR

To obtain the pricing equation we have to repeat the same process we used in the previous models, that is, find the pricing equation for a zero coupon bond, solving it according to CIR parameters and then add the corresponding payoff for each type of cat bond.

To solve the pricing equation, we propose once again a solution of the form

$$P(r,0) = A(T)\exp(-r_0 B(T))$$

where, in this particular case,

$$\begin{aligned} A(T) &= \left[\frac{\theta_1 \exp(\theta_2 T)}{\theta_2\left(\exp(\theta_1 T) - 1\right) + \theta_1}\right]^{\theta_3} \\ B(T) &= \frac{\exp(\theta_1 T) - 1}{\theta_2\left(\exp(\theta_1 T) - 1\right) + \theta_1} \end{aligned}$$

where

$$\theta_1 = \sqrt{(a+\lambda)^2 + 2\sigma^2}, \quad \theta_2 = \frac{a + \lambda + \theta_1}{2}, \quad \theta_3 = \frac{2ab}{\sigma^2}$$

And the corresponding pricing formulas are

$$\begin{aligned} IB_s(0) &= P(r,0)F_v[1 - \Phi(T)] \\ IB_p(0) &= P(r,0)E^Q[\nu_{IB_p(T,F_v)}] \end{aligned}$$

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