PRICING EQUITY DERIVATIVES WITH THE KRISTENSEN-MELE APPROACH

Carlos Catalán García

Trabajo de investigación 013/007

Master en Banca y Finanzas Cuantitativas

Tutores: Dr. Manuel Moreno
Dr. Javier F. Navas

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Contents

Abstract

This paper describes an approximation derivative pricing model recently introduced in Kristensen and Mele (2011) (K-M, from now on) and illustrates its applications to different derivatives and continuous-time models. This pricing model allows to obtain approximate asset prices for models that do not provide closed-form expressions in terms of the analytical formulas obtained for the price of these assets under a second (auxiliary) model.

Then, this approximation pricing model is based on the choice of an "auxiliary" pricing model for which a closed-form expression for prices is available. Given such auxiliary model, K-M derives an expression for the difference between the unknown price of the model of interest (true model) and that (known) in the auxiliary model. Such expression takes the form of a conditional expectation and can be developed in terms of a Taylor series expansion.

After describing this model, we illustrate its implementation to some equity derivatives and models that reflect stochastic volatility and jumps.

1 Introduction

During the last years, the use of continuous-time models have increased considerably and a large number of models have arisen. Continuous-time modeling has provided many benefits and it allows us to find elegant representations for the price of a variety of contingent claims.

While many models have arisen, new approaches have also been developed and they provided us new methods with applications to pricing, hedging, and spanning of derivatives contracts. In some cases, the approaches produce closed-form solutions to cope with prices. In some others cases where the closed-form does not exist, some other methodologies (as approximations or numerical approaches) have been developed to compute prices.

Although the benefits of the continuous-time modeling have been proved, an old issue remains:

How can we implement those methods that cannot be solved in closed-form?

Up to now, to deal with prices that are not given in a closed-form, several alternatives have been proposed. We can mention Fourier transformations, tree methods, numerical solutions to partial differential equations or Monte Carlo methods, among others. While the implementation of some of these methods can be cumbersome, a new approach has been recently proposed in Kristensen and Mele (2011) (K-M, from now on).

These authors have developed a new approach to approximate prices in multi-factor models. This approach develops what could be interpreted as a closed-form price formula for each multi-factor model. The key idea underlying this approach is to set as benchmark an auxiliary model for which a closed-form solution is known. Once the auxiliary model is set, the goal of the K-M approach is to relate the price under this auxiliary model to the price obtained from other model which will be considered as the target price.

This method will provide an expression for the difference between the model of interest (which price is not known in closed-form) and the auxiliary one (which is known in closed-form). Although such expression will take the form of a conditional expectation, K-M finds an expression that can be developed through a Taylor series expansion. Because this series expansion contains a infinite number of terms, we will need to truncate the series into a finite number of terms.

This paper is organized as follows. Section 2 introduces the K-M approach in the most general case and discusses its similarities with respect to the Yang's expansion (see [7]) and the risk-neutral probabilities. We also discuss the application of this method to compute sensitivities ("Greeks"). Section 3 applies the K-M approach to different derivatives and models. First we will implement the method when the model of interest contains new information respect to the auxiliary model. We also illustrate the K-M method and its properties under the most extreme conditions. We will illustrate how these conditions affect the percentage errors obtained through this approach and implement the K-M method using different number of terms in the series expansion. We also implement the model under a new and richer model than in the previous case, present an alternative to the K-M approach and, because of the complexity of this model, we will show its limitations. Finally, Section 4 summarizes our main pricing results and the robustness of this method. Proofs and technical details are deferred to a final Appendix.

2 The general Kristensen-Mele approach

2.1 Theoretical framework

To introduce the reader into our framework, our first benchmark is to set a true model, which produces our objective price. In this section we will consider our true model as the most general model. Let $x(t)$ be a d-dimensional multi-factor model. Under the risk-neutral probability, we assume $x(t)$, such that follows

$$dx(t) = \mu(x(t), t)dt + \sigma(x(t), t)dW(t), \tag{2.1.1}$$

where $W(t)$ is a standard Brownian motion under risk-neutral probability.

First of all, we set $w(x, t)$ as the option price $\forall t \in [0, T]$. Besides, the existence of the $\hat{L}$ for any d-dimensional multifactor model drives us into a differential equation, which is satisfied by the option price $w(x, t)$ and it takes the form

$$Lw(x, t) + c(x, t) = R(x, t)w(x, t), \tag{2.1.2}$$

where the option price is subject to the boundary condition at maturity time $w(x, T) = b(x)$, where $b(x)$ is the payoff function of our derivative.

After the true model is presented, it is the moment to introduce the auxiliary model, which has an option price given by $w_0(x_0, t)$. As we mentioned before, the option price for the auxiliary model also satisfies the equation 2.1.2, but with $w_0(x, t)$ replacing $w(x, t)$ and with $\hat{L}_0$ replacing $\hat{L}$ (in this case, the boundary condition at maturity time is $w_0(x, T) = b_0(x)$.

Once the models are set, if we define the instantaneous price difference between the two models as $\Delta w(x, t) = w(x, t) - w_0(x, t)$ and then taking thee difference between the equation 2.1.2 and the corresponding to the auxiliary model, we find:

$$L\Delta w(x, t) - r\Delta w(x, t) + \delta(x, t) = 0, \tag{2.1.3}$$

where the term $\delta(x,t)$ is known as the mispricing function and takes the form

$$\delta(x,t) = (L - L_0)w_0(x,t). \tag{2.1.4}$$

The above equation could be developed and the mispricing function turns into:

$$\delta(x,t) = \sum_{i=1}^{d} \Delta\mu_i(x,t)\frac{\partial w_0(x,t)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \Delta\sigma_{ij}^2(x,t)\frac{\partial^2 w_0(x,t)}{\partial x_i \partial x_j}, \tag{2.1.5}$$

where $\Delta\mu$ and $\Delta\sigma$ are the differences for the drift and diffusion terms between the true and the auxiliary model, respectively.

Now, we can apply the Feynman-Kac representation to find a solution for the price difference (see section A.4) and then the following identity holds:

$$\Delta w(x,t) = \mathbb{E}_{x,t}\left[exp\left(-\int_t^T R\left(x(s),s\right)ds\right) d\left(x(t)\right)\right] +$$
$$+ \int_t^T \mathbb{E}_{x,t}\left[exp\left(-\int_t^s R(x_u,u)du\right)\delta\left(x(s),s\right)\right] ds, \tag{2.1.6}$$

where $d(x) = b(x) - b_0(x)$.

The right-hand side of the above equation shows two terms of errors owing to the use of an auxiliary model. The first term arises due to the use of different payoffs between true and auxiliary models (that difference is discounted to the present value through the exponential function). The second term appears because of differences related to the underlying factors of the models.

Taking advantage of the Appendix A of the reference [3], the above equation can be written as a series expansion

$$w_N(x,t) = w_0(x,t) + \sum_{n=0}^{\infty}\frac{(T-t)^n}{n!}d_n(x,t) + \sum_{n=0}^{\infty}\frac{(T-t)^{n+1}}{(n+1)!}\delta_n(x,t). \tag{2.1.7}$$

In practice, this formula is truncated to a finite number of terms, yielding:

$$w_N(x,t) = w_0(x,t) + \sum_{n=0}^{N}\frac{(T-t)^n}{n!}d_n(x,t) + \sum_{n=0}^{N}\frac{(T-t)^{n+1}}{(n+1)!}\delta_n(x,t), \tag{2.1.8}$$

where $\delta_0 = \delta(x)$, $d_0 = d(x)$ and the δ_n function as well as the d_n function satisfy the recursive equations:

$$d_n(x,t) = Ld_{n-1}(x,t) - R(x,t)d_{n-1}(x,t), \tag{2.1.9}$$

$$\delta_n(x,t) = L\delta_{n-1}(x,t) - R(x,t)\delta_{n-1}(x,t). \tag{2.1.10}$$

Since w_0 is known in her closed form, we are able to approximate (through the equation 2.1.8) the price of any option or also partial derivatives respect to the variables of interest.

2.2 Yang's expansion

In this section, we will deal with the relationship with other expansion similar to Kristensen-Mele's expansion (see Yang's expansion through [7]). Under some conditions (see section A.5), the Yang's expansion suggests that the difference between the price in the true market and in the auxiliary one satisfies

$$w(x,t) = w^{(0)}(x,t) + \int_t^T \mathbb{E}^0_{x,t}\left[(L - L_0)\, w\,(x(u),u)\right] du. \tag{2.2.1}$$

On the other hand, under simplifying assumptions done in Yang's expansion to the Kristensen-Mele method ($R(x,t) = c(x,t) \equiv 0$ and $d(x) = 0$), then, the equation 2.1.6 drives into a similar but quite different price representation:

$$w(x,t) = w_0(x,t) + \int_t^T \mathbb{E}\left[(L - L_0)\, w_0\,(x(u),u)\right] du. \tag{2.2.2}$$

A quick look at both expressions might have some likenesses, but instead look over likenesses, we focus on the differences. Although both representations seem to be close, the Yang's expansion is based on the expectation under the base-model's probability, while K-M expansion is under the auxiliary model's probability. It is also quite important to note that Yang's expansion is based on a unknown closed-form function, $(L - L_0)w$, while that in K-M is based on a well known closed-form function, due to the use of an auxiliary model, $\delta = (L - L_0)w_0$. For those reasons, the K-M expansion method allow us to approximate directly the integrated expectation, while Yang's expansion needs the Feynman-Kac representations to be solved, which can be hard to implement.

2.3 Risk-neutral probabilities

Let p and p_0 be the risk-neutral conditional densities underlying the true model (p) and the auxiliary model (p_0) and considering that the payoff functions are the same in both models ($d(x) = 0$). Then the two prices are

$$\begin{aligned} w(x,t) &= \int_{\mathbb{R}^d} b(y)p(y,T|x,t)dy, \\ w_0(x,t) &= \int_{\mathbb{R}^d} b(y)p_0(y,T|x,t)dy. \end{aligned} \tag{2.3.1}$$

Taking the difference of these two different prices, we find

$$w(x,t) = w_0(x,t) + \int_{\mathbb{R}^d} b(y)\Delta p(y,T|x,t)dy. \tag{2.3.2}$$

Now, it is easy to see such relation with the K-M method. First, let assume $R = 0$, $c = 0$ and $d(X) = 0$. Those assumptions set the first term in the right side of equation 2.1.6 equal to zero, the exponential of the second term equal to one, and hence the Kristensen-Mele's method relates with that of probability densities as follows:

$$\int_{\mathbb{R}^d} b(y)\Delta p(y,t|x,t)dy = \int_t^T \mathbb{E}_{x,t}\left[\delta(x(s),s)\right], \tag{2.3.3}$$

where δ has the from of eq. 2.1.5. The right-hand of equation 2.3.3 can be solved through a power series expansion as shown in equation 2.1.8, while the risk-neutral method could be solved through Riemann integral, which can be cumbersome to implement (particularly when dimensionality of our model grows, the implementation becomes harder).

2.4 Greeks

Once equation 2.1.8 is given explicitly, the partial derivatives with respect to the variables of interest can be easily derived. Hence, the *k-th* order derivative of $w(x,t)$ is given by:

$$\frac{\partial w_N(x,t)}{\partial x^k} = \frac{\partial w_0(x,t)}{\partial x^k} + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n^{(k)}(x,t) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n^{(k)}(x,t), \tag{2.4.1}$$

where

$$d_n^{(k)}(x,t) = \frac{\partial^k d_n(x,t)}{\partial x^k},$$
$$\delta_n^{(k)}(x,t) = \frac{\partial^k \delta_n(x,t)}{\partial x^k}.$$

These terms have been developed until second order of derivative in [3].

3 Pricing Results

The main goal of the next sections will be to validate the K-M method through pricing results. We will analyze the method and the sensitivity under different contexts and we will depict the error for each context. It is also quite interesting how many terms we need in order to obtain good results.

First of all, we set our scenario for the auxiliary and the true models. In all our cases, the auxiliary model will be the Black-Scholes, while the true model will be selected with different purposes, but all of them adding new more information on a sequential basis. First we will consider a richer model than the Black-Scholes model, which adds new information in terms of stochastic volatility. The selected model has been the CEV model (analyzed in [4]) which has a similar process than the Black-Scholes model, but including a stochastic process for the volatility. This model has already been studied by K-M. Our objective will be to check our methodology and study new extreme scenarios that have not been studied by K-M, for example, when the option price is very close to zero.

After we test the approximation from Black-Scholes to CEV, a new richer model will be presented. We have considered the Merton model with stochastic volatility as our last experiment, which contains information about Jump Diffusion and therefore add new more information than Black-Scholes and even remaining the stochastic volatility information that was introduced by the CEV model. Although this method has been introduced in a theoretical framework by K-M, they have not studied this model empirically. We will study (empirically) this case under the theoretical framework proposed by K-M and under a new alternative that we will develop.

Once the structure has been explained, in the next section, our benchmark will be the auxiliary model and those properties that are related with our purpose. After the auxiliary model's presentation, we will test the K-M approach through the models we have mentioned above.

3.1 The auxiliary model

In all our futures scenarios, we will develop the K-M approach through the same auxiliary model, the Black-Scholes model (see [1]). We have chosen this one as the auxiliary model, because the Black-Scholes model provides a closed-form pricing formulas for a large number of derivatives and therefore this start point could be extended to any other derivatives and models.

The Black-Scholes model is a one-dimensional factor model where the state variable is the stock price and is the solution to

$$\frac{dS(t)}{S(t)} = rdt + \sigma_0 dW(t), \tag{3.1.1}$$

where $W(t)$ is a standard Brownian motion under the risk neutral probability, r and σ_0 are the short-term rate and the volatility respectively, and both taken to be constant.

The infinitesimal operator related to this model is given by

$$\hat{L}_0 = \frac{\partial}{\partial t} + rS(t)\frac{\partial}{\partial S} + \frac{1}{2}\sigma_0^2 S(t)^2 \frac{\partial^2}{\partial S^2}. \tag{3.1.2}$$

In addition to that, in the whole paper, we will refer to $w_0(x,t)$ as the price given by the auxiliary price, which should be known in closed form.

3.2 Models with stochastic volatility

3.2.1 Theoretical framework

In this subsection we will consider the CEV model (see [2]) as the true model and Black-Scholes remains as the auxiliary one. The CEV model is a 2-dimensional multi-factor model, so that x(t) is in this case, the vector $[\,S(t)\;\; v(t)\,]$. The explicit formula that follows the CEV model satisfies

$$\frac{dS(t)}{S(t)} = rdt + \sqrt{v(t)}dW(t), \tag{3.2.1}$$

where $W(t)$ is a standard Brownian motion under risk-neutral probability, r is the short-term rate (taken to be constant) and the variance follows the below stochastic process

$$dv(t) = \kappa(\alpha - v(t))dt + \omega|v(t)|^{\xi}dW_v(t), \tag{3.2.2}$$

where $W_v(t)$ is a standard Brownian motion correlated with $W(t)$ with instantaneous correlation ρ, $\xi < 0$ is the CEV parameter and (κ, α, ω) are three additional constants.

In terms of the infinitesimal generator, the CEV model has an operator which satisfies

$$\begin{aligned}\hat{L} = \frac{\partial}{\partial t} + S(t)\cdot r\frac{\partial}{\partial S} + \kappa(\alpha - v(t))\frac{\partial}{\partial v} + \frac{1}{2}S(t)^2 v(t)\frac{\partial^2}{\partial S^2} + \\ + \frac{1}{2}\omega^2 v(t)^{2\xi}\frac{\partial^2}{\partial v^2} + \rho\omega S(t)v(t)^{\xi+1/2}\frac{\partial^2}{\partial S\partial v}.\end{aligned} \tag{3.2.3}$$

The above expression redefines the "mispricing function", δ, (see eq. 2.1.4) which is given by

$$\delta(x, v, t; \sigma_0) = \frac{1}{2}(v - \sigma_0^2)S^2\frac{\partial^2 w_0}{\partial S^2}, \tag{3.2.4}$$

since $w_0(S, v, t; \sigma_0)$ is known in closed form, we can compute directly the mispricing function. As we mentioned before, the recursive function remains the same

$$\delta_n(x, t) = \hat{L}\delta_{n-1}(x, t) - R(x, t)\delta_{n-1}(x, t), \tag{3.2.5}$$

and the initial condition is $\delta_0 = \delta(x, t)$.

3.2.2 Pricing results

Once the code has been developed, the validation of the Kristense-Mele approach we will be analysed through different ways.

S	CEV MC	Kristensen-Mele N=0	Kristensen-Mele N=4
950	57.5661	58.0455	57.7312
960	62.1077	62.5759	62.2049
970	66.8486	67.3055	66.8835
980	71.7846	72.2332	71.7666
990	76.9181	77.3568	76.8532
1000	82.2477	82.6742	82.1418
1010	87.7666	88.1826	87.6300
1020	93.4722	93.879	93.3153
1030	99.3647	99.7601	99.1943
1040	105.443	105.822	105.263
1050	111.702	112.061	111.518

Table 1: European call option prices for the CEV model computed through Monte Carlo simulations and through the K-M method. Variations on stock price. The volatility parameter is set equal to $v(t) = 0,5172$.

Tables 1 and 2 compare European call option prices for the CEV model computed through Monte Carlo simulations and through the K-M method. European call options have a strike of $K = 1000$, time to expiration equals to one month ($T - t = 1/12$), and the parameters of the equations 3.2.1 and 3.2.2 are set equal to $\kappa = 0.1465$, $\alpha = 0,5172$, $\omega = 0.5786$, $\xi = 0.6$, $\rho = -0.0243$, and $r = 0$. Panel 1 produces option prices when the initial value of the instantaneous variance equals to $v(t) = \alpha$. Panel 2 produces option prices when the initial value of the underlying stock price equals $S(t) = 1000$. Both panels provide the prices of our method for different number of leading terms of eq. 2.1.8.

Tables 1 and 2 also show how the prices accurate as we reach high terms ($N = 4$) of our expansion (see eq. 2.1.8). As we can see, the price given by the Kristensen-Mele's method tends to converge to the true price[1] (CEV). These tables also show how the K-M approach captures the variations through different variables of interest, in our case, the volatility $v(t)$ and the stock price $S(t)$ (in both cases show a high level of confidence).

Tables 1 and 2 also show how the prices accurate as we reach high terms ($N = 4$) of our expansion (see eq. 2.1.8). As we can see, the price given by the K-M method tends to converge to the true price[2] (CEV). These Tables also

[1]The option price computed through Monte Carlo simulation have been designed with 50000 number of paths.

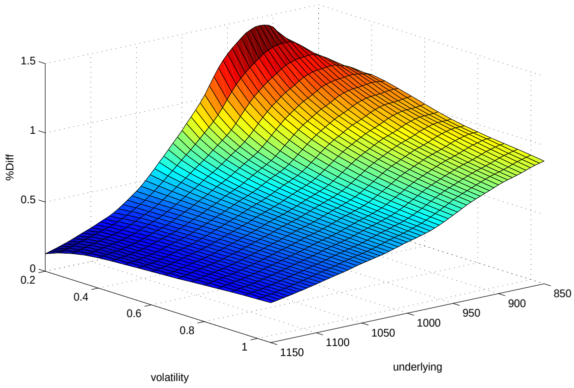
[2]The option price computed through Monte Carlo integration have been designed so that we

show how the K-M approach captures the variations through different variables of interest, in our case, the volatility $v(t)$ and the stock price $S(t)$ (in both cases show a high level of confidence).

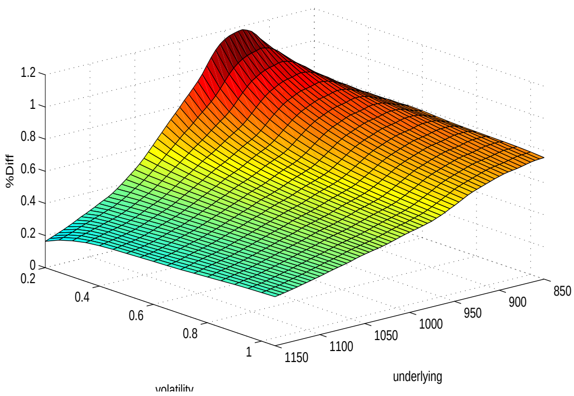
On the other hand, figures 1a, 1b and 1c show the joint variation of the percentage error of the K-M method respect to the stock price $S(t)$ and respect to the variance $v(t)$. In those figures, the behavior of the percentage pricing error is quite different for option with *in-the-money* stock price from those with *out-of-the-money* stock price. First, attending to *in-the-money* stock prices, we conclude that the K-M's method produces good results in a large range of volatility, being the percentage pricing errors around 0.5% for $N = 0$, a percentage error of 0.4% for $N = 1$ and lower than 0.1 for $N = 4$. In contrast to *in-the-money* options, the options with *out of the money* stock price produces percentage errors around 1% when $N = 0$, a percentage error of 0.8% when $N = 1$ and 0.5% when $N = 4$. A special case, which has not studied by [3], is when we attend to *far-out-of-the-money* options (stock prices between 850 and 950). The *further-out-of-the-money* options produces higher percentage errors, specially when the volatility is close to zero. It happens because the option prices are very close to zero and due to the definition of the percentage error.

From these results, we might confirm that the K-M approach produces a good closed pricing formula and we have also proved that the $\hat{L}$ operator add new information about the true model as the operator iterates upon the mispricing function so that only a very few terms of eq. 2.1.8 are needed to provide quite accurate approximations to the unknown price.

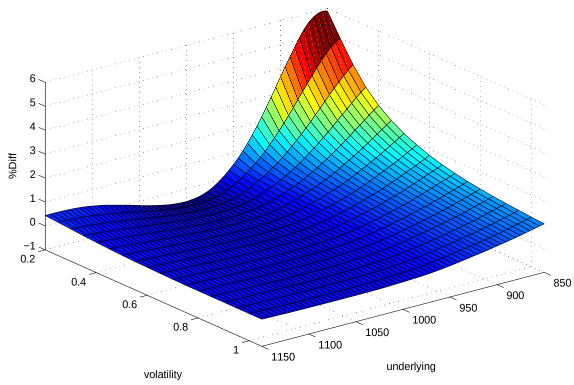
keep less than 0.5% discrepancy between the price computed through Monte Carlo integration and the prices obtained through Fourier transforms.



(a)



(b)



(c)

Figure 1: Joint variation of the percentage error of the Kristensen-Mele's method with respect to the stock price $S(t)$ and to the variance $v(t)$ when $N = 0$ (figure 1a), $N = 1$ (figure 1b) and $N = 4$ (figure 1c).

v	CEV MC	Kristensen-Mele N=0	Kristensen-Mele N=4
0.1	36.006	36.4058	35.5547
0.2	51.0693	51.4676	50.7666
0.3	62.6054	63.0128	62.389
0.4	72.3192	72.7357	72.1618
0.5	80.8676	81.2927	80.755
0.6	88.5887	89.0208	88.511
0.7	95.6825	96.1201	95.6327
0.8	102.279	102.721	102.252
0.9	108.470	108.914	108.462
1.0	114.320	114.766	114.327
1.1	119.885	120.326	119.899

Table 2: European Call Option prices for the CEV model computed through Monte Carlo and through the K-M method. Variations in volatility. The stock price is set to $S(t) = 1000$.

3.3 Models with jumps and stochastic volatility

Finally, a richer model will be studied. The K-M approach will be tested with a model which includes jump diffusion as well as stochastic volatility.

3.3.1 Theoretical framework

The jump-diffusion model with stochastic volatility that will be analyzed is the Merton jump-diffusion (76) model (see [5] and [6]) and the stochastic volatility follows a CEV process.

$$\frac{dS(t)}{S(t)} = (r - \lambda \bar{j})dt + \sqrt{v(t)}dW(t) + jdP(t), \tag{3.3.1}$$

where $W(t)$ is a Brownian motion, $dP(t)$ is a Poisson process with a bounded intensity parameter of λ, j is a random bariable with probability of measure on $[-1, \infty)$, density p and expectation $\bar{j}$. The stochastic process that our volatility follows is a CEV model, that is, $v(t)$ evolves according to

$$d(v) = \kappa(\alpha - v(t))dt + \omega|v(t)|^{\xi}dW_v(t), \tag{3.3.2}$$

where $W_v(t)$ is a Brownian motion correlated with $W(t)$ with correlation ρ, $\xi > 0$ is the CEV parameter and (κ, α, ω) are three additional constants.

This model has been developed through two similar ways, so that we could test the robustness of the K-M approach and the limits of this method.

- **First method**

  First, we set the infinitesimal generator related to the Merton-CEV model as K-M suggests in [3]. In this case, the $\hat{L}$ operator is given by the following expression

$$\hat{L}^J f(x,v,t) = \hat{L} f(x,v,t) + \lambda \int_{-1}^{\infty} [f(x(1+j),v,t) - f(x,v,t)] p(dj), \quad (3.3.3)$$

  where the $\hat{L}$ operator takes the form of the equation 3.2.3.

  Since eq. 3.3.3 split the expression of the $\hat{L}^J$ operator into the $\hat{L}$ operator, eq. 3.2.3 (which contains stochastic volatility information) and a second term (which contains jump-diffusion information through the integral), our mispricing function function is simply the eq. 3.2.4, but adding the right hand term of eq. 3.3.3. The mispricing function is now

$$\delta(x,v,t;\sigma_0) = \frac{1}{2}(v-\sigma_0^2)S^2 \frac{\partial^2 w_0}{\partial S^2} + \lambda \int_{-1}^{\infty} [w_0(x(1+j),v,t) - w_0(x,v,t)] p(dj), \quad (3.3.4)$$

  where δ_n follows the recursive function

$$\delta_n(x,t) = \hat{L}^J \delta_{n-1}(x,t) - R(x,t)\delta_{n-1}(x,t). \quad (3.3.5)$$

- **Second method**

  As a second method, we present an alternative to the previous method which is related to the $\hat{L}$ operator through the Itô's lemma, and takes the form of:

$$\begin{aligned}\hat{L}^J &= (r - \lambda\bar{j})S(t)\frac{\partial}{\partial S} + \kappa(\alpha - v(t))\frac{\partial}{\partial v(t)} + \frac{\partial}{\partial t} + \\ &+ \frac{1}{2}S(t)^2(v(t) + j^2\lambda)\frac{\partial^2}{\partial S^2} + \frac{1}{2}\omega^2|v(t)|^{2\xi}\frac{\partial^2}{\partial v^2} + \\ &+ \rho S(t) v(t)^{1/2}\omega|v(t)|^{\xi}\frac{\partial^2}{\partial S \partial v}, \end{aligned} \quad (3.3.6)$$

  where j^2 is $j^2 = E[j^2] = var[j] + E[j]^2 = \sigma_j^2 + \bar{j}^2$.

  Once the explicit expression for the L operator is known and considering Black-Scholes as the auxiliary model as we did in previous section, we are able to compute the recursive function

$$\delta_n(x,t) = \hat{L}^J \delta_{n-1}(x,t) - R(x,t)\delta_{n-1}(x,t), \quad (3.3.7)$$

where $\hat{L}^J$ behaves liking in equation 3.3.6 and the mispricing function is the solution to

$$\delta_0 = (\hat{L} - \hat{L}_0)w_0 = -\lambda\bar{j}S(t)\frac{\partial w_0}{\partial S} + \kappa(\alpha - v(t))\frac{\partial w_0}{\partial v(t)} +$$

$$+\frac{1}{2}S(t)^2(v(t) + (\sigma_j^2 + \bar{j}^2)\lambda - \sigma_0^2)\frac{\partial^2 w_0}{\partial S^2} + \frac{1}{2}\omega^2|v(t)|^{2\xi}\frac{\partial^2 w_0}{\partial v^2} +$$

$$+\rho S(t)v(t)^{1/2}\omega|v(t)|^{\xi}\frac{\partial^2 w_0}{\partial S \partial v},$$

where $\hat{L}_0$ takes the form of 3.1.2. Because of the Black-Scholes model does not depend on the v parameter (it depends on the σ_0), all the partial derivatives with respect to v are equal to zero, that is $\partial w_0/\partial v = \partial^2 w_0/\partial v^2 = \partial^2 w_0/\partial S v = 0$. Under these assumptions, the mispricing function formula for the Merton-CEV model is given by

$$\delta_0 = -\lambda\bar{j}S(t)\frac{\partial w_0}{\partial S} + \frac{1}{2}S(t)^2\left(v(t) + (\sigma_j^2 + \bar{j}^2)\lambda - \sigma_0^2\right)\frac{\partial^2 w_0}{\partial S^2}. \qquad (3.3.8)$$

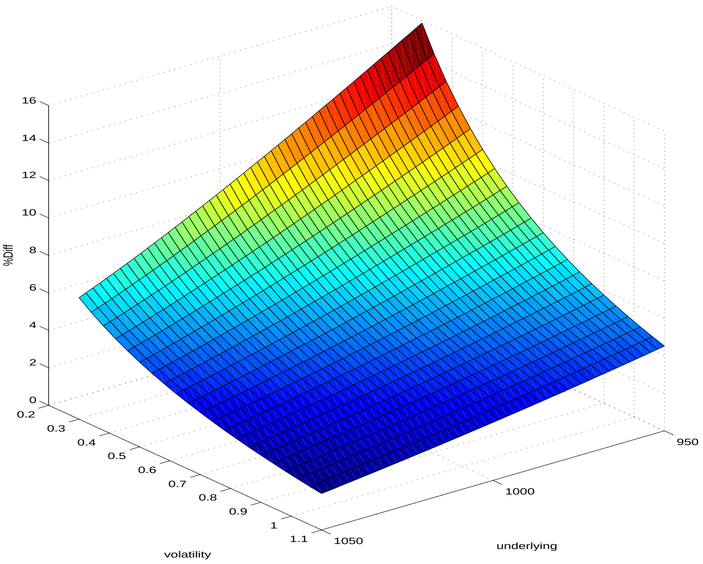
Once our objectives have been set, we will compare our method with that of the K-M approach.

3.3.2 Pricing results

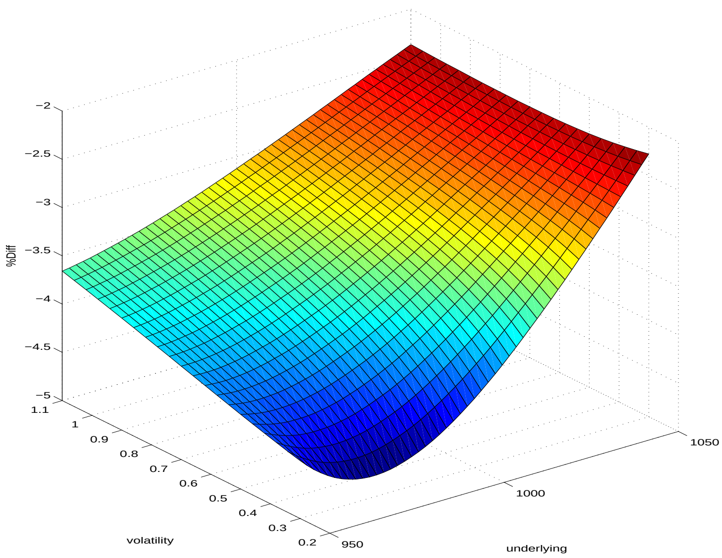
As in Section 3.2.2, we will analyze the K-M approach through different methods. First we will depict the percentage approximation error as a function of the stock price and the volatility in 3D plots. Later on, a table will provide explicit prices given by our method for many different leading terms. This Table compares the performance of our approximations with those of Merton with stochastic volatility.

In figures 2 and 3, options are European call options. They have a strike price of $K = 1000$, time to maturity equals one month ($T - t = 1/12$), and the parameters of equations 3.3.1 and 3.3.2 are set equal to $\kappa = 1$, $\omega = 0.5786$, $\xi = 0.6$, $\rho = -0.5$, $\lambda = 1$, $\bar{z} = 0$ and $\sigma_z = 0.4$. Where z is distributed as a Normal distribution and therefore $j = (\exp(z) - 1)$ is distributed with probability of measure $[-1, \infty)$. We have also set α equals to the instantaneous variance $v(t)$ and the volatility of the Black-Scholes model equals to $\sigma_0 = \sqrt{v}$.

As we can see, Figures 2 and 3 depict the approximation errors resulting from our method for the two alternatives we presented in the previous section. Once again, both methods provides better results as we add new more terms. In this case, the results could only be tested until first leading term, and hence, we can not confirm a strong evidence of convergence to the Merton with stochastic

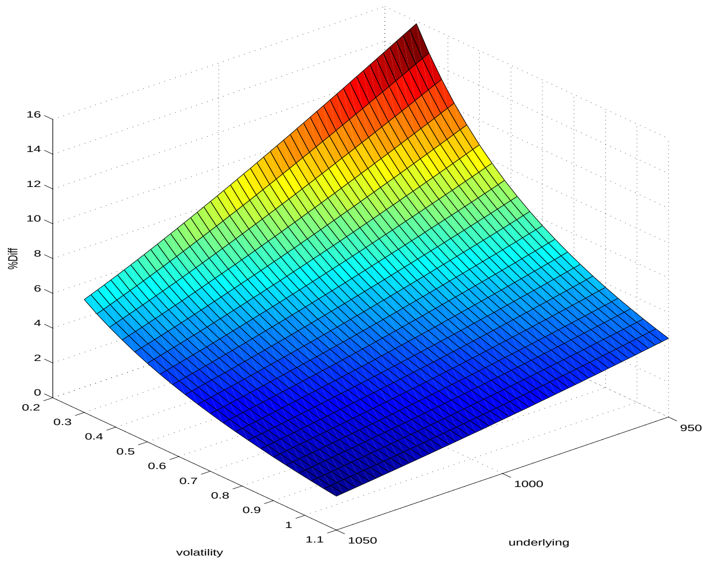


(a)

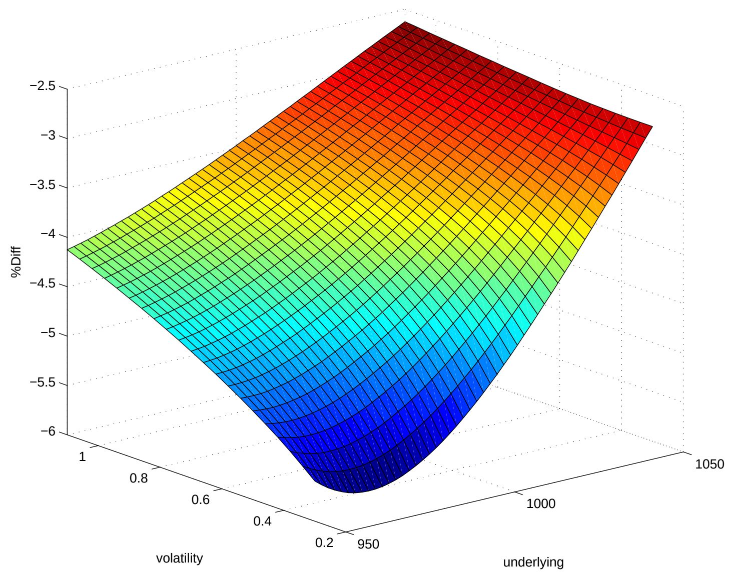


(b)

Figure 2: Joint variation of the percentage error of our method respect to the stock price $S(t)$ and respect to the variance $v(t)$ when $N = 0$ (figure 1a), $N = 1$ (figure 1b).



(a)



(b)

Figure 3: Joint variation of the percentage error of our method respect to the stock price $S(t)$ and respect to the variance $v(t)$ when $N = 0$ (figure 1a), $N = 1$ (figure 1b).

S	Merton-CEV MC	Our alternative N=1	Kristensen-Mele N=1
950	59,5270	57,0572	56,5877
960	64,1573	61,4851	61,0106
970	68,9898	66,1588	65,6809
980	74,0289	71,078	70,5982
990	79,2761	76,2412	75,7609
1000	84,7222	81,6453	81,166
1010	90,3661	87,2864	86,8096
1020	96,2039	93,1595	92,6867
1030	102,235	99,2588	98,7912
1040	108,456	105,577	105,117
1050	114,854	112,108	111,655

Table 3: European call option prices for the Merton-CEV model computed through Monte Carlo and through the K-M method. Variations in stock price. The variance parameter is set to $v(t) = 0.58$.

volatility prices. Nevertheless, the percentage errors are substantially cutting down when only a new one leading term is added.

With the purpose of explaining why we only could develop our methods until first order, we might take a look to the number of partial derivatives our $\hat{L}^J$ operator contains in eq. 3.3.3 as well as in eq. 3.3.6. These partial derivatives have been computed through central finite differences until fourth order because the second order approximation bounded us into errors (see A.6). Attending to the central finite differences method until fourth order, each single partial derivatives iterates four times upon the recursive function, each second partial derivative iterates five times, the cross partial derivatives iterates eight times and the independent term iterates once. All in all, when the recursive function acts, it makes our method cumbersome, and in some cases, when N becomes higher, we are limited by the computer's memory. For instance, if we want to develop the eq. 2.1.8 until first order, we calculate 27 evaluations over the mispricing function, for $N = 2$ we need $27^2 = 729$ and in general, for $N = i$ we need 27^i evaluations. Under this assumptions, with only a very few leading terms in our eq. 2.1.8, we reach millions of evaluations.

In addition to that, eq. 3.3.6 and the mispricing function (see eq. 3.3.8) contain integrals, which needs to be calculated by numerical method (in our case computed by the rectangle rule). Although the integral term seems to be a simple integral, we might attend when $\hat{L}^J$ acts over the recursive function in eq. 3.3.7

v	Merton-CEV MC	Our Alternative N=1	Kristensen-Mele N=1
0,3	72,0123	68,9270	68,3378
0,42	77,7557	74,7045	74,1695
0,54	83,0434	79,9767	79,4849
0,66	87,9711	84,8695	84,4132
0,78	92,6028	89,4613	89,0349
0,90	96,9868	93,8053	93,4045
1,02	101,159	97,9398	97,5613
1,10	103,838	100,594	100,229

Table 4: European call option prices for the Merton-CEV model computed through Monte Carlo simulations and through the Kristensen-Mele's method. Variations on volatility. Parameter value of stock price is $S(t) = 1000$.

as well as when it does over the mispricing function (see eq. 3.3.8). When the integral needs to be calculated, we have to choose between high precision, which means a high number of iterations, or either a lower number of iterations, which might lead to numerical errors.

In both scenarios we built, we have tried to test the K-M approach, and the results are quite interesting. Although both methods seems to reduce the percentage price errors as we add new more terms, our method performs better than K-M suggestion, unless when first order of approximation are developed.

On the other hand, Tables 3 and 4 compares the performance of the K-M approach. The option prices for Merton-CEV model have been computed through Monte Carlo simulation. Table 3 provides option price variations respect to the stock price, while table 4 does respect to the variance variable.

In both Tables, our alternative through eq. 3.3.6, provides better results than those of K-M and the percentage difference error have been reduced when options are *in-the-money* as well as *far-out-of-the-money*. As we saw with the CEV model, the percentage error tends to increase as volatility tends to lower values, and also as prices go *far-out-of-the-money* (due to the low value of price).

4 Conclusions

This paper has analyzed the K-M approach and illustrated its implementation for different derivatives and models.

In our first case, where the CEV model was chosen as true model, we have studied how new terms in the series expansion of eq. 2.1.8 add new more information about the true model and they capture the stochastic behavior of our objective model. We have also confirmed that we can obtain quite accurate approximations to the unknown price, with only a very few additional corrective terms to eq. 2.1.8.

For this scenario, with four leading terms, the K-M approach includes enough additional information about the stochastic volatility and also seems to be the best choice in terms of computational time.

We have also shown that the percentage errors increase when options are very *far-out-of-the-money*. This behavior is due to the definition of the percentage error and the fact that prices are very close to zero.

In our second scenario, with the Jump Diffusion model and stochastic volatility as true model, the results are quite interesting, despite we have been forced to face off some numerical issues because the numerical computation becomes harder.

One of them was the error propagation through the iterations when the recursive function is applied. As central finite differences until second order does not produce accurate prices and the errors propagate through the iterations, we have extend our approximation to derivatives through central finite differences until fourth order. They have minimized the propagation of the errors and prices seem to converge to the true price.

Because of computational limitations, the computation of the integral in eq. 3.3.3 does not allow us to calculate more terms in the series expansion than the two first leading terms. Although we only could develop our results until the two first leading term, some interesting conclusions could be obtained. As we add these two leading terms (δ_0 and δ_1), prices seem to converge to the price of interest.

We have introduced an alternative to the approximation suggested by the K-M approach. This alternative is also computationally cumbersome but we do not deal with the propagation of errors through the iterations of the integral. Our alternative decreases the computational time with respect to the K-M suggestion. Since we can avoid the errors related to the integral, the results might be more accurate.

In this section, we were computationally limited and the choice of another model as the auxiliary model could make it lighter. It might be for instance, the Merton model, which only needs to add information about stochastic volatility.

A Appendix

A.1 The infinitesimal generator of the option price

Let our model be associated with equations 3.2.1 and 3.2.2. Considering the Itô's lemma and let $w(t) = w(S, v, t)$ be the option price function, then the infinitesimal process of the option price follows

$$dw(t) = \big[S(t) \cdot r\frac{\partial w}{\partial S} + \kappa(\alpha - v(t)) + \frac{1}{2}S(t)^2 v(t)\frac{\partial^2 w}{\partial S^2}$$
$$+\frac{1}{2}\omega^2 v(t)^{2\xi}\frac{\partial^2 w}{\partial v^2} + \rho\omega S(t)v(t)^{\xi+1/2}\frac{\partial^2 w}{\partial S\partial v} + \frac{\partial w}{\partial t}\big]dt+$$
$$+S(t)\sqrt{v(t)}\frac{\partial w}{\partial S}dW(t) + \omega|v(t)|^{\xi}\frac{\partial w}{\partial v}dW_v(t).$$

In operator's notation holds:

$$dw(t) = \hat{L}wdt + \hat{M}wdW(t) + \hat{N}wdW_v(t), \tag{A.1.1}$$

where the $\hat{L}$ operator is considered as the Heston's infinitesimal generator.

Any other model could be proved in the same way.

A.2 Option Pricing Differential Equation

Let π be our portfolio and assuming it is a hedge portfolio composed by options of interest and the underlying

$$d\pi = k_1 dw + k_2 dS. \tag{A.2.1}$$

If we suppose Heston as our model and therefore

$$dS = rSdt + \sqrt{v(t)}SdW(t).$$

As shown in the previous section

$$dw(t) = \hat{L}wdt + \hat{M}wdW(t) + \hat{N}wdW_v(t).$$

Replacing dS and $dw(t)$ from the equation A.2.1, and applying discretization methods,

$$\Delta\pi = [k_1 Lw + k_2 rS]\,\Delta t + \left[k_1 Mw + K_2\sqrt{v(t)S}\right]\Delta W(t) + k_1 Nw\Delta W_v(t),$$

where we set $k_1 = -1$. Assuming that risk can be protected against with a hedge,

$$k_2 = \frac{Mw}{\sqrt{v(t)}S} \equiv w_s. \tag{A.2.2}$$

Assuming our portfolio is a hedge portfolio, then $\Delta\pi = r\pi\Delta t$, and also assuming that the diffusion term of the variance can not be hedge, we get:

$$\begin{aligned} r\left[-w + w_s S\right] &= -Lw + w_s r S, \\ Lw(S,v,t) &= rw(S,v,t). \end{aligned} \tag{A.2.3}$$

The differential equation has been proved for the CEV model. Any other model follows the same process.

A.3 Mispricing function

As we obtained the differential equation A.2.3, for any other models

$$L_0 w_0(S,v,t) - rw_0(S,v,t) = 0.$$

Taking the difference between the above equation and the equation A.2.3, it is trivial to find the explicit form of the mispricing function

$$Lw - rw - (L_0 w_0 - rw_0) = L\Delta w - r\Delta w + (L - L_0)\, w_0 = 0,$$

and therefore

$$L\Delta w(S,v,t;\sigma_0) - r\Delta w(S,v,t;\sigma_0) + \delta(S,v,t;\sigma_0) = 0, \tag{A.3.1}$$

where $\Delta w = w(S,v,t) - w_0(S,t;\sigma_0)$ and $\delta = (L - L_0) w_0$.

A.4 The Feynman-Kac Theorem

Stochastic differential equations are closely linked to partial differential equations. The Feynman-Kac theorem is key to Financial modeling.

The Feynman-Kac Representation 1 *Let $f \in C^{1,2}$ be a solution to the deterministic partial differential equation:*

$$\frac{\partial}{\partial t} f(t,x) + \mu(t,x)\frac{\partial}{\partial x} f(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2} f(t,x) - v(x)f(t,x) + g(t,x) = 0, \tag{A.4.1}$$

for all $(t,x) \in [0,T]$ subject to the terminal condition $f(T,x) = h(x)$. Let the solution to the stochastic differential equation:

$$dX_t = \mu(s,X_s)ds + \sigma(s,X_s)dW_s,$$

where $s \in [t,T]$ and $X_t = x$. Then,

$$\begin{aligned} f(T,x) = E\left[exp\left(-\int_t^T v(s,X_s)ds\right) h(X_T) | X_t = x\right] + \\ + \int_t^T E\left[exp\left(-\int_t^s v(u,X_u)du\right) g(s,X_s) | X_t = x\right] ds \end{aligned} \tag{A.4.2}$$

A.5 Yang's expansion

For the purpose of simplifying these section, we will assume $R(x,t) = c(x,t) \equiv 0$ and that the payoff function are the same for both models (implying $d(x) = 0$). Yang's expansion starts with a base-model, which satisfies $L_0 w^{(0)} = 0$. Using the equation A.2.3, the unknown price can be written as the solution to

$$0 = Lw(x,t) \equiv L_0 w(x,t) + (L - L_0)w(x,t). \tag{A.5.1}$$

The fact that $L_0 w^{(0)} = 0$ allows us to rewrite the above equation to a new one

$$L_0 \Delta w(x,t) + (L - L_0)w(x,t) = 0, \tag{A.5.2}$$

and its solutions satisfies:

$$w(x,t) = w^{(0)}(x,t) + \int_t^T \mathbb{E}^0_{x,t}\left[(L - L_0)w(x(u),u)\right] du. \tag{A.5.3}$$

A.6 Finite Difference

The approximation of derivatives by finite differences plays a central role in finite difference methods for the numerical solution of differential equations. We will develop the central finite differences for first and second derivatives until fourth order.

1. Finite differences for first derivatives.

   Central finite difference scheme second order:

   $$\frac{\partial f(x)}{\partial x} = \frac{f(x(1+h)) - f(x(1-h))}{2xh} + \epsilon(2).$$

   Central finite difference scheme fourth order:

   $$\frac{\partial f(x)}{\partial x} = \frac{-f(x(1+2h)) + 8f(x(1+h)) - 8f(x(1-h)) + f(x(1-2h))}{12xh} + \epsilon(2).$$

2. Finite differences for second derivatives.

   Central finite difference scheme second order:

   $$\frac{\partial^2 f(x)}{\partial x^2} = \frac{f(x(1+h)) - 2f(x) + f(x(1-h))}{x^2h^2} + \epsilon(4).$$

   Central finite difference scheme fourth order:

   $$\frac{\partial f(x)}{\partial x} = \frac{-f(x(1+2h)) + 16f(x(1+h)) - 30f(x)}{12x^2h^2} + $$
   $$+ \frac{16f(x(1-h)) - f(x(1-2h))}{12x^2h^2} + \epsilon(4).$$

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