CORRELATED DEFAULT PROBABILITY IN CVA

Fernando Villalba Chaves

Trabajo de investigación 019/015

Master en Banca y Finanzas Cuantitativas

Tutores: Dr. Manuel Moreno
Dr. Federico Platania

Universidad Complutense de Madrid

Universidad del País Vasco

Universidad de Valencia

Universidad de Castilla-La Mancha

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July 1, 2015

Abstract

This master thesis introduces a method to estimate the probability of default based on debts and assets for a company. the intuitionis that when the assets value fall behind that of debts the company defaults. We start considering constant debts and stochastic assets and assume no correlation between both. In this framework, we derive closed-form expressions for this probability. Then we introduce correlation in assets and bilateral CVA. In this case we obtain some results by simulations. We study the sensitivity of these results to different parameters. Finally, we analyze the case of stochastic debts.

CVA; Probability of default; Default Correlation; Monte Carlo simulations;

*Special thanks to Manuel Moreno and Federico Platania for the orientation, motivation and observations. Manuel Moreno is from University of Castilla La-Mancha, Department of Economic Analysis and Finance, Cobertizo San Pedro Mártir s/n, 45071 Toledo, Spain. E-mail: manuel.moreno@uclm.es. Federico Platania is from Universidad Complutense de Madrid, Department of Quantitative Economics, Madrid, Spain, and University of Liège, HEC Management School, Department of Finance, Liège, Belgium. E-mail: federico.platania@ulg.ac.be.

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1 Introduction

Nowadays the computation of unilateral or bilateral CVA is very common, but it has some restrictions. One of the most problematic parts of this calculation is the probability of default. In the literature, this probability of default has been calculated in diferent ways. Collin-Dufresne (2002) [3] present a deterministic process to the hazard rate aiming to model the time to default. Brigo and Chourdakis (2008) [2] present a stochastic process called CIR++ . It can be hard to calibrate because of the low market liquidity in some CDS markets. Another problem is that ignoring correlation usually overvalues or undervalues this probability. Brigo-Alfonsi (2005) [1] study in detail these limitations. Jarrow-Turnbull (1995) [4] model the state of solvency instead of the time to default as in the previous cases.

In this master thesis we will develop a model for the probability of default that is in the spirit of that introduced in Merton (1974) [5]. We assume that the company has a certain amount of coupon-zero debt with maturity T. If at any time the value of the company assets falls below this debt level, the firm defaults. We will assume the assets follow a Geometric Brownian Motion (GBM). Under this setup, we will derive some closed-form expression for the probability of default, and the price of a zero-coupon bond with credit risk. Generalizing this framework to allow correlation makes it more realistic but requires using Monte Carlo simulations. We can introduce correlation using the Brownian Motions, so the default of two companies can be correlated through their assets. This will give us better estimations of the probability of default, giving us different values for the same probability according to this correlation. We can go one step further and relax the hypothesis ofconstant debts. Continuing with the same idea of assets/debts, we can use stochastic barriers. We will study three different models for our barriers: the specification in Vasicek (1977), Geometric Brownian Motion, and Arithmetic Brownian Motion.

This master thesis is organized as follows. Section 2 intoduces the definitions and formulas that we are interested in. Section 3 presents the model for the probability of default in unilateral CVA and derives some closed-form expressions for this probability and the price of a zero-coupon bond with credit risk. Section 4 extends the model for the probability of default in bilateral CVA introducing correlation with different models. In section 5 we summarize the main results and conclude.

2 Previous definitions

CVA (Credit Valuation Adjustemt) is the difference between the risk-free portfolio value and the true portfolio value that takes into account the possibility of a counterparty's default. If we have two counterparties A, B, and we only assume credit risk in one direction, for example, only B has credit risk, we can define the **unilateral** CVA of A as:

$$CVA_A = LGD_B \int_0^T E[V_A^+(t) \times DF(0,t)] dPD_B(0,t) \tag{1}$$

where

- LGD_B (Loss Given Default) is the loss rate of the counterparty B in case of default.
- $V_A^+(t)$ (exposure) is the positive value of the product for counterparty A in the moment t.
- $DF(0,t)$ is the discount factor in (0, t).
- dPB_B is the risk-neutral probability of default of the counterparty B.

It is usual to write it as a function of the *time to default* and the formula is:

$$CVA_A = E[1_{\{t<\tau_B<T\}} V_A^+(\tau_B) LGD_B DF(0,\tau_B)] \tag{2}$$

$$CVA_B = E[1_{\{t<\tau_A<T\}} V_B^+(\tau_A) LGD_A DF(0,\tau_A)] \tag{3}$$

where τ_A and τ_B are the time to default of A and B.

Alternatively, if both counterparties have credit risk, we can define the **bilateral** case, using CVA and DVA (Debit Valuation Adjustment) and, in this case, the CVA expressions are given by:

$$CVA_A = E[1_{\{t<\tau_B<T, \tau_B<\tau_A\}} V_A^+(\tau_B) LGD_B DF(0,\tau_B)] \tag{4}$$

$$CVA_B = E[1_{\{t<\tau_A<T, \tau_A<\tau_B\}} V_B^+(\tau_A) LGD_A DF(0,\tau_A)] \tag{5}$$

In practice, the observation time is not continuous. Then, these expressions are discretized leading to:

$$CVA_A = \sum P(t_i < \tau_B < t_{i+1}, \tau_B < \tau_A) E[V_A^+(t_i)] LGD_B DF(0, t_i) \tag{6}$$

$$CVA_B = \sum P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B) E[V_B^+(t_i)] LGD_A DF(0, t_i) \tag{7}$$

We will focus on the probability of default since CVA is the product of this probability of default (which we want to estimate) and other factors like LGD (given by the counterparty) and the discount factor (that can be estimated with a certain term structure model).

In the unilateral, case we will study:

$$P(t_i < \tau_B < t_{i+1}) \tag{8}$$

$$P(t_i < \tau_A < t_{i+1}) \tag{9}$$

while, in the bilateral case, we focus on:

$$P(t_i < \tau_B < t_{i+1}, \tau_B < \tau_A) \tag{10}$$

$$P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B) \tag{11}$$

3 Probability of default: Unilateral case

In this section we are interested in the probability of default defined in (8). First we have to model the time to default.

3.1 Probability density function

We define the "first passage time" of the Brownian Motion Z_t as:

$$\tau_m = \min\{t \geqslant 0 : Z_t = m\}$$

We are interested in $P(\tau_m \leqslant T)$, where m is the barrier. By considering complementary events, we have:

$$P(\tau_m \leqslant T) = P(\tau_m \leqslant T; Z_t \leqslant -m) + P(\tau_m \leqslant T; Z_t \geqslant -m)$$

Now we simplify each term of the sum:

- If $Z_t \leqslant -m$, then $\tau_m \leqslant T$, so we can write $P(\tau_m \leqslant T; Z_t \leqslant -m) = P(Z_t \leqslant -m)$.
- Using the reflection principle we can write $P(\tau_m \leqslant T; Z_t \geqslant -m) = P(Z_t^s \leqslant -2m + x) = P(Z_t^s \leqslant -m) = P(Z_t \leqslant -m)$

Now using these simplifications and the normal distribution of Z_t we have:

$$P(\tau_m \leqslant T) = 2P(Z_t \leqslant -m) = 2\int_{-\infty}^{-m} \frac{e^{\frac{-x^2}{2t}}}{\sqrt{2\pi t}} dx$$

The change of variable $\frac{x}{\sqrt{t}}$ leads to:

$$P(\tau_m \leqslant T) = 2\int_{-\infty}^{\frac{-m}{\sqrt{t}}} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy$$

Using the Leibniz differentiation rule (see Appendix A) we finally get the probability density function:

$$\rho(\tau_m = t) = \frac{m}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}$$

Take into account that the relation between this Brownian Motion and the underlying is not one to one. Then, their minimum value may not match, yhat is,

$$H \neq S_0 e^{(r - \frac{\sigma^2}{2})\tau_m + \sigma m}$$

In this case we define a new Brownian Motion: $\overline{Z}_t = \theta t + Z_t$, where $\theta = (r - \frac{\sigma^2}{2})\frac{1}{\sigma}$ so, we have:

$$H = S_0 e^{\sigma \overline{Z}_{\tau_m}}$$

We relate both Brownian motions using the Girsanov theorem:

$$P(\Delta\overline{Z}_t \in [x, x+dx]) = e^{\theta x - \frac{\theta^2 \Delta t}{2}} P_{Brownian}(\Delta Z_t \in [x, x+dx])$$

Then,

$$\rho(\tau_m = t) \ = \ e^{\theta m - \frac{\theta^2 t}{2}} \frac{m}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}} \ = \ \frac{m}{t\sqrt{2\pi t}} e^{-\frac{m^2+\theta^2 t^2 - 2\theta m t}{2t}}$$

Finally, the probability density function is given by:

$$\rho(\tau_m = t) = \frac{m}{t\sqrt{2\pi t}} e^{-\frac{(m-\theta t)^2}{2t}} \tag{12}$$

3.2 Probability of default

Now we are only interested in the interval [0, T]. By repeating the process described in Appendix A we obtain:

$$\begin{aligned}
P(\tau < T) &= \int_0^\infty H(T-t)\rho(\tau_m = t)dt \\
&= \int_0^T \rho(\tau_m = t)dt \\
&= \ldots = \\
&= -e^{m\theta}\left\{-e^{\theta m}\int_{-x(0)}^{-x(T)} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx - e^{-\theta m}\int_{-y(0)}^{-y(T)} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy\right\} \\
&= N(-x(T)) + e^{2m\theta}N(-y(T))
\end{aligned}$$

This formula is the probability of default $P(0 < \tau < T)$. If we want a specific formula for $P(t_i < \tau < t_{i+1})$ we can use this formula to obtain:

$$\begin{aligned}
P(t_i < \tau < t_{i+1}) &= P(\tau < t_{i+1}) - P(\tau < t_i) \\
&= N(-x(t_{i+1})) - N(-x(t_i)) + e^{2m\theta}[N(-y(t_{i+1})) - N(-y(t_i))]
\end{aligned}$$

3.3 Example: Pricing of zero-coupon bonds with credit risk

With this formula we can price a zero-coupon bond with credit risk:

$$\begin{aligned} P(0,T) &= E[FD(0,T)1_{\tau>s}] \\ &= e^{\int_0^T r_s ds} P(\tau > T) \\ &= e^{\int_0^T r_s ds}(1 - P(\tau < T)) \\ &= e^{\int_0^T r_s ds}(1 - N(-x(T)) - e^{2m\theta} N(-y(T))) \end{aligned}$$

4 Probability of default: Bilateral case

Now we have to estimate the probabilities defined in (10) - (11). If we assume independence between A and B, we can write a double integral with the product of the density functions:

$$\begin{aligned} P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B) &= \int_{t_i}^{t_{i+1}} \int_u^{\infty} \frac{m_B}{v\sqrt{2\pi v}} e^{-\frac{(m_B-\theta v)^2}{2v}} \frac{m_A}{u\sqrt{2\pi u}} e^{-\frac{(m_A-\theta u)^2}{2u}} dvdu \\ &= \int_{t_i}^{t_{i+1}} \frac{m_A}{u\sqrt{2\pi u}} e^{-\frac{(m_A-\theta u)^2}{2u}} \int_u^{\infty} \frac{m_B}{v\sqrt{2\pi v}} e^{-\frac{(m_B-\theta v)^2}{2v}} dvdu \\ &= \int_{t_i}^{t_{i+1}} \frac{m_A}{u\sqrt{2\pi u}} e^{-\frac{(m_A-\theta u)^2}{2u}} [e^{2m\theta} - e^{2m\theta} N(-x(u)) - N(-y(u))] du \end{aligned}$$

The computation of this integral requires numerical methods as it does not provide a closed-form expression. In addition, the assumption of independence is not realistic. Then, from now on, we will use Monte Carlo simulations to analyze assets and debts enhanced with correlation.

4.1 Case 1: Constant debts

We start assuming that our barrier (debts) is started as a constant level while assets are assumed to follow a Geometric Brownian Motion. We need values for the following parameters:

- X0A : Initial value for the assets of A. A positive number
- barrA : Barrier. A positive number, lower than X0A.
- sigmaA : Instantaneous volatility of A.
- corr : Different values of correlation
- r : Risk free (anual) interest rate
- T : Maturity (in years)
- X0B : Initial value for the assets of B. A positive number.
- barrB : Barrier. A positive number lower than X0B.
- sigmaB : Instantaneous volatility B.

4.1.1 Changes in the PD respect to correlation: Making A riskier than B

In this case, we choose the same parameters for A and B except the volatility. As A is more volatile than B, now, we make A riskier than B:

$$X0A = X0B = 15;\ barrA = barrB = 13;\ sigmaA = 0.3;\ sigmaB = 0.1;\ T = 1;\ r = 0.01$$

Figure 1 illustrates that the probability of default of B is strictly decreasing in the level of correlation arriving at zero. If A is riskierthan B, with perfect positive correlation, always A will default earlier, i.e. the event $\tau_B < \tau_A$ is impossible. That makes $P(t_i < \tau_B < t_{i+1}, \tau_B < \tau_A) = 0$. However, in A we have a different behaviour. We need to zoom into it to apreciate the form it has.

Figure 2 shows that this probability starts decreasing but, for null correlation in this example, we find a turning point and the probability starts to increase. This is caused when the correlation increases, the event $\tau_A < \tau_B$ is more likely than $\tau_B < \tau_A$ since it is easier for A to hit the barrier. That makes $P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B)$ increase when the correlation is increasing from a point, instead of being always decreasing because of the joint default event.

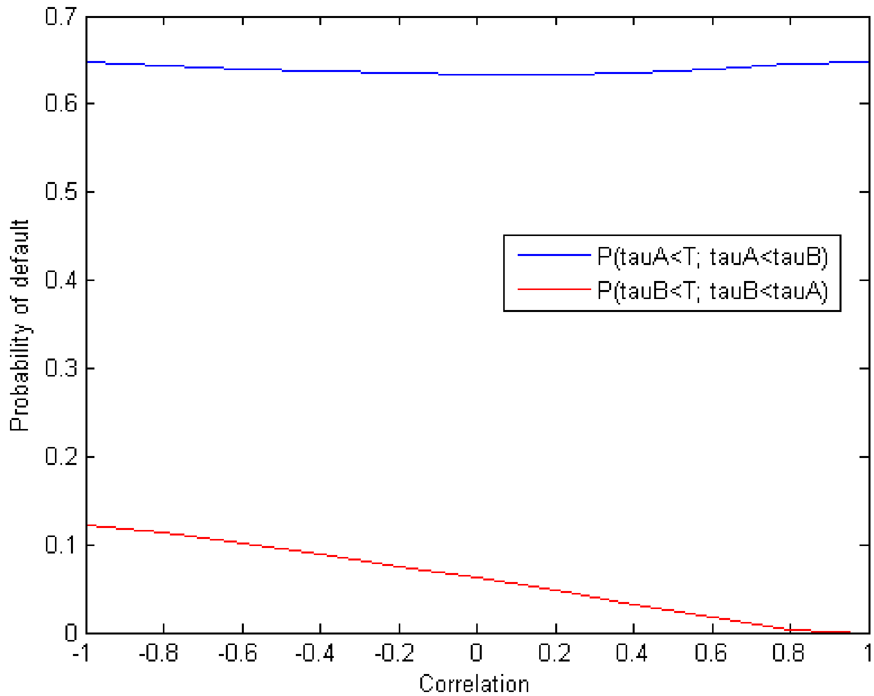

Figure 1: Probability of default of A and B with more risk in A.

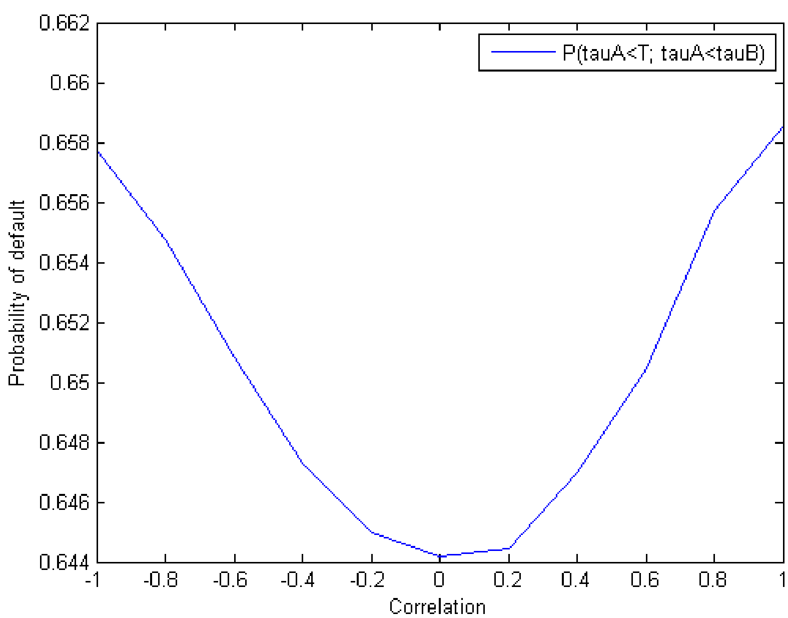

Figure 2: Probability of default of A when A has more risk than B.

4.1.2 Sensitivity analysis of the parameters and correlation

We work now with two counterparties, A and B, both with same assets and debts:

$$X0A = X0B = 15;\ barrA = barrB = 13;\ sigmaA = sigmaB = 0.2;\ T = 1;\ r = 0.01$$

For each parameter, we will consider a grid of values maintaining the other parameters constant.

- X0A : Initial value for A assets. Values between 14 and 16

  As we expected, in Figure 3, we can observe that the higher initial value, the less probability of default for A. This is because assets begin further debts, so it is harder to hit the barrier. So the effect in the probability of default for A of the initial value of A is monotone decreasing. Looking at the correlation, we can see two different regions. When the initial value is 15 or more, we have the case where this counterparty has less risk than the other one, so we have the same behaviour, monotonic decreasing, as the red line in Figure 1. When the initial value is lower than 15 we are in the other case, where the probability starts decreasing but later it increases.

  However, in Figure 4 we can see the opposite effect. If A has more risk (it happens when its initial value is lower), then the event $\tau_A < \tau_B$ is more likely than the event $\tau_B < \tau_A$ and this produces a lower probability of default for B than when the initial value for A is higher, where the event $\tau_B < \tau_A$ is more likely than $\tau_A < \tau_B$. So we now have that the effect in the probability of default for B is monotone increasing. Looking at the correlation, we again have a symmetric result. Now the monotone decreasing region is for values between 14 and 15. When the initial value is between 15 and 16 we are in the other case where the probability starts decreasing but later it increases.

  We do not analyze the sensitivity of the parameters in B as it is completely similar.

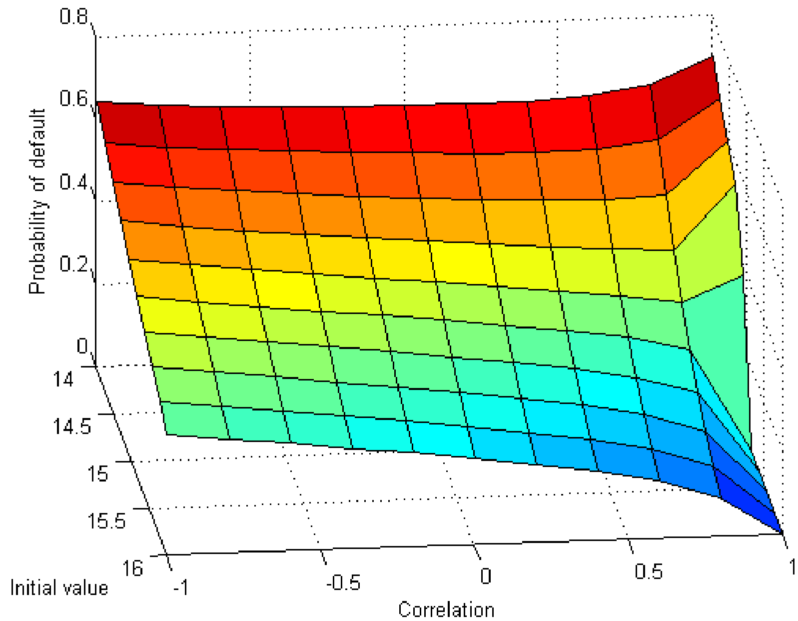

Figure 3: Initial value sensitivity for A

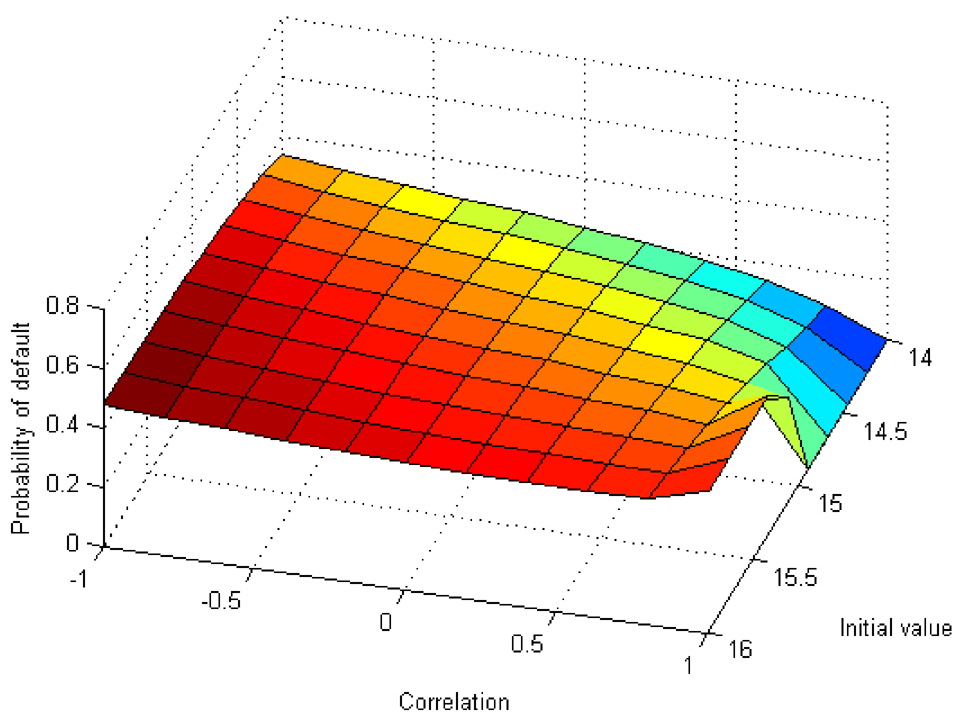

Figure 4: Initial value sensitivity for B

- barrA : Constant value for debts. Values between 12 and 14.

  We now have the same idea. If debts begin further assets, it is harder to hit the barrier for assets. But it is now in the opposite direction: when debts are lower, the distance between debts and assets is higher. As we can see in Figure 5, the effect in the probability of default of the debts value is monotone increasing.

  Looking at the correlation, again we can see two different regions. When the debts value is 13 or less, we have the case where this company has less or equal risk than the other one, so we have the same behaviour than the red line in Figure 1, monotonically decreasing. When the initial value is lower than 13 we are in the other case, it starts decreasing but then increases at some point.

  For B we have symmetric results as we explained when we studied the sensitivity to the initial value of the assets of A, so we will not repeat it.

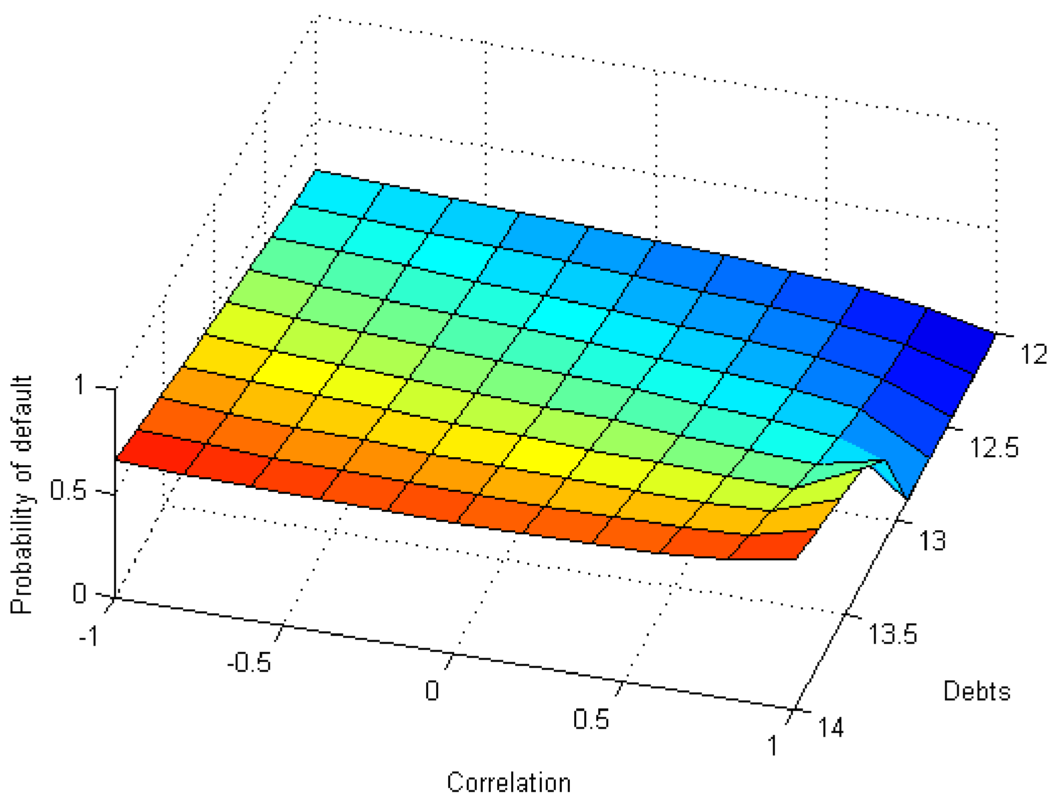

Figure 5: Debts sensitivity for A

- sigmaA : Instantaneous volatility of A. Values between 0.1 and 0.3

  Now the idea is different but the result is similar, as we can see in Figure 6. When assets have more volatility, the probability to hit debts increases. So the effect of the volatility on the probability of default is strictly increasing.

  Looking at correlation, again we can see two different regions. When volatility is 0.2 or less, we have the case where this company has less or equal risk than the other one, so we have the same behaviour as in the red line in Figure 1, strictly decreasing. When the volatility is higher than 0.2 we are in the other case, it starts decreasing but then increases at some point.

  Again, we obtain symmetric results for B.

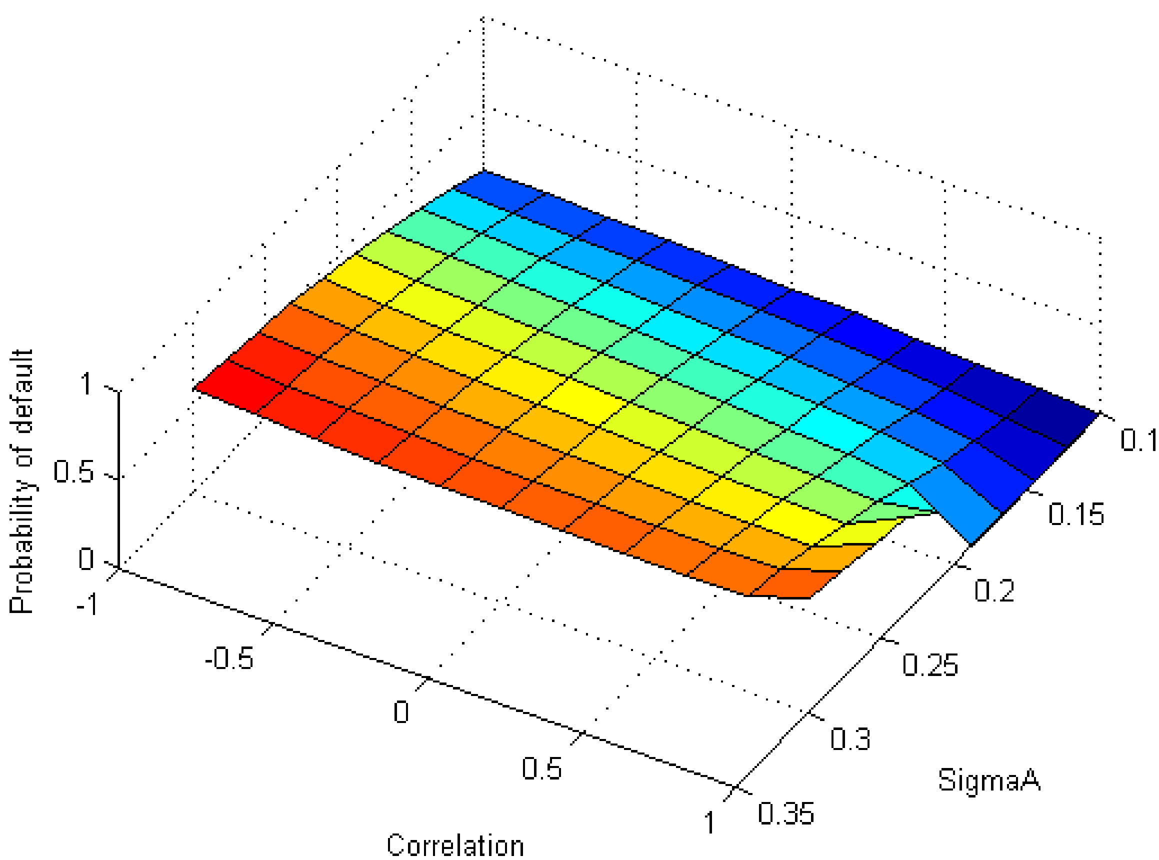

Figure 6: Sensitivity to the volatility of A

- r: Risk-free interest rate.

In this case we can see how the probability of default decreases when the interest rate increases. It is because r has a positive effect on the Geometric Brownian Motion, so if r is high, the Geometric Brownian Motion will grow faster and will be furher from the debts level, so the probability of default will decrease.

Looking at the correlation we have the case where both companies have the same risk, so we obtain again the same (decreasing) behavior as in the red line in Figure 1. That is caused by the joint default event.

In this case we can find the same behaviour for A and B.

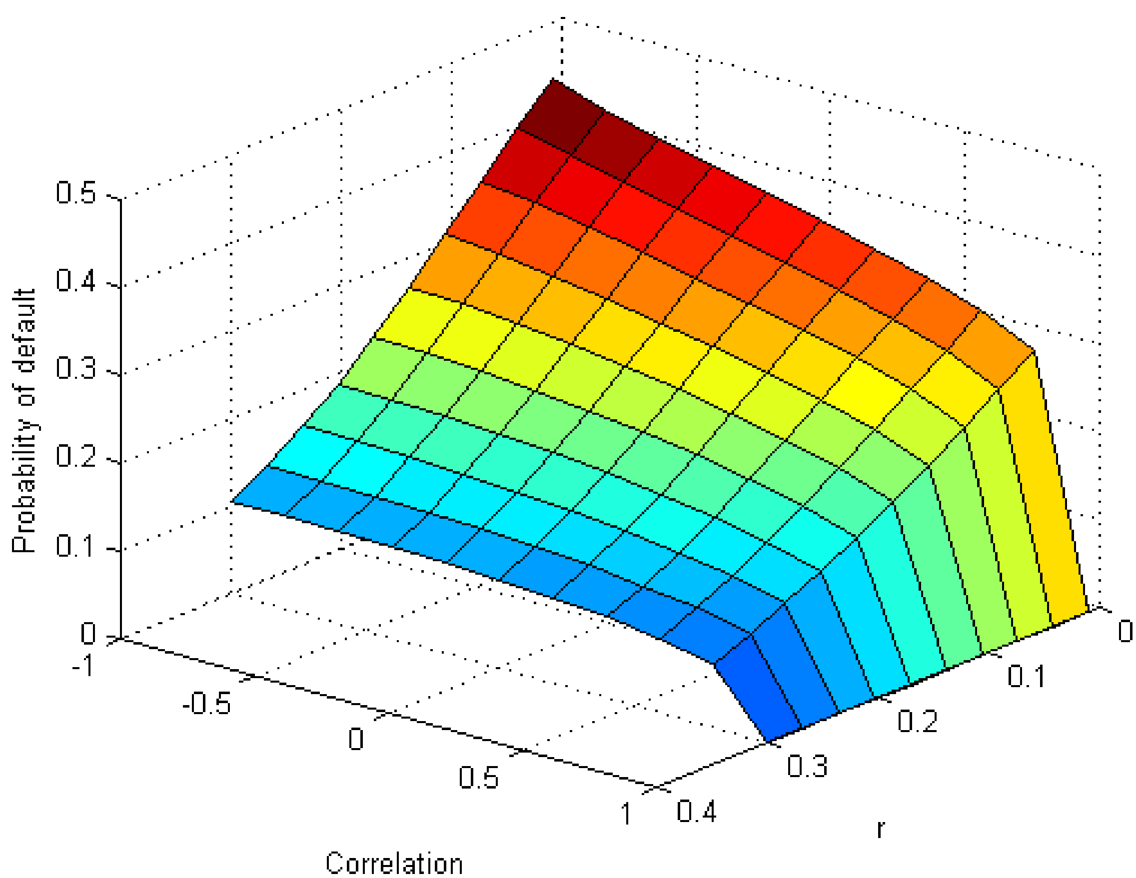

Figure 7: Sensitivity to interest rate in A and B

- T: Maturity (in years). Values between 0.1 and 5.

Figure 8 illustrates that the probability of default increases with maturity. Them, the probability of default is very small when we are very close to expiration. In a similar way, the probability of default should be higher in 5 years than in 1 year as there are more chances to default. As in the previous case, looking at correlation, both companies has the same risk. Then, once again, we get a strictly decreasing behaviour, derived from the joint default event.

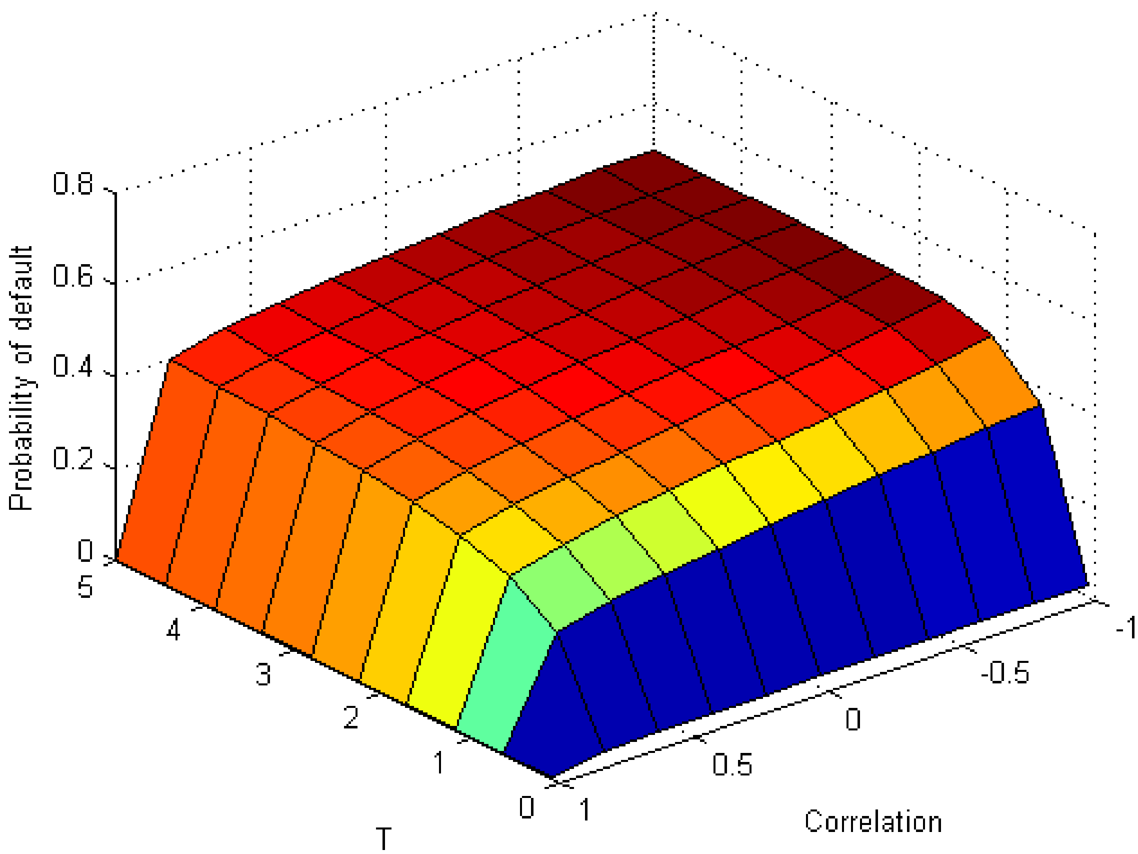

Figure 8: Sensitivity to maturity

Finally, Table 1 summarizes the effects of each parameter on the probability of default:

Table 1: Sensitivity

	X0A	BarrA	sigmaA	r	T
Change in PD for A	negative	positive	positive	negative	positive
Change in PD for B	positive	negative	negative	negative	positive

4.2 Case 2: Stochastic debts

We relax now the hypothesis of constant debts. Moreover, we assume correlation between assets of A and B, and between debts of A and B, but we assume no correlation between assets and debts.

4.2.1 Debts follow the Vasicek (1977) model

The first model that we are going to assume for debts is that introduced in Vasicek (1977).

$$dS_t \;=\; k(\theta - S_t)dt + \sigma dW_t$$

Where k indicates the speed of mean reversion, θ is the long-term value of debts and σ denotes the volatility of the debts-. We use now the same parameters as in Section 4.1 but we need extra parameters for the Vasicek specification:

- barrA0 : Initial value for barrier. A positive number lower than X0A
- DsigmaA : Instantaneous volatility for debts of A.
- thetaA: Long-term mean level of debts.
- kA: Speed of mean reversion of A debts to their long-term value.
- Similar parameters for debts of B.

We choose arbitrarily the following parameters:

$$X0A = X0B = 15;\; barrA0 = barrB0 = 12;\; sigmaA = 0.1;\; sigmaB = 0.3;\; T = 1;\; r = 0.01;$$

$$thetaA = thetaB = 15;\; DsigmaA = 0.1;\; DsigmaB = 0.3;\; kA = kB = 1$$

Then, we have the same parameters for A and B except the volatility. In shor, we assume that B is riskier than A.

Figure 9 shows that the probability of default increases with the correlation of the assets. As in Section 4.1, this probability becomes null for a perfect positive correlation. In addition, this probability does not depend on the level of the correlation in debts.

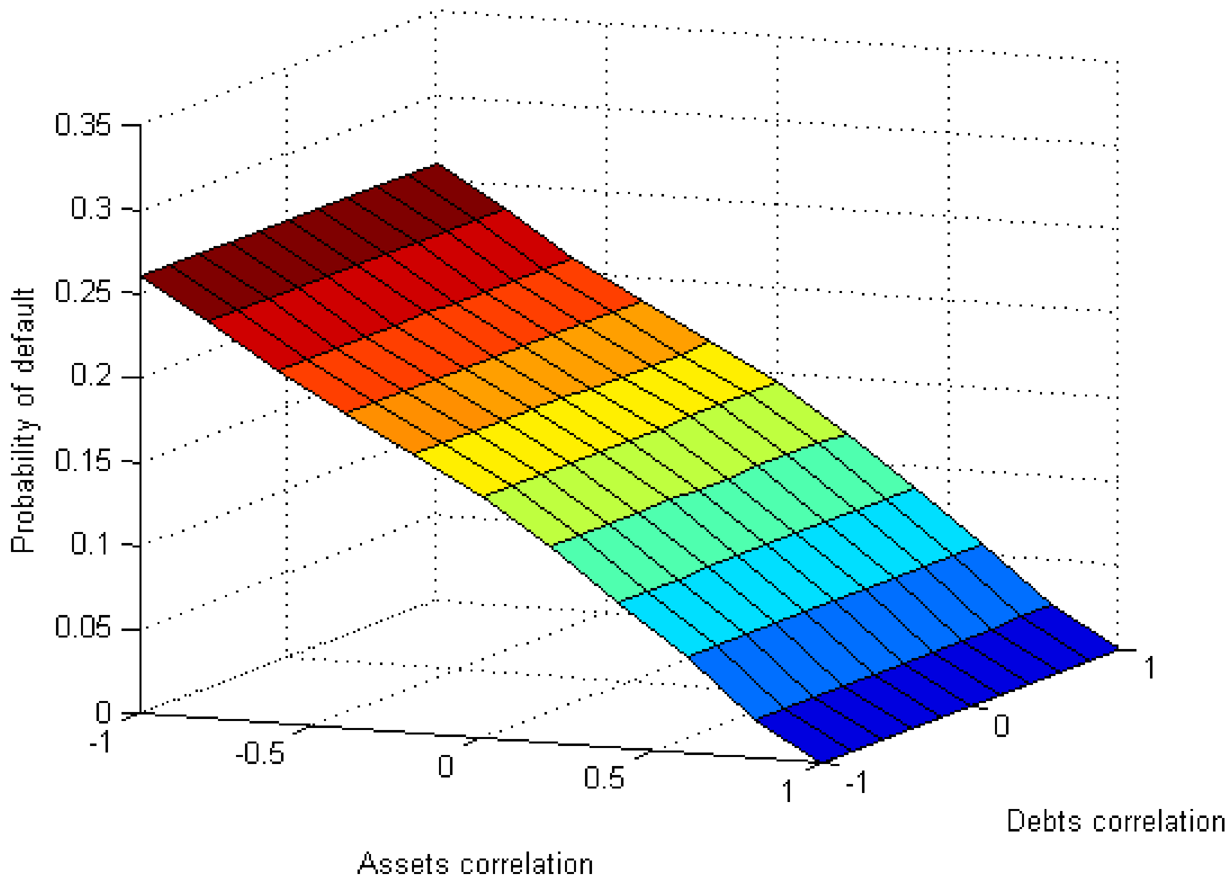

Figure 9: Probability of default of A.

In Figure 10 we can see that this probability starts decreasing, but when assets correlation becomes positive (in this example) we find a turning point, and the probability starts to increase. Again we have the same result as in Section 4.1. As in the previous case, the debts correlation has no effect on the probability of default.

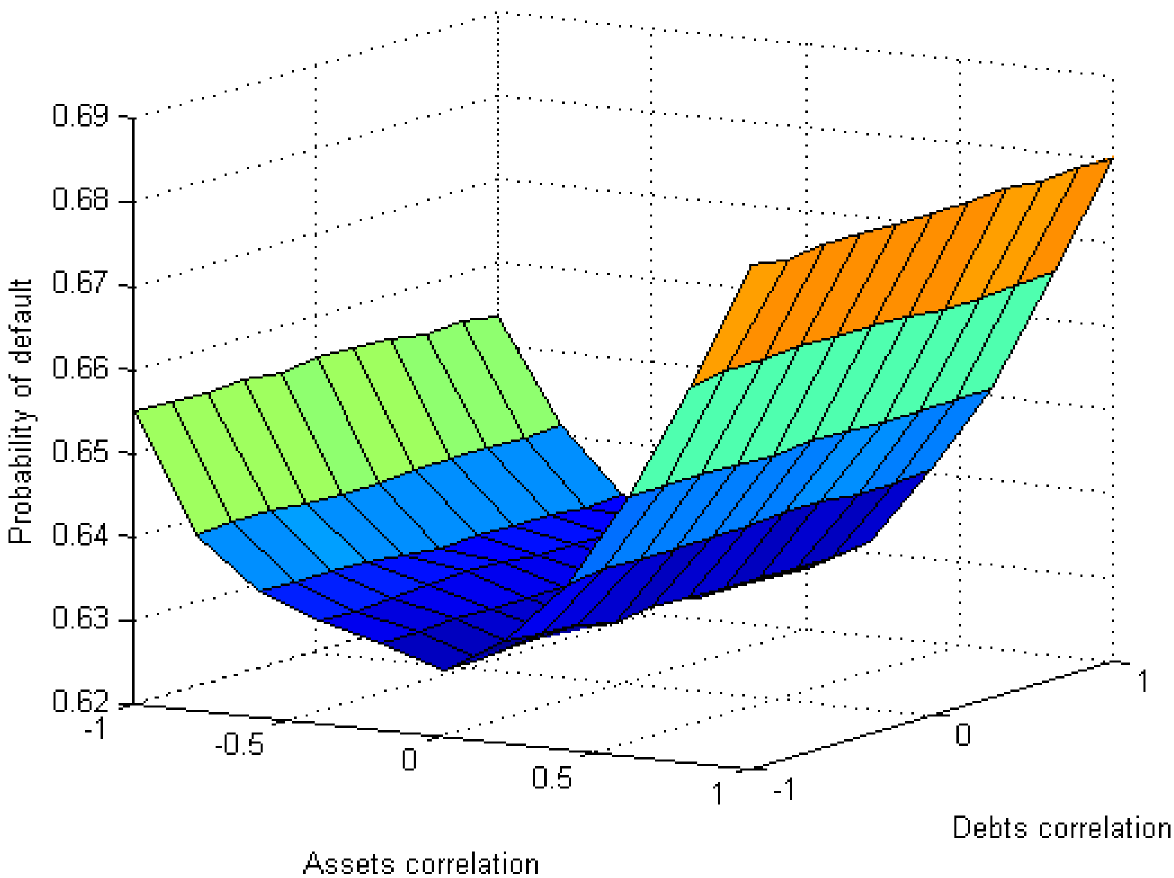

Figure 10: Probability of default of B.

4.2.2 Debts follow a Geometric Brownian Motion

We assume now that debts are given by a Geometric Brownian motion.

$$dS_t \;=\; \mu S_t dt + \sigma S_t dW_t$$

where μ and σ denote, respectively, the espected return of debts and their volatility. We use the same parameters as in section 4.1,assuming again that B is risker than A. We propose the following parameters:

$$X0A = X0B = 15;\; barrA0 = barrB0 = 12;\; sigmaA = 0.1;\; sigmaB = 0.3;$$

$$T = 1;\ r = 0.01;\ DsigmaA = 0.05;\ DsigmaB = 0.1$$

Figure 11 illustrates that the probability of default decreases with the assets correlation. When this correlation becomes 1, the probability of default goes to 0, except when debts correlation is close to -1. We observe a similar behaviour with respect to the debts correlation. We now have the minimum (maximum) probabilities of default are obtained when both correlations are 1 (-1). When the debts correlation is one, both barriers grow in the same direction, and B will default earlier than A, so the event $\tau_A < \tau_B$ is impossible. That makes $P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B) = 0$. When debts correlation is -1, it is still hard to have $\tau_A < \tau_B$, but is not impossible, and that is why $P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B)$ is greater when debts correlation is lower. This effect is monotone in the intermedium values.

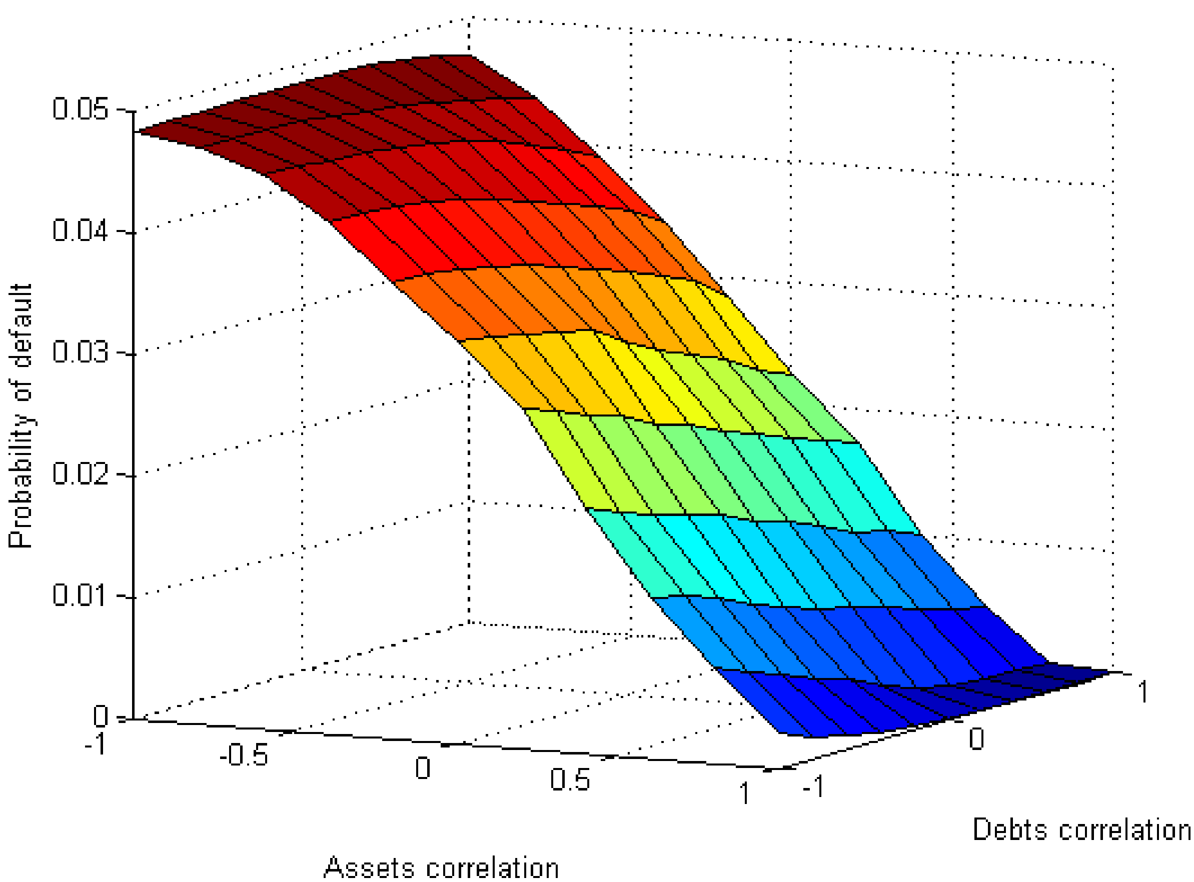

Figure 11: Probability of default of A as a function os correlation in assets and debts.

In Figure 12 we can see that this probability starts decreasing, but when assets correlation becomes positive we find a turning point and the probability starts to increase. We see the same effect looking at debts correlation. When debts correlation is 1, both barriers grow in the same direction and B will default earlier than A, so the event $\tau_B < \tau_A$ makes $P(t_i < \tau_B < t_{i+1}, \tau_B < \tau_A)$ increase.

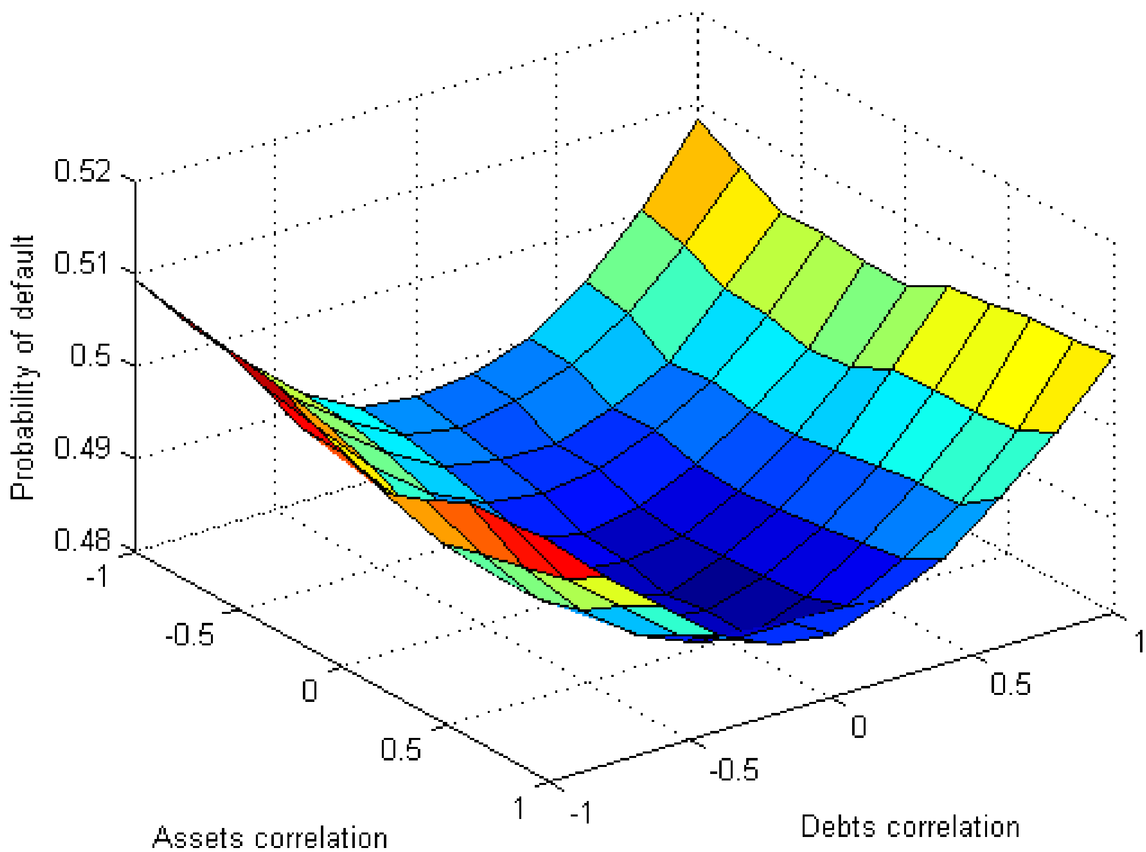

Figure 12: Probability of default of B as a function of correlation in assets and in debts.

Aiming to analyze these effects deeper, we consider more correlation values. In Figure 13 we can see how the greatest value is reached when both correlations are -1. The lowest area is situated in the centre, when debts correlation is close to 0 and assets correlation near 0.4. In every portion, if we take each debts correlation as a constant, we can observe the same shape as in Figure 2 but now we have another degree of freedom to move over debts correlation and we have different levels.

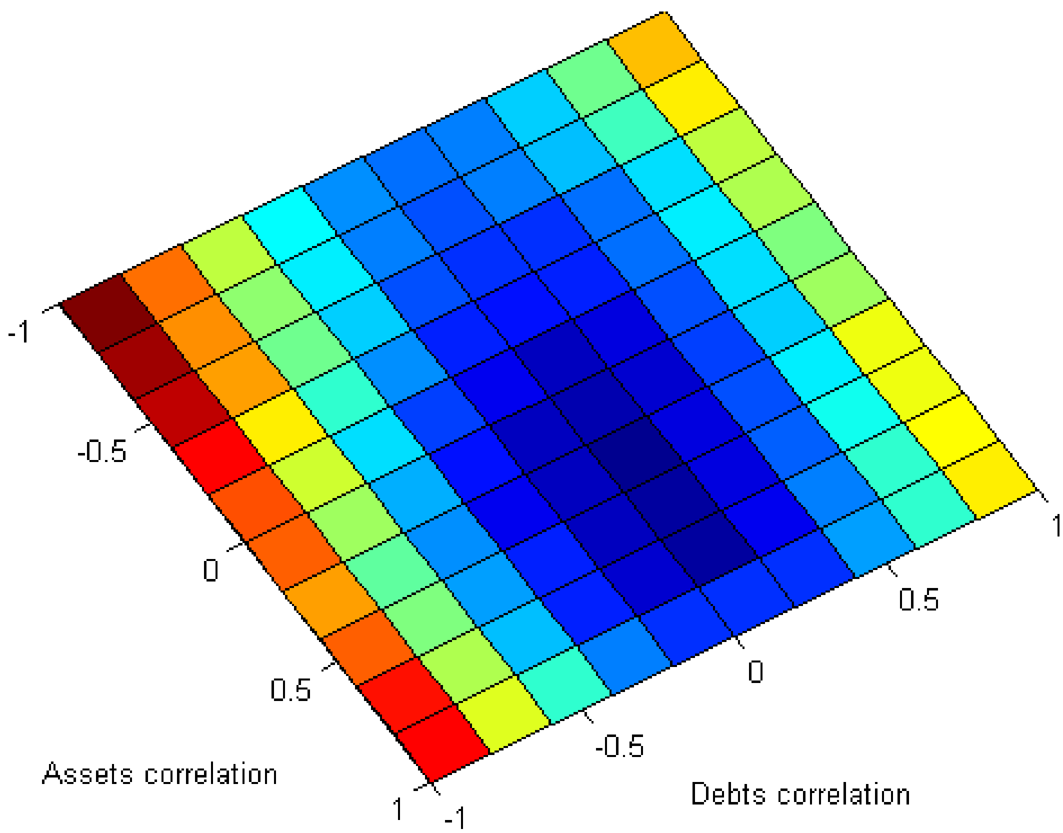

Figure 13: Probability of default of B as a function of correlation in assets and in debts.

4.2.3 Debts follow an Arithmetic Brownian motion

Finally, we assume that debts are given by an Arithmetic Brownian motion, that is,

$$dS_t \;=\; \mu dt + \sigma dW_t$$

where μ and σ denote, respectively, the expected infinitesimal change and variability of debts. We use use the same parameters than in section 4.2.2, except DsigmaA=0.5; DsigmaB=0.8. So B has a higher risk than A since its assets and debts have more volatility (it is easier to hit the barrier). Now we have increased debts volatility with respect to Section (4.2.2) in order to have similar levels in drift and diffusion terms.

In Figure 14, we see that the default probability decreases with the assets correlations. When this correlation equates one, the probability of default goes to zero. Changes in the debts correlation imply small changes of opposite sign in the probsbility of default. This effect is less evident than in Section 4.2.2. When debts correlation is 1, it makes both barriers grow in the same direction, and B will default earlier than A, so the event $\tau_A < \tau_B$ is impossible. That makes $P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B) = 0$. When debts correlation is -1, it is still hard but possible to have $\tau_A < \tau_B$, but it is not impossible, and that is why $P(t_i < \tau_A < t_{i+1}, \tau_A < \tau_B)$ is greater when debts correlation is lower. This effect is monotone in the intermedium values.

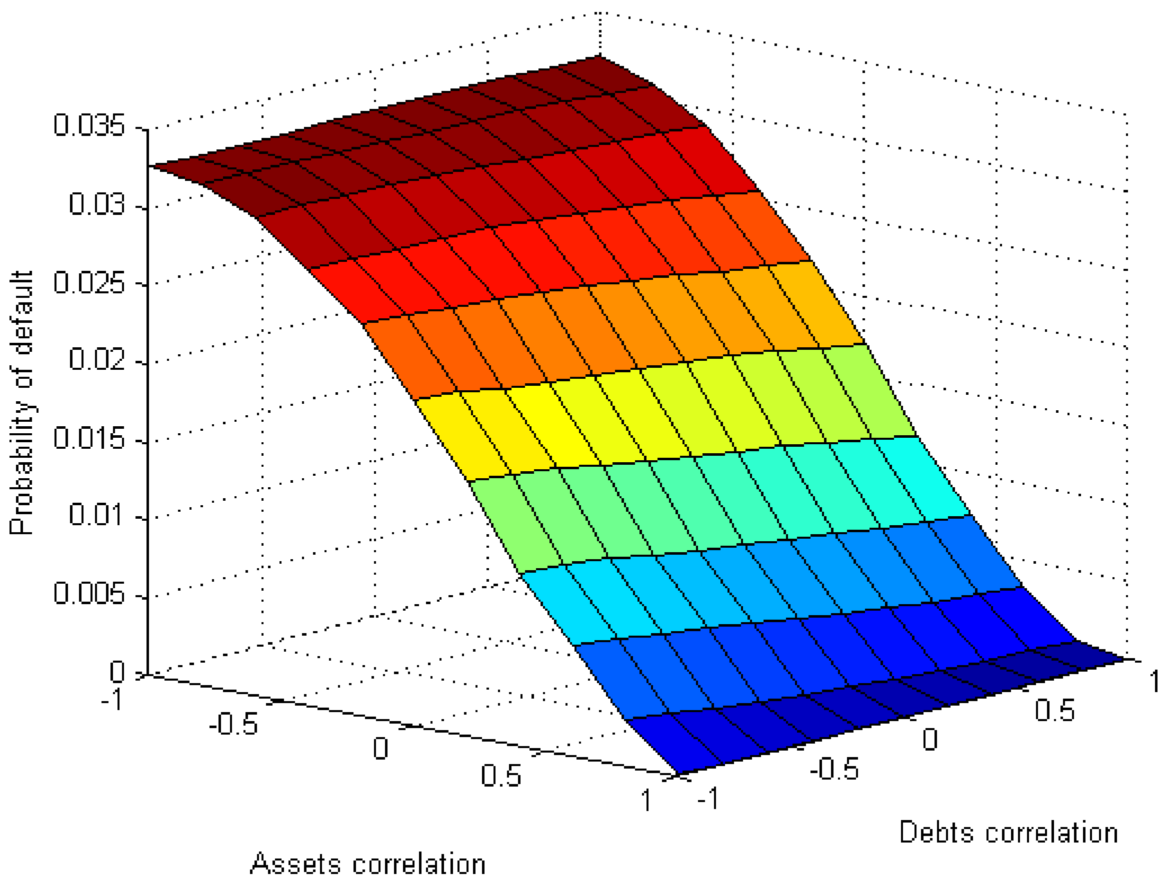

Figure 14: Probability of default of A as a function of correlation in assets and debts.

Finally, Figure 15 provides similar qualitative effects as in Section 4.2.2, that is, we obtain a similar shape although, in this case, the slope with respect to both correlation is lower.

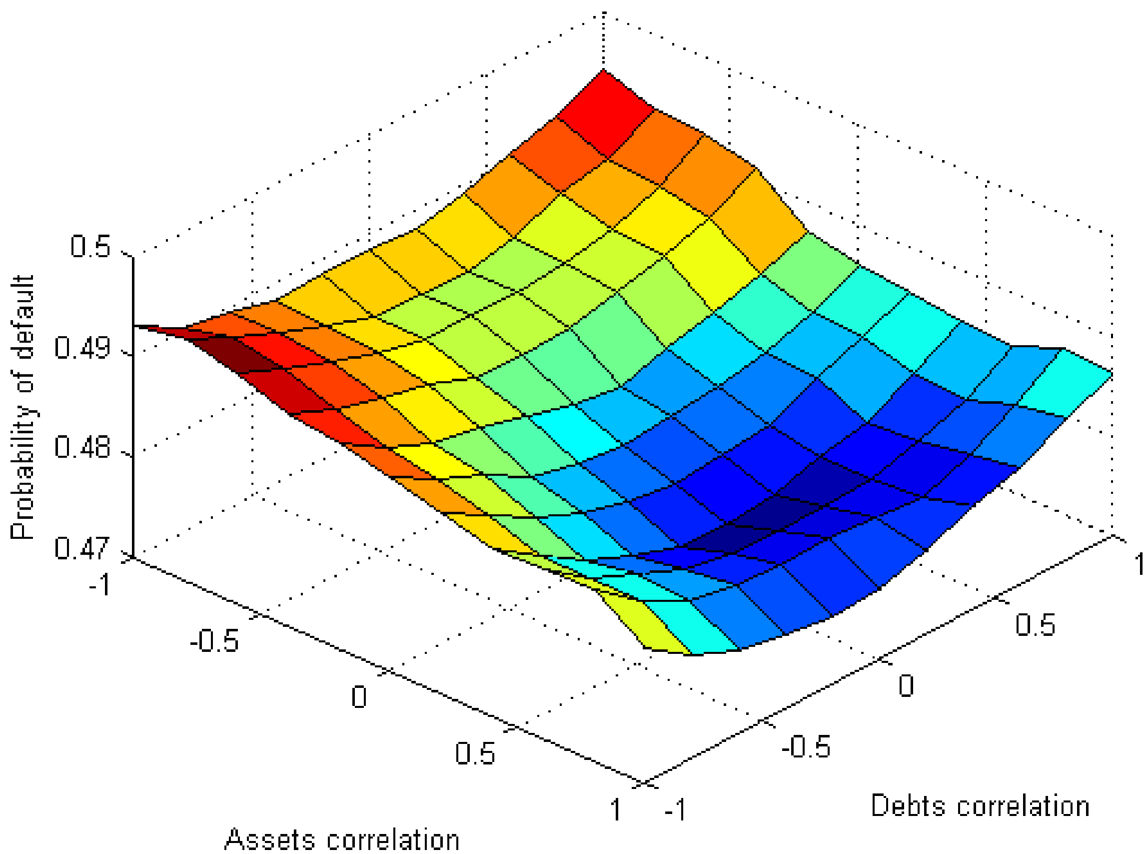

Figure 15: Probability of default of B as a function of correlation in assets and in debts.

5 Conclusions

In this work, we have analyzed several estimations for the probability of default under both unilateral and bilateral cases. In the first case, we have obtained a closed-form expression for the default probability and we have priced zero-coupon bonds very easily by applying this expression. We only need the level of the barrier, the volaitlity of the assets and the risk-free interest rate. In the bilateral case we started with constant debts and we found than the counterparty with less risk has less probability to default the higher the correlation is between assets, become null for a positive perfect correlation. This fact was explained by the joint default event. However, the company with more risk has the same behaviour at the beginning but, at some point (correlationclose to zero in our example) it changes and it increases again instead of approaching to zero. The joint default event is impossible with correlation equal to one and the counterparty with more risk will always default earlier. When we analyze the sensitivity of our parameters we can find reasonable results, as shown in Table 1.

When we introduce stochastic debts, the results described in the previous paragraph are the same at every constant level of debts correlation but, if we look at that debts correlation, our model gives us more alternatives. The correlation in debts was irrelevant underr the Vasicek (1977) model. This is because of the mean reversion of this model. Every path was converging to the long-term level and it makes this barrier "similar" to deterministic since the variance around the mean reversion level was too low. A different pattern arises under the GBM and ABM specifications. Even when we increased debts volatility in ABM, the difference through debts correlation was more evident in GBM but both cases present a similar behaviour.

If we look at the counterparty with less risk, we can find how debts correlation has a bit of impact on the probability of default. The main idea is the same as with assets correlation. When the correlation is 1, the other counterparty will always default earlier, so the probability of default is lower. When the debts correlation is -1 this security does not exist, so the probability of default is higher. In the intermedium points it is monotone. So we have, in both correlations, a monotone decreasing function. The lowest value will be when both correlations are 1 and the highest value when both correlations are -1.

If we look at the counterparty with more risk, we have the same idea. Both correlations have similar behaviour: the lower level in an intermedium point and the highest value in the extremes. In our example, the probability of default has a higher slope through debts correlation and the highest value appears when both correlations are -1, but when debts correlation is -1 in general it has a higher value. The lower value is in the centre, a little displaced, corresponding to "small" debts correlationand assets correlation=0,4 (as we saw in the first bilateral case when debts were constant).

So with this model, we are able to make a difference in the probability of default if we have correlation with the counterparties. This correlation can be implemented through their assets and debts. That allows us not to missvalue this probability in particular cases.

The main problem of this model is the computational cost. Using stochastic debts, stochastic assets, and we can even use stochastic parameters for example for the risk-free interest rate or volatility. Since our probability is a positive number lower than 1, we need to make a good estimation. In short, our fluctuation error should be bounded by 10^{-5}. In our example we used 20,000 simulations with 500 time steps. Sometimes, it was not enough and we needed to perform 50000 simulations (for example in the ABM case) and the surface is still not perfectly soft.

A Appendix: Proofs and previous results

A.1 The Leibniz differentiation rule

Let $f(x,\theta)$ be a function so that $f_\theta(x,\theta)$ exists and is continuous. Then:

$$\frac{d}{d\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)dx + f(b(\theta),\theta)b'(\theta) - f(a(\theta),\theta)a'(\theta)$$

A.2 Checking that our density function integrates one in its domain

We need to check that the probability density function (12) integrates one.

$$\begin{aligned}
\int_0^\infty \rho(\tau_m = t)dt &= \int_0^\infty \frac{m}{t\sqrt{2\pi t}} e^{-\frac{(m-\theta t)^2}{2t}} dt \\
&= \int_0^\infty \frac{m}{t\sqrt{2\pi t}} e^{-\frac{1}{2t}(m^2+\theta^2 t^2 - 2m\theta t)} dt \\
&= \int_0^\infty \frac{m}{t\sqrt{2\pi t}} e^{-\frac{1}{2t}(m^2+\theta^2 t^2)} e^{m\theta} dt \\
&= \frac{e^{m\theta}}{\sqrt{2\pi}} \int_0^\infty \frac{m}{t\sqrt{t}} e^{-\frac{1}{2t}(m^2+\theta^2 t^2 + 2m\theta t - 2m\theta t)} dt \\
&= \frac{e^{m\theta}}{\sqrt{2\pi}} \left[\int_0^\infty \frac{m+\theta t}{2t\sqrt{t}} e^{-\frac{1}{2t}(m^2+\theta^2 t^2 - 2m\theta t)} e^{-\theta m} dt + \int_0^\infty \frac{m-\theta}{2t\sqrt{t}} e^{-\frac{1}{2t}(m^2+\theta^2 t^2 + 2m\theta t)} e^{\theta m} dt\right] \\
&= \frac{e^{m\theta}}{\sqrt{2\pi}} \left[\int_0^\infty \frac{m+\theta t}{2t\sqrt{t}} e^{-\frac{1}{2}(\frac{m-\theta t}{\sqrt{t}})^2} e^{-\theta m} dt + \int_0^\infty \frac{m-\theta t}{2t\sqrt{t}} e^{-\frac{1}{2}(\frac{m+\theta t}{\sqrt{t}})^2} e^{\theta m} dt\right]
\end{aligned}$$

We propose a change of variable for each integral:

$$\begin{aligned}
x = \frac{m-\theta t}{\sqrt{t}} \quad &\Rightarrow \quad dt = \frac{-2t\sqrt{t}}{m+\theta t}dx \\
y = \frac{m+\theta t}{\sqrt{t}} \quad &\Rightarrow \quad dt = \frac{-2t\sqrt{t}}{m-\theta t}dy
\end{aligned}$$

So now we have the limits:

$$\begin{aligned}
x(0) = \lim_{t\to 0} \frac{m-\theta t}{\sqrt{t}} = \infty, \quad x(\infty) = \lim_{t\to\infty} \frac{m-\theta t}{\sqrt{t}} = -\infty \\
y(0) = \lim_{t\to 0} \frac{m+\theta t}{\sqrt{t}} = \infty, \quad y(\infty) = \lim_{t\to\infty} \frac{m+\theta t}{\sqrt{t}} = \infty
\end{aligned}$$

Finally, the formula is:

$$
\begin{aligned}
P(\tau < T) &= -e^{m\theta}\left[-e^{-\theta m}\int_{-x(0)}^{-x(\infty)} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx - e^{\theta m}\int_{-y(0)}^{-y(\infty)} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy\right] \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx - e^{2m\theta}\int_{\infty}^{\infty} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy \\
&= 1 + 0 = 1
\end{aligned}
$$

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