CONTINUOUS TIME KALMAN FILTER MODELS FOR THE VALUATION OF COMMODITY FUTURES AND OPTIONS

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To all people who make the world the marvellous place it is. Let them find the happiness they give and deserve.

“A habit of basing convictions upon evidence, and of giving to them only that degree or certainty which the evidence warrants, would, if it became general, cure most of the ills from which the world suffers” Bertrand Russell

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INTRODUCTION

A HISTORICAL BACKGROUND

The history of Kalman filter is long and broad, and so is the literature of its applications to the field of Economics. It was first derived by Kalman in a celebrated article in 1960, following a previous and more theoretical work of Stratonovich (1959). Its importance was recognized in the Engineering literature from the very start.

Economics lagged a few years in following this approach, as it was dominated by a more antique ARIMA approach. However, as early as 1989, Andrew J. Harvey, in his now classical book "Forecasting, Structural Time Series and the Kalman Filter" already exposes practically all now mainstream techniques in dealing with Kalman filter estimation.

Continuous-time Finance, being a rather more recent field (we can not even speak properly of Continuous-time Finance until the seventies, with the pioneer works of Black and Scholes) had to wait a bit more. We can establish the time when this approach became dominant in the influential work of Schwartz (1997).

However, since this date, the field has really become exuberant. Kalman Filter deals routinely, in the blackboards of academics and the workstations of practitioners with thousands of real world financial series and its implications seem to be far from exhausted. This thesis tries to be a contribution, humble as may be, to this research.

GENERAL SETUP

The framework where all these thesis' results are set is a continuous-time state space system that exhibits a dynamics given by:

$$\begin{cases} dX_t = (b + AX_t)dt + RdW_t \\ S_t = \exp(cX_t) \end{cases} \tag{MR}$$

where S_t is the spot price of a given financial asset commodity, X_t is a vector of n states which are usually not observable, W_t is unitary Brownian motion and b and A,R and C are matrices of appropriate size, that in most applications need to be identified.

Following Schwartz (1997), in the spirit of the Black-Scholes risk neutral valuation, another fictitious dynamics is introduced via a vector called risk premium. We thus obtained a risk neutral dynamics, which is used to value options and futures contracts:

$$\begin{cases} dX_t = (b - \lambda + AX_t)dt + RdW_t \\ S_t = \exp(cX_t) \end{cases} \tag{MN}$$

It is worth remarking why models exhibit hidden dynamics. In fact, classical continuous-time financial models are directly observable. In the black-Scholes world, dynamics is just given by:

$$\begin{cases} dX_t = \left(\mu - \dfrac{\sigma^2}{2} + X_t\right)dt + \sigma dW_t \\ S_t = \exp(X_t) \end{cases}$$

And we just have to take logarithms to recover state from spot price. However, as noted by Schwartz (1997), this model implies perfect correlation among different futures, which is contrary to existing evidence. As a result, he proposed a particular version of general model (MR)-(MN) where the spot price was the sum of two hidden components, one continuous-time random walk (the classical model for financial assets) and transitory short run component. A number of generalizations following model structure (MN)-(MR) have been proposed since. As examples, the reader can consult Cortazar and Naranjo (2003) or García, Población and Serna (2012).

Going back into the equations, we shall see that they can be solved explicitly, giving a complete discrete time model to be identified directly from observable data.

Although full details will be given in the thesis, let us briefly outline how this is done. A direct application of the results in Oksendal (1992) gives us the solution of equation

(MR) as $X_{t+\Delta t} = e^{A\Delta t}\left[X_t + \int_0^{\Delta t} e^{-As} b ds + \int_0^{\Delta t} e^{-As} R dW_{t+s}\right]$ which means we can exactly compute state dynamics. Defining $b_D = e^{A\Delta t}\left[\int_0^{\Delta t} e^{-As} b ds\right]$, $A_D = e^{A\Delta t}$ and $\eta_t = e^{A\Delta t}\int_0^{\Delta t} e^{-As} R dW_{t+s}$ we have a fully specified equation $X_{t+\Delta t} = b_D + A_D X_t + \eta_t$.

However, we do not usually (and never in the models considered in this work) observe spot prices but instead have data on futures or options. Regarding futures, which is the data we shall use to estimate models (options are taken into account later for valuation purposes), in the Black Scholes world they are simply the risk neutral expectation of spot prices or, in symbols, $F_{t,T} = E_Q\left[S_{t+T} / I_t\right]$ where $F_{t,T}$ is the future contracted at *t* with maturity *T* (i.e. with delivery time $t+T$), *Q* is the risk neutral measure and I_t is the information available at *t*.

Under risk neutral measure, we have to use equations (MN) and therefore, conditional to *t*, $F_{t,T}$ is lognormal and $ce^{AT}\left[X_t + \int_0^T e^{-As}(b-\lambda)ds\right]$ is its logarithm's mean while $c\left(\int_0^T e^{-A(T-s)}RR'\left[e^{-A(T-s)}\right]' ds\right)c'$.

The bottom line is that $\log F_{t,T} = d(T) + c(T)X_t$ for known matrices $d(T)$ and $c(T)$ whereas X_t has a known discrete dynamics so we arrive to a fully specified discrete model that can be estimated from real data via Kalman filter :

$$\begin{cases} X_{t+\Delta t} = b_D + A_D X_t + \eta_t \\ \log F_{t,T} = d(T) + c(T)X_t + \varepsilon_t \end{cases}$$

Different chapters of this thesis describe different aspects of this model, using it to estimate parameters and value options in different commodities.

SUMMARY OF CHAPTER ONE

This chapter deals with a mathematically general version of (MR) and (MN). As financial data are never observed in continuous time (even ultra high frequency data is observed at intervals of tens of milliseconds), in order to estimate parameters a discrete time version of the model has to be achieved.

In the literature, the dominant approach was to develop discrete time formulae from ad hoc procedures, involving limit steps and partial differential equations. We have shown that these ideas are unnecessary and have developed a general method to achieve discrete time forms which is applicable to all models proposed in the literature. Moreover, we have also establish a general, directly programmable, computer efficient method to obtain this formulae, which we have contrasted against theoretical alternatives, reducing computation time in an order of magnitude.

In this part, we have also used our formulae to contrast our approach with Schwartz (1997) formulae using West Texas Intermediate (WTI) futures data. We show that his method was an approximation that tends to (slightly) overestimate the parameters and increase error.

SUMMARY OF CHAPTER TWO

This chapter treats a modification of model (MN)-(MR) where risk premium is allowed to vary over time, that is:

$$\begin{cases} dX_t = (b - \lambda_t + AX_t)dt + RdW_t \\ S_t = \exp(CX_t) \end{cases} \tag{MN'}$$

This problem was very appealing, as seemed very reasonable to assume that the state of world economy should have a direct implication in the premium an investor demands to purchase a risky asset.

Estimating this premium via a Kolos and Ronn (2008) algorithm and a moving window we obtained a time series, which we compared with several economic indicators. Results were very interesting as we observed, among other findings fully described in the chapter, that there was a positive relation between the estimated long-term market price of risk and the average NAPM index, the average S&P 500 index and an indicator of economic expansion. This relation was reversed when we compared these economic indicators with short term risk premium.

In addition, we proposed a model with time varying risk premium, and showed how it could be estimated via exactly the same discrete Kalman filter, by modifying the way discrete time equations were obtained. This model was estimated (separately) with real WTI Oil, Heating Oil, Gasoline and Henry Hub (HH) Natural Gas outperforming constant risk premium models.

Finally, we applied the new model was used to valuate a sample of American WTI options, obtaining better results than more standard approaches.

SUMMARY OF CHAPTER THREE

This final chapter studies convenience yield dynamics. Convenience yield can be defined as the value of owing a commodity physically instead of having a financial asset that guarantees its possession in a certain date.

More formally, remember that in a Black-Scholes world, futures prices are given by risk neutral expectation of spot prices or $F_{t,T} = E^*\left[S_{t+T} / I_t\right]$. Convenience yield ($\delta_{t,T}$) is the difference, in continuous time between this price and the spot price increased due to real interest rate, that is $F_{t,T} \cdot e^{\delta_{r,T} \cdot T} = S_t \cdot e^{r_{t,T} \cdot T}$.

What we did in this part was to derive the distribution of convenience yield from first principles when spot prices followed a stochastic seasonal model. We showed that this implies, in convenience yield series, a seasonal component directly related to the spot price original. Moreover, this finding was confirmed when estimating a model for convenience yield directly from real world (WTI Oil, Heating Oil, Gasoline and HH Natural Gas) data.

In addition, we also showed that our seasonal model was maximal in a sense related to Dai-Singleton (2000) and gave a canonical, globally identifiable form for this model, which can actually be applied to all constant volatility models in the literature.

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CHAPTER 1: ANALYTIC FORMULAE FOR COMMODITY CONTINGENT VALUATION

1.1. INTRODUCTION

Itô calculus has become the main approach in derivatives valuation theory since it was first used in Finance (Black and Scholes, 1972). The same methodology was first used in the valuation of commodity contingent claims (see for example Brennan and Schwartz, 1985, Paddock et al., 1988, among others), i.e. by assuming that asset prices follow a geometric Brownian motion, the classical Black-Scholes formulae can be used with slight modifications (if any). Subsequently several authors, such as Laughton and Jacobi (1993) Ross (1997) or Schwartz (1997), have considered that a mean-reverting process is more appropriate to model the stochastic behaviour of commodity prices, pointing out that the geometric Brownian motion hypothesis implies a constant rate of growth in the commodity price and a variance of futures prices increasing monotonically with time, which are not realistic assumptions. The idea behind mean-reverting processes is that the supply of the commodity, by increasing or decreasing, will force its price towards an equilibrium (or long-term mean) price level[1].

In spite of their attractiveness, these one-factor mean-reverting models are not very realistic since they generate a constant volatility term structure of futures returns, instead of a decreasing term structure, as observed in practice. Gibson and Schwartz (1990) and Schwartz (1997) propose a two-factor model, where the second factor is the convenience yield, which is also assumed to follow a mean-reverting process. Schwartz and Smith (2000) propose a two-factor model allowing for mean reversion in short-term prices and uncertainty in the equilibrium (long-term) price to which prices revert, which

[1] See Schwartz (1997) and Schwartz and Smith (2000) for an excellent discussion of these issues.

is equivalent to the Schwartz (1997) one. Schwartz (1997) also considers a three-factor model, extending the Gibson-Schwartz (1990) model to include stochastic interest rates. Cortazar and Schwartz (2003) propose a three-factor model, which is an extension of the Schwartz (1997) two-factor model, where all three factors are calibrated using only commodity prices. More recently Cortazar and Naranjo (2006) extend two and three factor models to an arbitrary number of factors (N-factor model).

Unfortunately, the application of the standard Black-Scholes valuation framework is not easy in the context of commodity contingent valuation, given the complex dynamics of commodity prices. This is the reason why the studies on commodity contingent valuation usually present very complex ad-hoc solutions and sometimes include approximations or limit steps. In this article we show how to simplify formulae and deductions, computing the explicit, directly implementable general formula, based on well known results in stochastic calculus.

Specifically, after describing the general theoretical model for commodity contingent valuation, we present two specific applications. Firstly, we show how this general framework can be implemented in the context of the two-factor model by Schwartz (1997), obtaining simpler expressions and more precise estimates than the approximations given by the author. It is also shown that the approximations by Schwartz tend to overestimate the parameters, a fact that, as we will see, becomes important in the valuation of commodity contingent claims. Secondly, we shall show how to obtain the expression for the futures price and volatility of futures returns given by Schwartz (1997) and Schwartz and Smith (2000) in a simpler way, avoiding unnecessary partial differential equations or limit steps.

This chapter is organized as follows. The general methodology for commodity contingent valuation and volatility estimation is presented in Section 2. Section 3 describes how these formulae can be used in practice and proposes a ready-to-implement algorithm to estimate any linear model which is evaluated in terms of computer time. Section 4 shows how to obtain more precise estimators of the parameters in the two-factor model by Schwartz (1997). Section 5 shows how to simplify the deduction of the futures price in the two-factor model by Schwartz and Smith (2000), avoiding unnecessary limit steps. Finally, section 6 concludes with a summary and discussion.

1.2. THEORETICAL MODEL

Contract Valuation

Most of the models proposed in the literature for the stochastic behaviour of commodity prices can be summarized by means of the following system:

$$\begin{cases} dX_t = (b + AX_t)dt + RdW_t \\ Y_t \quad = cX_t \end{cases} \tag{1}$$

where Y_t is the commodity price (or its log), b, A, R and c are deterministic matrices[2] independent of t ($b \in \Re^n, A, R \in \Re^{n\,x\,n}$, $c \in \Re^n$) and W_t is a n-dimensional canonical Brownian motion (i.e. all components uncorrelated and its variance equal to unity). Usually, the estimation of these matrices can be simplified, as they can be assumed to depend in a predefined way of some estimable values, called structural parameters or hyperparameters (for example, if A is 2x2, instead of computing four values one may assume, as in Schwartz, 1997, that $A = \begin{pmatrix} 0 & -1 \\ 0 & -\kappa \end{pmatrix}$ where κ is the hyperparameter to be estimated).

[2] R does not have to be computed, as all formulae shall use RR'.

As it shall be proven in appendix B the solution of this problem is:

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As} b ds + \int_0^t e^{-As} R dW_s\right] \tag{2}$$

We shall assume now that A is diagonalizable with $A = PDP^{-1}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}$ diagonal[3]. Let us define the auxiliary quantities:

$$J(t) = P\begin{pmatrix} It & 0 \\ 0 & D_1^{-1}[\exp(D_1 t) - I] \end{pmatrix} P^{-1} \tag{3}$$

$$G(t) = \exp(At) P vec^{-1}\left\{\left[\int_0^t \exp(Ds) \otimes \exp(Ds) ds\right] vec\left(P^{-1} RR' P'\right)\right\}\left(P^{-1}\right)' \exp(At)' \tag{4}$$

This integral can be computed explicitly, but depends on the eigenvalues (see appendix A).

Using (2) and the results in Appendix A about integrals, it is evident that, given X_0, X_t is Gaussian, with mean and variance:

$$E[X_t] = e^{At} X_0 + J(t)b \ , \ Var[X_t] = G(t). \tag{5}$$

Which yields that Y_t is also Gaussian with $E[Y_t] = cE[X_t]$, $Var[Y_t] = cVar[X_t]c'$

Under the risk-neutral measure, the dynamics are exactly the same as in (1) but changing b into a different b^* which contains the risk premia (all other matrices stay the same) so, using this measure and conditional to X_0, X_t is Gaussian. To compute the risk-neutral mean and variance of X_t and Y_t we must substitute b for b^* in (5), thus providing a valuation scheme for all sorts of commodity contingent claims such as financial derivatives on commodity prices, real options, investment decisions, etc.

[3] To the best of the authors' knowledge all models in the existing literature fulfil this restriction, most of them directly by imposing A to be diagonal. Notable exceptions where A is not diagonal but diagonalizable are the Schwartz (1997) model or the cycles in Harvey (1991).

If Y_t is the log of the commodity price (S_t), it is easy to prove (just by the properties of the log-normal distribution) that the price of a futures contract traded at time "*t*" with maturity at time "*t+T*" is:

$$F(t,T) = \exp\left(ce^{AT} X_t + cJ(T)b^* + \frac{1}{2} cG(T)c' \right) \tag{6}$$

This methodology is general, feasible for all kind of problems, at least when the parameters in (1) are independent of *t*, and much simpler than the ad-hoc solutions presented in the literature, that can only be used in the concrete problem for which they were developed and need complex procedures such as partial differential equations (Schwartz 1997) or limit steps (Schwartz-Smith 2000). Even more, these formulae can be implemented directly in any mathematical oriented computer language, such as Matlab or C++ *regardless on the size of the matrices or their dependence of the hyperparameters, using the matrices directly as inputs.* So there is no need to compute explicit formulae each time we use a different model. It possible to use the same script (changing the way the matrix depend on the hyperparameters) for any model.

Volatility of Future Returns

We can define the squared volatility of a futures contract traded at time "*t*" with maturity at time "*t+T*" as[4]: $\lim_{h \to 0} \frac{Var\left[\log F_{t+h,T} - \log F_{t,T}\right]}{h}$. In appendix C it is proved that it is the expected value of the square of the coefficient of the Brownian motion (σ_t) in the expansion $d\log\left(F_{t,T}\right) = \mu_t ds + \sigma_t dW_t^F$, where W_t^F is a scalar canonical Brownian

[4] The same results would be obtained if the volatility were defined as: $\lim_{h \to 0} \frac{Var\left[\log F_{t+h,T-h} - \log F_{t,T}\right]}{h}$.

motion, as long as μ_t is mean squared bounded in an interval containing t (it does not matter whether it is a function of $F_{t,T}$ or not) and $E[\sigma_t^2]$ is continuous in t.

In the general problem of this article these conditions are satisfied. Therefore, after taking logarithms and differentials on both sides of Equation (6), we can obtain that:

$d(\log F_{t,T}) = ce^{AT}dX_t = ce^{AT}[b + AX_t]dt + ce^{AT}R\,dW_t$

So, the squared volatility is simply[5]:

$$ce^{AT}RR'e^{AT}{}'c'. \tag{7}$$

1.3. DISCRETIZATION AND ESTIMATION ISSUES

This section is devoted to provide a practitioner's guide to the use of the above results. Suppose that we observe a forward curve $F(t,T)$ of N futures prices and wish to estimate a linear multifactor model as in (1). First of all, we need a discrete version of (1). Let Δt be the interval of discretization.

As stated above $E[X_t] = e^{At}X_0 + J(t)b$ and $Var[X_t] = G(t)$. Consequently, it is easy to prove that:

$$\begin{cases} X_{t+\Delta t} & = b_D + A_D X_t + \eta_t \\ y_t & = d + c_d X_t + \varepsilon_t \end{cases} \tag{8}$$

where $y_t = [\log(F(t,T_1)),...,\log(F(t,T_N))]'$ is the log of the full forward curve, $A_D = \exp(A\Delta t)$, $b_D = J(\Delta t)b$, $E[\eta_t] = 0$, $Var(\eta) = G(\Delta t)$,

$d_i = cJ(T_i)b^* + \frac{1}{2}cG(T_i)c' \quad i = 1...N$ and $c_D = \begin{pmatrix} c\exp(AT_1) \\ ... \\ c\exp(AT_N) \end{pmatrix}$.

---

[5] Note again that R does not need to be computed as RR' is the noise covariance matrix.

Of course, the measurement noise (ε_t) is user-defined. The most usual convention, followed by Schwartz (1997), Schwartz and Smith (2000), Cortazar and Naranjo (2006) among others, is $E[\varepsilon_t] = 0$, $Var[\varepsilon_t] = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \sigma_N \end{pmatrix}$

The process to estimate a model is as follows:

1. Given a set of hyperparameters ϕ, make explicit the dependence of the continuous time system matrices $A(\phi), c(\phi)$ and so on in (1)

2. Compute the discrete-time system (8). This can be done using the formulae (3) and (4) or directly via the integrals in appendix B. The easiest way is obviously to compute them by hand and insert them in the program. However, the computer can do it, using the formulae (3) and (4) each iteration at a moderate additional computational cost (thus allowing the user to write a single program for all models, instead of changing it each time).

3. Estimate the parameters in the models by a log-likelihood algorithm. See Hamilton (1994) for details on estimating a state-space model.

From the authors' point of view, unless the user always deals with the same kind of model, the increasing complexity of using formulae (3) and (4) in each iteration is a price worth paying by having a single general program.

We would like to stress the importance of formulae (3) and (4). Without them, unless the practitioner writes a separate script for each model, he would have to compute (via a symbolic processor such as Matlab Symbolic Toolbox) an integral in each iteration. The computational cost of that is burdensome, approximately 100 times the one with the formulae, which is two orders of magnitude higher.

To proof this, we have estimated the Schwartz and Smith (2000) and Cortazar and Schwartz (2003) models with different data sets, representative of the kind of series a practitioner is likely to work with. Here, it suffices to say that they are a two factor (Schwartz and Smith, 2000) and a three factor (Cortazar and Schwartz, 2003) model with 8 and 13 identifiable hyperparameters respectively. The data set employed consists on weekly observations of Henry Hub natural gas, WTI crude oil futures prices (both of them traded at NYMEX) and Brent crude oil futures prices (traded at ICE). The data set for Henry Hub natural gas is made of contracts F1, F5, F9, F13, F17, F21, F25, F29, F33, F37, F41, F44 and F48 where F1 is the contract closest to maturity, F2 is the second contract closest to maturity and so on. This data set contains 330 quotations of each contract from 12/03/2001 to 03/24/2008. The data set for WTI crude oil is made of contracts F1, F4, F7, F10, F13, F16, F19, F22, F25 and F28. This data set contains 654 quotations of each contract from 9/18/1995 to 03/24/2008. The data set for Brent crude oil is made of contracts F1, F4, F7, F10, F12, F16-18, F22-24 and F31-36. This data set contains 537 quotations of each contract from 12/15/1997 to 03/24/2008. These data sets have been chosen taking into account that futures contracts with long-term and short-term maturities are necessary to estimate properly the parameters of the long-term and the short-term factors.

In Table 1 a brief summary of the time needed for an evaluation of the log-likelihood function is given, specifying the data and model used (two factors means Schwartz and Smith, 2000, model, three factors means Cortazar and Schwartz, 2003). Note that, as all quantities are given in milliseconds, a 30% less for the formulae (implementing each case separately) is not a big reward. All experiments were made with an x86 Intel Celeron (Family 6 Model 8 Stepping 3, 261.616 Kb RAM).

In order to illustrate this fact, we have also included another Table (number 2) where the estimation time is given for the general case and the estimation for each case separately (using the theoretical formulae for integrals would be too slow). As the reader can see, the difference is small enough and, from the authors' point of view, it is not worth the effort to compute formulae by hand case by case instead of using matrix forms. Note that the difference is estimating a model in a minute and a minute and a half, even with a rather old computer.

1.4. PRECISE ESTIMATION OF THE SCHWARTZ (1997) TWO-FACTOR MODEL

Let us consider the two-factor model in Schwartz (1997). Let S_t and δ_t be the spot price of a commodity and its instantaneous convenience yield at time t. The model can be expressed as:

$$\begin{aligned} dS_t &= (\mu - \delta_t)S_t dt + \sigma_1 S_t dz_1 \\ d\delta_t &= \kappa(\alpha - \delta_t)dt + \sigma_2 dz_2 \end{aligned}$$

The standard Brownian motions, dz_1 and dz_2, are assumed to be correlated, i.e. $dz_1 dz_2 = \rho dt$. The parameter μ is the long-term total return on the commodity, κ is the mean-reverting coefficient, α is the long-term convenience yield, and finally σ_1 and σ_2 are the volatilities of the spot price and the convenience yield respectively.

Defining $Y_t = \ln(S_t)$ and applying Itô's Lemma, the model, under the risk-neutral measure, can be expressed as:

$$\begin{aligned} dY_t &= \left(r - \delta_t - \sigma_1^2 / 2\right)dt + \sigma_1 dz_1^* \\ d\delta_t &= \left[\kappa(\alpha - \delta_t) - \lambda\right]dt + \sigma_2 dz_2^* \end{aligned}$$

Where $d\,z_1^*$ and $d\,z_2^*$ are the Brownian motions under the equivalent martingale measure, which are assumed to be correlated, i.e. $d\,z_1^* dz_2^* = \rho\, dt$, λ is the market price of risk associated to the convenience yield and r is the risk-free interest rate.

If we define the state vector as $X_t = (Y_t\,, \delta_t)'$ and after applying the results in section 2, it is easy to prove that X_t is normally distributed with a mean and variance given by the following expressions[6,7]:

$$E^*[X_t] = \begin{pmatrix} (r - \sigma_1^2/2 - \alpha)t + \alpha(1 - e^{-kt})\lambda/k \\ \alpha(1 - e^{-kt}) \end{pmatrix} + \begin{pmatrix} 1 & -(1 - e^{-kt})/k \\ 0 & e^{-kt} \end{pmatrix} X_0$$

$$Var^*[X_t] =$$

$$\begin{pmatrix} \sigma_1 t + 2\sigma_1\sigma_2\rho(1 - e^{-kt} - kt)/k^2 - \sigma_2^2(3 - 4e^{-kt} + e^{-2kt} - 2kt)/2k^3 & \sigma_1\sigma_2\rho(1 - e^{-kt})/k + \sigma_2^2(1 - 2e^{-kt} + e^{-2kt})/2k^2 \\ \sigma_1\sigma_2\rho(1 - e^{-kt})/k + \sigma_2^2(1 - 2e^{-kt} + e^{-2kt})/2k^2 & \sigma_2^2(1 - e^{-kt})/k \end{pmatrix}$$

Therefore, $Y_t = \ln(S_t)$ is also Gaussian, under the risk-neutral measure, with mean:

$$Y_0 - \delta_0(1 - e^{-kt})/k + (r - \sigma_Y^2/2 - \alpha^*)t + \alpha^* k(1 - e^{-kt})/k^2$$

where $\alpha^* = \alpha - \lambda/\kappa$, and variance:

$$(\sigma_1^2 + \sigma_2^2/k^2 - 2\sigma_1\sigma_2\rho/k)t + (1 - e^{-2\kappa t})\sigma_2^2/2\kappa^3 + 2(\rho\sigma_1\sigma_2 - \sigma_2^2/k)\left(1 - e^{-\kappa t}\right)/\kappa^2 .$$

Finally, given that the spot price S_t is lognormal, the futures price can be expressed as:

$$F_{0,T} = E^*[S_T] = \exp\left(E^*[Y_T] + \frac{1}{2}Var^*[Y_T]\right) =$$
$$\exp\{Y_0 - \delta_0(1 - e^{-kt})/k + (r - \alpha^* + \sigma_2^2/2k^2 - \sigma_1\sigma_2\rho/k)T$$
$$+ (1 - e^{-2\kappa t})\sigma_2^2/4\kappa^3 + (\alpha^* k + \rho\sigma_1\sigma_2 - \sigma_2^2/k)\left(1 - e^{-\kappa t}\right)/\kappa^2\}$$

---

[6] E*[] and Var*[] are the mean and variance under the risk neutral measure.
[7] Here, in this section, we shall use the formulas in integral form, without resorting to (3) and (4).

This is the result already obtained in Schwartz (1997), equation 20, but avoiding unnecessary partial differential equations.

Using the results in section 2, the squared volatility of futures returns can be expressed as:

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & \left(e^{-\kappa T}-1\right)/\kappa \\ 0 & e^{-\kappa T}\end{pmatrix}\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\begin{pmatrix}1 & 0 \\ \left(e^{-\kappa T}-1\right)/\kappa & e^{-\kappa T}\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix}=$$

$$\sigma_1^2+\left(1-e^{-\kappa T}\right)^2\sigma_2^2/\kappa^2-2\left(1-e^{-\kappa T}\right)\rho\sigma_1\sigma_2/\kappa$$

which is the same formula as in Schwartz (1997), equation 40.

Now let us express the model in its discrete-time version. Following Schwartz's notation the model can be expressed as[8]:

$$X_t=c_t+M_tX_{t-1}+\psi_t$$

where:

$$c_t=\begin{pmatrix}(\mu-\sigma_1^2/2-\alpha)\Delta t+\alpha(1-e^{-k\Delta t})\lambda/k \\ \alpha(1-e^{-k\Delta t})\end{pmatrix}, M_t=\begin{pmatrix}1 & \left(e^{-\kappa\Delta t}-1\right)/\kappa \\ 0 & e^{-\kappa\Delta t}\end{pmatrix} \tag{9}$$

and the error term vector, denoted as ψ_t, is a n-vector of serially uncorrelated Gaussian disturbances with zero mean and variance given by the following expression:

$$Var[\psi_t]=$$
$$\begin{pmatrix}\sigma_1\Delta t+\dfrac{2\sigma_1\sigma_2\rho(1-e^{-k\Delta t}-k\Delta t)}{k^2}-\dfrac{\sigma_2^2(3-4e^{-k\Delta t}+e^{-2k\Delta t}-2k\Delta t)}{2k^3} & \dfrac{\sigma_1\sigma_2\rho(1-e^{-k\Delta t})}{k}+\dfrac{\sigma_2^2(1-2e^{-k\Delta t}+e^{-2k\Delta t})}{2k^2} \\ \dfrac{\sigma_1\sigma_2\rho(1-e^{-k\Delta t})}{k}+\dfrac{\sigma_2^2(1-2e^{-k\Delta t}+e^{-2k\Delta t})}{2k^2} & \dfrac{\sigma_2^2(1-e^{-k\Delta t})}{k}\end{pmatrix} \tag{10}$$

[8] Note that these expressions are just the discrete-time counterpart of expressions (8) with $A_D=M_t$ and $d=c_t$ in our notation.

If we perform a Taylor expansion when Δt tends to zero and drop all terms of order higher than one, we get expressions 35 in Schwartz (1997):

$$c_t = \begin{pmatrix} (\mu - \sigma_1^2/2)\Delta t \\ \alpha k \Delta t \end{pmatrix},\; M_t = \begin{pmatrix} 1 & -\Delta t \\ 0 & 1-\kappa\Delta t \end{pmatrix} \text{ and } Var[\psi_t] = \begin{pmatrix} \sigma_1^2\Delta t & \sigma_1\sigma_2\rho\Delta t \\ \sigma_1\sigma_2\rho\Delta t & \sigma_2^2\Delta t \end{pmatrix}$$

Therefore, we can conclude that Schwartz (1997) uses a discrete-time version of the model which is an approximation to the precise one presented above, which is given by expressions (9) and (10). As we will see below, these divergences, specially the more accurate estimator of the variance of the residual, $Var[\psi_t]$, given by expression (10), are important in the valuation of commodity contingent claims.

Next we are going to compare the empirical performance of both estimation procedures, i.e. the precise version of the estimates given in this chapter and the approximate version in Schwartz (1997), using the same data set as in Schwartz (1997). Specifically, the data set is composed of weekly observations of NYMEX WTI crude oil futures contracts, with maturity 1, 3, 5, 7, and 9 months, from 1/1/1985 to 02/13/1995. We have a total of 529 observations[9]. WTI futures prices with one month to maturity are depicted in Figure 1.

The results of the estimation of the two factor model by Schwartz obtained with both estimation procedures are contained in Table 3. The main differences between the results obtained with both procedures are found in the values of κ (the mean-reverting parameter), σ_2 (the volatility of the convenience yield) and λ (the market price of risk associated to the convenience yield). Specifically, the value of κ found with the precise version, 1.5433, is considerable lower than the value found with the Schwartz approximation, 1.8855. Moreover, the value of λ found with the precise version is also

[9] This is one of the data sets used in Schwartz (1997). However in that paper the data set includes 510 observations, instead of 529. That is the reason why the results presented here for Schwartz approximation are not exactly the same as the ones presented in Schwartz (1997).

lower than the value found with the Schwartz approximation (0.2181 and 0.2558 respectively). Finally, the value of σ_2 obtained with the precise and approximate versions is 0.3967 and 0.4622 respectively. In general looking at the Table we can appreciate that all the values found with the approximate version used by Schwartz (1997) are higher than the corresponding values found with the precise version. Therefore, we can conclude that, at least with this data set, the approximate version by Schwartz (1997) tends to overestimate the parameters.

Figures 2 and 3 present the differences between one month WTI futures prices and the spot price calculated with both the precise and the approximated estimates[10]. Specifically, Figure 2 compares the predictive ability of both estimates in terms of the mean error (ME), defined as the average of the series of one month futures price minus estimated spot prices, whereas in Figure 3 it is used the root mean squared error (RMSE).

In the full sample period, 1985-1995, the precise estimates outperform the approximation by Schwartz (1997), using the two metrics. This is also the case in all the annual periods considered in the Figures. However, it is interesting to note that the best performance of the precise estimates is found in 1985 and 1990, years which are characterized by high volatility, as can be appreciated in Figure 1. This fact is not surprising since, as pointed out above, one of the main advantage of the precise methodology is that it provides a more accurate estimation of the variance of the residual, $Var[\psi_t]$, which is given by expression (10). Finally, it is worth noting that the mean error is negative in the whole sample period, implying that both estimates tend to

[10] To the best of our knowledge, there is no reliable index which reflects the WTI crude oil spot price. Therefore, the best available approximation for it, NYMEX WTI crude oil futures contracts with one month to maturity, is used.

overestimate spot prices. It is also the case in all the annual periods, except for 1986, 1993 and 1994.

Figures 4 and 5 show the differences between one month WTI futures and spot prices calculated with both the precise and the approximated estimates, by month. The results are similar to those obtained in Figures 2 and 3, i.e. the precise estimates outperform the approximation by Schwartz (1997), using the two metrics (mean error and root mean squared error), in all months, except for March with the mean error measure.

Finally, Table 4 compares the improvement[11] (expressed in percentage) in the RMSE and the standard deviation of one-month futures price, by month. Interestingly, the highest improvement in the RMSE is obtained in October and November, which are that the months characterized by the highest degree of variance. As pointed out above, this result can be related with the fact that one of the main advantages of the precise estimation procedure is that it provides a more accurate estimation of the variance of the residual, $Var[\psi_t]$, which is given by expression (10). It should be noted, however, that there are also months with no such high variance showing a high improvement in the RMSE (January and December).

1.5. SIMPLIFIED DEDUCTION OF THE FUTURES PRICES IN THE TWO-FACTOR MODEL BY SCHWARTZ AND SMITH (2000)

Let us consider the two-factor model in Schwartz and Smith (2000). They assume that the spot log-price of a commodity at time t, $\ln(S_t)$, can be decomposed as the sum of a short-term deviation, χ_t, and the equilibrium price level, ξ_t: $\ln(S_t) = \chi_t + \xi_t$.

[11] Defined as the RMSE computed with the Schwartz approximation minus the RMSE computed with the precise version of the estimates.

The short-term deviation and the equilibrium level are assumed to follow a mean-reverting process (toward zero) and a standard Brownian motion respectively, i.e.:

$$\begin{cases} d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \\ d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi \end{cases}$$

Where dz_χ and dz_ξ are standard Brownian motions with correlation ρ, i.e. $dz_\chi dz_\xi = \rho dt$, κ represents the rate at which the short-term deviations revert toward zero (the mean-reverting coefficient), μ_ξ is the equilibrium total return and σ_χ and σ_ξ are the volatilities of the short-term deviation and the equilibrium level respectively.

The risk-neutral version of their model is given by the following SDE:

$$\begin{cases} d\chi_t = (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dz_\chi^* \\ d\xi_t = \mu_\xi^* dt + \sigma_\xi dz_\xi^* \end{cases}$$

Where dz_χ^* and dz_ξ^* are again standard Brownian motions with correlation ρ, i.e. $dz_\chi dz_\xi = \rho dt$, $\mu_\xi^* = \mu_\xi - \lambda_\xi$, and λ_χ and λ_ξ are the market prices of risk associated to the short-term deviation and the equilibrium level respectively.

Defining the state vector as $X_t = (\chi_t, \xi_t)'$, the model can be expressed as[12]:

$$dX_t = \left[\begin{pmatrix} -\lambda_\chi \\ \mu_\xi^* \end{pmatrix} + \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix} X_t \right] dt + R dW_t$$

where R is the Choleski decomposition of the noise covariance matrix[13]:

$$\begin{pmatrix} \sigma_\chi^2 & \rho\sigma_\chi\sigma_\xi \\ \rho\sigma_\chi\sigma_\xi & \sigma_\xi^2 \end{pmatrix}$$

[12] See Appendix B.

[13] Note again that R does not need to be calculated as RR' is the noise covariance matrix.

Now, we will use expressions (3) and (4). Note that, as A is diagonal, $P = I$ so we can safely drop P and P^{-1} from all expressions.

It is easy to see that (note that, in order to comply with Schwartz and Smith's notation, $D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$, the null part is in the bottom of the matrix):

$$J(t) = \begin{pmatrix} \dfrac{1-e^{-\kappa t}}{\kappa} & 0 \\ 0 & t \end{pmatrix} \quad \exp(At) = \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix}$$

$$G(t) = \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} vec^{-1}\left[\begin{pmatrix} \dfrac{e^{2\kappa t}-1}{2\kappa} & 0 & 0 & 0 \\ 0 & \dfrac{e^{\kappa t}-1}{\kappa} & 0 & 0 \\ 0 & 0 & \dfrac{e^{\kappa t}-1}{\kappa} & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \begin{pmatrix} \sigma_\chi^2 \\ \rho\sigma_\chi\sigma_\xi \\ \rho\sigma_\chi\sigma_\xi \\ \sigma_\xi^2 \end{pmatrix} \right] \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{e^{2\kappa t}-1}{2\kappa}\sigma_\chi^2 & \dfrac{e^{\kappa t}-1}{\kappa}\rho\sigma_\chi\sigma_\xi \\ \dfrac{e^{\kappa t}-1}{\kappa}\rho\sigma_\chi\sigma_\xi & t\sigma_\xi^2 \end{pmatrix} \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{1-e^{2\kappa t}}{2\kappa}\sigma_\chi^2 & \dfrac{1-e^{\kappa t}}{\kappa}\rho\sigma_\chi\sigma_\xi \\ \dfrac{1-e^{\kappa t}}{\kappa}\rho\sigma_\chi\sigma_\xi & t\sigma_\xi^2 \end{pmatrix}$$

Now, the mean and variance of X_t are:

$$E^*[X_t] = \begin{pmatrix} -(1-e^{-kt})\lambda_\chi / k \\ \mu_\xi^* \end{pmatrix} + \begin{pmatrix} e^{-\kappa t} & 0 \\ 0 & 1 \end{pmatrix} X_0$$

$$Var^*[X_t] = G(t) = \begin{pmatrix} \left(1-e^{-2\kappa t}\right)\sigma_\chi^2 / 2\kappa & \left(1-e^{-\kappa t}\right)\rho\sigma_\chi\sigma_\xi / \kappa \\ \left(1-e^{-\kappa t}\right)\rho\sigma_\chi\sigma_\xi / \kappa & \sigma_\xi^2 t \end{pmatrix}$$

In this model, the log of spot price, $Y_t = \ln(S_t)$, is given by $\chi_t + \xi_t$. Thus, $\ln(S_t)$ is a Gaussian variable with mean:

$$e^{-\kappa t}\chi_0 + \xi_0 + \mu_\xi^* t - (1 - e^{-kt})\lambda_\chi / k$$

and variance:

$$\left(1 - e^{-2\kappa t}\right)\sigma_\chi^2 / 2\kappa + 2\left(1 - e^{-\kappa t}\right)\rho\sigma_\chi\sigma_\xi / \kappa + \sigma_\xi^2 t .$$

Finally, the spot price, S_t, is lognormal distributed, and, therefore, the futures price can be written as:

$$F_{0,T} = E^*[S_T] = \exp\left(E^*[Y_T] + \frac{1}{2}Var^*[Y_T]\right) =$$

$$= \exp\left\{e^{-\kappa t}\chi_0 + \xi_0 + \mu_\xi^* t - (1 - e^{-kt})\lambda_\chi / k + \frac{\left(1 - e^{-2\kappa t}\right)\sigma_{\chi_\chi}^2 / 2\kappa + 2\left(1 - e^{-\kappa t}\right)\rho\sigma_\chi\sigma_\xi / \kappa + \sigma_\xi^2 t}{2}\right\}$$

We have obtained the same result as in Schwarz and Smith (2000), Equation 9, but in a simpler way, avoiding unnecessary limit steps.

1.6. CONCLUSIONS

The stochastic behaviour of commodity prices has been a common topic of research during the last years. However, the application of the standard Black-Scholes analysis is not straightforward, due to the complex dynamics of commodity prices. This is the reason why most of these studies present ad-hoc solutions, which are very complex and sometimes include approximations.

This article shows how to simplify formulae and deductions, and even compute an explicit matrix general formula, using well known techniques and results in stochastic calculus. This formula has been tested on real data and is a real alternative to programming each model separately.

Concretely, we show how to obtain more precise estimators of the parameters in the Schwartz (1997) two-factor model context, than the approximations given by the author. It is found that, in general, the approximations by Schwartz tend to overestimate the parameters. These divergences are important in the valuation of commodity contingent claims. Moreover, we have shown how to obtain the expression for the futures price given by Schwartz and Smith (2000) in a simpler way, avoiding unnecessary limit steps.

APPENDIX A: MATHEMATICAL REFERENCE RESULTS

In order to understand the results, it is necessary to introduce some mathematical preliminaries. All the concepts and formulae here shall be presented in an intuitive way, stressing the practical implementation.

First of all, we remind the reader some well known concepts. For an extensive review of matrix algebra and matrix derivatives, we recommend Magnus and Neudecker (1999).

- The derivative and integral of a time-dependent matrix (which we shall denote $A(t)$ or A_t indistinctly) are given element by element:

$$\frac{d}{dt}A(t)=\begin{pmatrix}\frac{d}{dt}a_{11}(t) & \dots & \frac{d}{dt}a_{1n}(t)\\ \dots & \dots & \dots \\ \frac{d}{dt}a_{m1}(t) & \dots & \frac{d}{dt}a_{mn}(t)\end{pmatrix}, \qquad \int_r^s A(t)dt=\begin{pmatrix}\int_r^s a_{11}(t)dt & \dots & \int_r^s a_{1n}(t)dt\\ \dots & \dots & \dots \\ \int_r^s a_{m1}(t)dt & \dots & \int_r^s a_{mn}(t)dt\end{pmatrix}.$$

  Indefinite integrals $\int A_t dt$ are defined in the same way. Linear properties, such as $\frac{d}{dt}(BA_t)=B\frac{d}{dt}A_t$, are easy to prove and shall be used without explicitly mentioning them.

- The matrix exponential of a diagonalizable matrix $A=PDP^{-1}$ with D diagonal is: $\exp(A)=P\begin{pmatrix}\exp(d_1) & 0 & 0\\ \dots & \dots & \dots \\ \dots & \dots & \exp(d_n)\end{pmatrix}P^{-1}$. It is not hard to see the equality $\frac{d}{dt}\exp(At)=A\exp(At)$

- Given two matrices $A\in\Re^{pxq}, B\in\Re^{mxn}$ their Kronecker product is a *pm* x *qn* matrix defined as: $A\otimes B=\begin{pmatrix}a_{11}B & a_{12}B & \dots & a_{1q}B\\ a_{21}B & a_{22}B & \dots & a_{2q}B\\ \dots & \dots & \dots & \dots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B\end{pmatrix}$.

- The *vec* operator is defined as: $vec\begin{pmatrix} a_{11} & ... & a_{1q} \\ ... & ... & ... \\ a_{p1} & ... & a_{pq} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{p1} \\ a_{12} \\ ... \\ a_{p2} \\ ... \end{pmatrix}$.

- Integrals with a single product: We shall calculate $\int_r^s \exp(At)H\,dt$ where H is an arbitrary constant matrix. Let $A = PDP^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}P^{-1}$ with D diagonal and D_1 non-singular. The previous integral is therefore easily computed explicitly as: $\int_r^s \exp(At)Hdt = P\left(\int_r^s \exp(Dt)dt\right)P^{-1}H = \left[P\begin{pmatrix} tI & 0 \\ 0 & \exp(D_1 t) \end{pmatrix}P^{-1}H\right]_r^s =$ $P\begin{pmatrix} (s-r)I & 0 \\ 0 & D_1^{-1}(\exp(D_1 s) - \exp(D_1 r)) \end{pmatrix}P^{-1}H$

- Integrals with double product: We shall calculate $\int_r^s U\exp(At)H\exp(At)'V\,dt$, where *U, H, V* are arbitrary constant matrices. As before: $A = PDP^{-1} = P\begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}P^{-1}$,

  $\int_r^s U\exp(At)H\exp(At)'V\,dt = UP\left(\int_r^s \exp(Dt)P^{-1}H\left(P^{-1}\right)'\exp(Dt)'\,dt\right)P'V$ so we shall focus on the middle part. Using the vec operator: $\int_r^s \exp(Dt)H\exp(Dt)'\,dt = vec^{-1}\left[vec\left(\int_r^s \exp(Dt)\left(P^{-1}H\left(P^{-1}\right)'\right)\exp(Dt)'\,dt\right)\right] =$ $vec^{-1}\left[\int_r^s (\exp(Dt)\otimes\exp(Dt))vec\left(P^{-1}H\left(P^{-1}\right)'\right)dt\right] =$ $= vec^{-1}\left\{\left[\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right]vec\left(P^{-1}H\left(P^{-1}\right)'\right)\right\}$ The only thing left is to compute the central integral. However, if D is diagonal, let

$D = \begin{pmatrix} 0 & ... & ... & 0 \\ 0 & d_1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & d_n \end{pmatrix}$. Then $\exp(Dt) = \begin{pmatrix} 0 & ... & ... & 0 \\ 0 & e^{d_1 t} & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & e^{d_n t} \end{pmatrix}$. The Kronecker product is thus given by: $\exp(Dt) \otimes \exp(Dt) = \begin{pmatrix} 0 & ... & ... & 0 \\ 0 & e^{(d_1 I + D_1)t} & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & e^{(d_n + D_1)t} \end{pmatrix}$. If no eigenvalue is exactly the opposite of another eigenvalue the integral is given by $\int_r^s \exp(Dt) \otimes \exp(Dt) = \begin{pmatrix} (r-s)I & ... & ... & 0 \\ 0 & (d_1 I + D_1)^{-1} e^{(d_1 I + D)t} & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & (d_k I + D_1)^{-1} e^{(d_n + D)t} \end{pmatrix}$ If two eingenvalues are one the opposite of the other, matters are not much more difficult. Let $D = \begin{pmatrix} \mu_1 & 0 & ... & 0 \\ 0 & \mu_2 & ... & 0 \\ ... & ... & ... & 0 \\ 0 & 0 & ... & \mu_k \end{pmatrix}$ including all zero and nonzero eigenvalues. If we just let $\gamma_{ij} = \mu_i + \mu_j$ and substitute in the formula, we have $\exp(Dt) \otimes \exp(Dt) = \exp\left[\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 & ... & 0 \\ 0 & \gamma_{12} & ... & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & \gamma_{1k} & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & ... & \gamma_{kk} \end{pmatrix} t \right]$ and its integral is:

$$\int_r^s \exp(Dt)\otimes\exp(Dt) = \begin{pmatrix} \int_r^s e^{\gamma_{11}t}dt & 0 & 0 & 0 & \ldots & 0 \\ 0 & \int_r^s e^{\gamma_{12}t}dt & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \int_r^s e^{\gamma_{1k}t}dt & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & \int_r^s e^{\gamma_{kk}t}dt \end{pmatrix}.$$ Where

obviously $\int_r^s e^{\gamma_{ij}t}dt = \begin{cases} s-r & \text{for } \gamma_{ij}=0 \\ \dfrac{e^{\gamma_{ij}s}-e^{\gamma_{ij}r}}{\gamma_{ii}} & \text{for } \gamma_{ij}\neq 0 \end{cases}$.

- Note that the expression $vec^{-1}\left\{\left[\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right]vec\left(P^{-1}H\left(P^{-1}\right)'\right)\right\}$ can be done in a different way, using the Hadamard product instead of the Kronecker one and thus avoiding the use of diagonal matrices. To do so, remember that the Hadamard product of A and B denoted $A\bullet B$ is defined each element at a time: $(A\bullet B)_{ij} = A_{ij}B_{ij}$. If we just define $Z = vec^{-1}\left(\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right)$ or equivalently $Z_{ij} = \int_r^s e^{\gamma_{ij}t}dt$, then it is easy to notice, just by substitution, that $vec^{-1}\left\{\left[\int_r^s \exp(Dt)\otimes\exp(Dt)dt\right]vec\left(P^{-1}H\left(P^{-1}\right)'\right)\right\}$ equals $ZP^{-1}H\left(P^{-1}\right)'$. The reader should note, however, that due to the fact that our Kronecker product is diagonal, it does not have to be stored in full, so an efficient implementation of the algorithm will use only the diagonal

All operations are easily implemented in any mathematically adapted computer language such as Matlab.

APPENDIX B: FUTURES CONTRACT VALUATION

Most of the models proposed in the literature assume that the risk-neutral dynamics of a commodity price (or its log) is given by a linear stochastic differential system:

$$\begin{cases} dX_t = \left(b + AX_t\right)dt + RdW_t \\ Y_t \quad = cX_t \end{cases}$$

where Y_t is the commodity price (or its log), b, A, R and c are deterministic parameters[14] independent of t ($b \in \Re^n$, $A, R \in \Re^{n\,x\,n}$, $c \in \Re^n$) and W_t is a n-dimensional canonical Brownian motion (i.e. all components uncorrelated and its variance equal to unity) under the risk-neutral measure.

Let us see that the solution of that problem is[15]:

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As} b ds + \int_0^t e^{-As} R dW_s\right] \tag{B1}$$

In order to proof it, we shall apply the general rule for the derivation of the product of stochastic components (Oksendal, 1992):

$$dX_t = \left(de^{At}\right)\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] + e^{At} d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] + \\ + \left(de^{At}\right) d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right]$$

It is easy to show that:

$$d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] = e^{-At} bdt + e^{-At} RdW_t$$

The first differential only has elements of type dt, hence the product of the first differential times the second differential is zero.

Thus:

[14] Again note that R does not need to be computed.

[15] Even in the case that b, A and R were function of t, if A_t and $\int_0^t A_s ds$ commute, the solution of that problem is (B1).

$$dX_t = Ae^{At}dt\left[X_0 + \int_0^t e^{-As}bds + \int_0^t e^{-As}RdW_s\right] + e^{At}\left[e^{-At}bdt + e^{-At}RdW_t\right] = A_tX_tdt + bdt + RdW_t$$

Consequently we obtain expression (B1):

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As}bds + \int_0^t e^{-As}RdW_s\right].$$

It is easy to prove that the solution is unique (Oksendal, 1992).

An elementary rule of the stochastic calculus states that if J_s is a deterministic function, $\int_0^t J_s dW_s$ is normally distributed with mean zero and variance:

$$Var\left(\int_0^t J_s dW_s\right) = \int_0^t J_s J_s^{\,T} ds \text{ (Itô´s isometry).}$$

Accordingly, X_t is normally distributed with mean and variance[16]:

$$E^*[X_t] = e^{At}\left[X_0 + \int_0^t e^{-As}\, bds\right] \tag{B2}$$

$$Var^*[X_t] = e^{At}\left[\int_0^t e^{-As} RR' e^{-As'} ds\right]e^{At'} \tag{B3}$$

Therefore, Y_t, under the risk-neutral measure, is also Gaussian and it easily follows that its mean and variance are: $E^*[Y_t] = cE^*[X_t]$, $Var^*[Y_t] = cVar^*[X_t]c'$, providing a valuation scheme for all sorts of commodity contingent claims as financial derivatives on commodity prices, real options, investment decisions and other more.

If Y_t is the log of the commodity price (S_t), the price of a futures contract traded at time t with maturity at time $t+T$, $F_{t,T}$, can be computed as:

$$F_{t,T} = E^*\left[S_{t+T} \| I_t\right] = \exp\left\{E^*\left[Y_{t+T} \| I_t\right] + \frac{1}{2}Var^*\left[Y_{t+T} \| I_t\right]\right\} \tag{B4}$$

where I_t is the information available at time t.

[16] E*[] and Var*[] are the mean and variance under the risk neutral measure.

This methodology can be used in all kind of problems (even if b, A and R are functions of t, although, in this case the explicit formulae for the integrals, given in appendix A, do not apply). Moreover, this methodology is much simpler than the ad-hoc solutions presented in the literature that can only be used in the concrete problem for which they were developed and need complex procedures like limit steps (Schwartz and Smith, 2000) or partial differential equations (Schwartz, 1997).

APPENDIX C: VOLATILITY OF FUTURES RETURNS

The squared volatility of a futures contract traded at time t with maturity at time $t+T$ is defined as[17]:

$$\lim_{h\to 0}\frac{Var\left[\log F_{t+h,T}-\log F_{t,T}\right]}{h}.$$

We will prove that it is the expected value of the square of the coefficient of the Brownian motion (σ_t) in the expression $d\log\left(F_{t,T}\right)=\mu_t ds+\sigma_t dW_t^F$, where W_t^F is a scalar canonical Brownian motion, as long as μ_t is mean squared bounded in an interval containing t (it does not matter whether it is a function of $F_{t,T}$ or not) and $E\left[\sigma_t^2\right]$ is continuous in t[18].

Expressing $d\log F_{t,T}=\mu_t dt+\sigma_t dW_t$ in the equivalent integral form:

$$\log F_{t+h,T}-\log F_{t,T}=\int_t^{t+h}\mu_s ds+\int_t^{t+h}\sigma_s dW_s\ ,$$

its expected value is $\int_t^{t+h}E\left[\mu_s\right]ds$. Therefore, its variance is given by:

$$Var\left[\log F_{t+h,T}-\log F_{t,T}\right]=E\left[\left(\int_t^{t+h}\mu_s-E\left[\mu_s\right]ds+\int_t^{t+h}\sigma_s dW_s\right)^2\right].$$

Using standard properties:

$$E\left[\left(\int_t^{t+h}\mu_s-E\left[\mu_s\right]ds+\int_t^{t+h}\sigma_s dW_s\right)^2\right]=E\left[\left(\int_t^{t+h}\mu_s-E\left[\mu_s\right]ds\right)^2\right]+E\left[\left(\int_t^{t+h}\sigma_s dW_s\right)^2\right]$$

as μ_t is non-anticipating.

By Itô's isometry: $E\left[\left(\int_t^{t+h}\sigma_s dW_s\right)^2\right]=\int_t^{t+h}E\left[\sigma_s^2\right]ds$

[17] The same results are going to be obtained if the volatility is defined as:
$\lim_{h\to 0}\frac{Var\left[\log F_{t+h,T-h}-\log F_{t,T}\right]}{h}$.

[18] In the general problem of this article these conditions are satisfied.

Taking limits and using the mean value theorem of the integral calculus:

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} E\left[\sigma_s^2\right] ds = E\left[\sigma_t^2\right].$$

For the other term it can be seen that:

$$E\left[\left(\int_t^{t+h} \mu_s - E[\mu_s]ds\right)^2\right] = \left\|\int_t^{t+h} \mu_s - E[\mu_s]ds\right\|_2^2 \leq \left(\int_t^{t+h} \|\mu_s - E[\mu_s]\|_2 ds\right)^2$$

As for some $\delta>0$, μ_t is mean squared bounded in the interval (t-δ, t+δ), when $h\to 0$, this integral is less or equal than $h^2 \sup\left\{\|\mu_s - E[\mu_s]\|_2 : s\in(t-\delta, t+\delta)\right\}$, and $\sup\left\{\|\mu_s - E[\mu_s]\|_2 : s\in(t-\delta, t+\delta)\right\}\leq M$ for some M. Hence,

$$\frac{1}{h}E\left[\left(\int_t^{t+h} \mu_s - E[\mu_s]ds\right)^2\right] \leq \frac{1}{h}h^2 M$$

which converges to 0 when $h\to 0$.

Therefore:

$$\lim_{h\to 0}\frac{Var\left[\log F_{t+h,T} - \log F_{t,T}\right]}{h} = E\left[\sigma_t^2\right].$$

Hence, taking logarithms and differentials on both sides of Equation (B4), it follows that:

$$d\left(\log F_{t,T}\right) = ce^{AT}dX_t = ce^{AT}\left[b + AX_t\right]dt + ce^{AT}R\,dW_t$$

Therefore, the squared volatility is[19]:

$$ce^{AT}RR'e^{AT}{}'c'.$$

[19] Again note that R needs not to be computed as RR' is the noise covariance matrix.

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TABLES AND FIGURES

TABLE 1

TIME (MILISECONDS) NEEDED FOR AN EVALUATION OF THE LOG-LIKELIHOOD FUNCTION

Integral stands for using a symbolic processor to compute the integral each step. General means using the same script (formulae (3) and (4) in matrix form) for all models and Particular means writing down the formulae for each case.

Data	Brent		Heating oil		WTI	
Factors	2	3	2	3	2	3
Integral	2785.00	7881.34	3316.16	14774.04	5404.36	3916.64
General	61.48	64.28	55.48	56.08	75.52	89.12
Particular	47.08	49.88	33.06	34.64	57.48	70.10

TABLE 2

TIME (SECONDS) FOR A FULL ESTIMATION OF A MODEL

General means using the same script (formulae (3) and (4) in matrix form) for all models and Particular means writing down the formulae for each case. Integrating symbolically each step would be computationally burdensome.

Data	Brent		Heating oil		WTI	
Factors	2	3	2	3	2	3
General	74.10	250.02	59.39	180.23	91.70	210.33
Particular	60.26	220.97	39.31	128.06	69.53	234.42

FIGURE 1

WTI FUTURES PRICE WITH ONE MONTH TO MATURITY

TABLE 3

THE TWO-FACTOR MODEL BY SCHWARTZ (1997). PRECISE AND APPROXIMATE ESTIMATES

The Table shows the parameter estimates obtained with the Schwartz (1997) approximation and with the precise method described in this chapter. Standard errors in parenthesis.

Parameter	Precise Method	Schwartz Approximation
μ	0.1629 (0.0725)	0.1678 (0.0732)
k	1.5433 (0.0318)	1.8855 (0.0356)
α	0.1458 (0.0558)	0.1496 (0.0545)
σ_1	0.3278 (0.0073)	0.3293 (0.0072)
σ_2	0.3967 (0.0113)	0.4622 (0.0119)
ρ	0.8073 (0.0104)	0.8084 (0.0107)
λ	0.2181 (0.0864)	0.2558 (0.1029)

FIGURE 2

MEAN ERROR BY YEAR

The Figure shows the differences (mean error) between the one month futures price and the spot price calculated with precise and approximated estimates, by year.

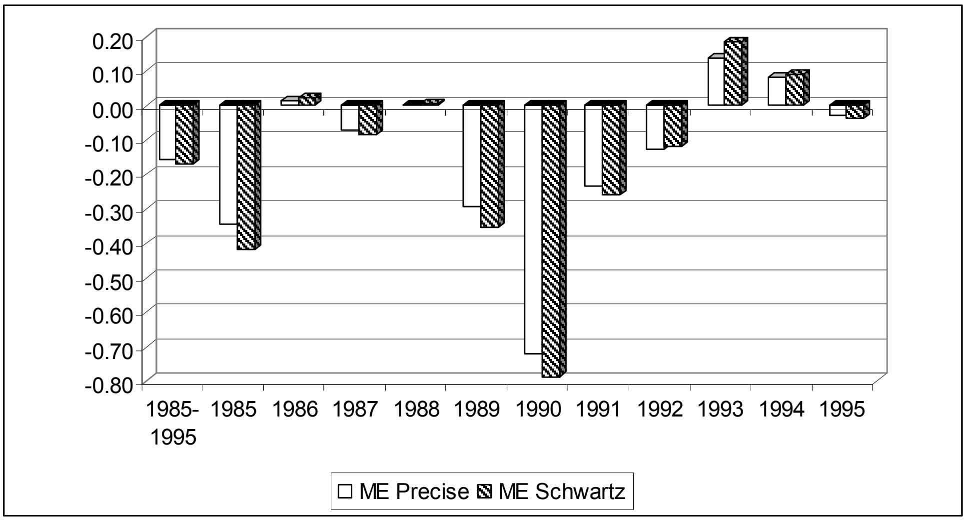

FIGURE 3

ROOT MEAN SQUARED ERROR BY YEAR

The Figure shows the differences (root mean squared error) between the one month futures price and the spot price calculated with precise and approximated estimates, by year.

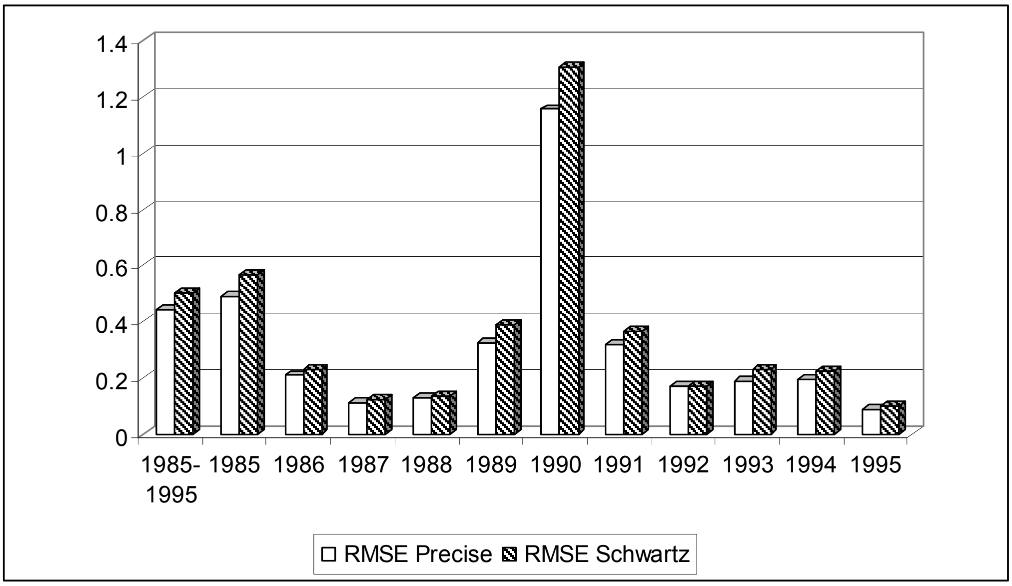

FIGURE 4

MEAN ERROR BY MONTH

The Figure shows the differences (mean error) between the one month futures price and the spot price calculated with both precise and approximated estimates, by month.

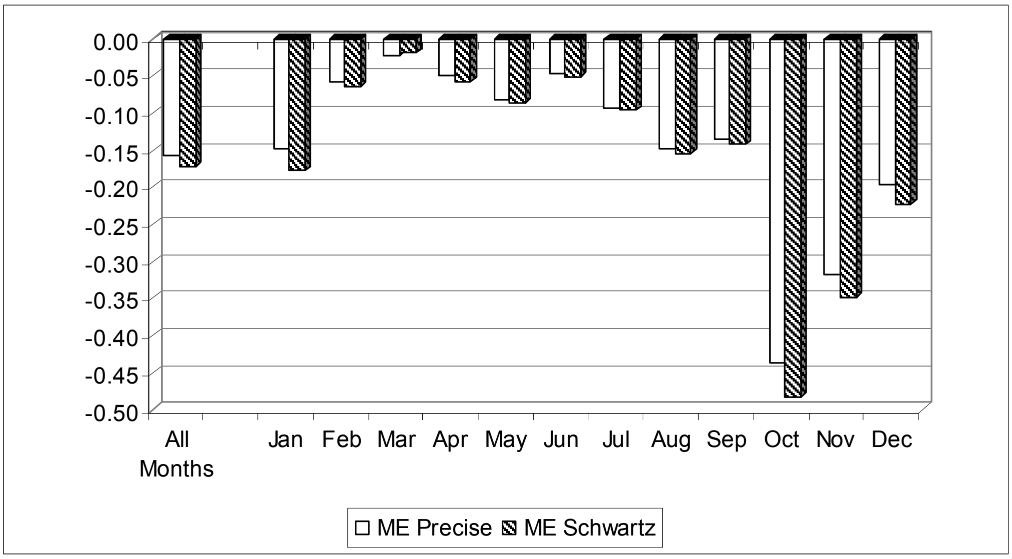

FIGURE 5

ROOT MEAN SQUARED ERROR BY MONTH

The Figure shows the differences (root mean squared error) between the one month futures price and the spot price calculated with both precise and approximated estimates, by month.

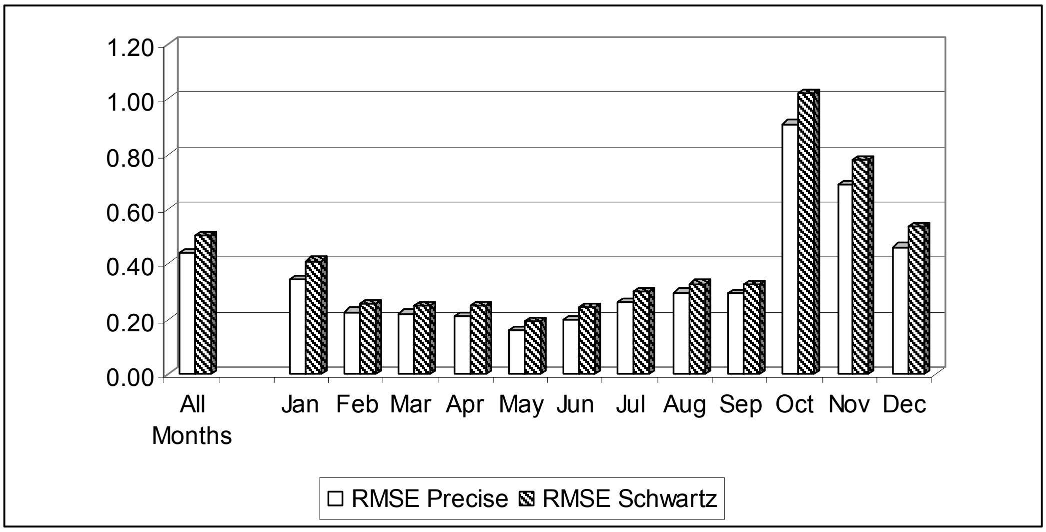

TABLE 4

COMPARISON OF THE IMPROVEMENT IN THE RMSE AND ONE-MONTH FUTURES PRICE STANDAR DEVIATION BY MONTH

The Table shows the improvement (expressed in percentage) in the RMSE, defined as the RMSE computed with the Schwartz approximation minus the RMSE computed with the precise version of the estimates, and one-month futures price standard deviation, by month.

	Improvement RMSE (%)	Volatility
All Months	6.06341562	4.5066963
January	6.69700526	3.45920263
February	2.90069147	3.43375304
March	2.86456161	3.9271667
April	3.82981177	3.88082312
May	3.20130602	3.37948674
June	4.20386706	3.61776438
July	4.02239618	4.05271984
August	3.25451898	4.14305907
September	3.37241986	4.25738991
October	11.0467666	6.73405967
November	8.73584998	5.73730612
December	7.36128089	4.18504435

CHAPTER 2: COMMODITY DERIVATIVES VALUATION UNDER A FACTOR MODEL WITH TIME-VARYING RISK PREMIA

2.1 INTRODUCTION

In equity markets, the market price of risk is the excess return over the risk-free rate per unit standard deviation $\left((\mu - r)/\sigma\right)$ that investors want as compensation for taking risk, which is also called the Sharpe ratio. This ratio plays an important role in derivatives valuation. If the underlying asset is a traded asset, it is possible to build a risk-free portfolio by buying the derivative and selling the underlying asset or vice versa. Consequently, the market price of risk does not appear in the derivatives valuation model.

However, if the underlying asset is not a traded asset, there is no way of building a riskless portfolio by buying the derivative and selling the underlying asset or vice versa; therefore, we must know how much return is needed to compensate the unhedgeable risk. This is why the market price of risk must be estimated to obtain a theoretical value for the derivative asset.

In commodity markets, the market price of risk has a slightly different definition. As noted by Kolos and Ronn (2008), equities require a costly investment and, consequently, return the risk-free rate under the risk-neutral measure. In the case of commodities, it should be noted that sometimes there is a storage cost associated with storing the commodity and also a convenience yield associated with holding the commodity rather than the derivative asset. Nevertheless, futures contracts are costless to enter into; therefore, their risk-neutral drift is zero. Thus, the market price of risk in commodity markets is defined as the ratio of the asset return to its standard deviation(μ/σ). Additionally, whereas the market price of risk must be positive in equity markets, it can be negative in commodity markets.

There have been several papers that have analyzed the properties of market prices of risk in commodity markets and their relation with other variables. Fama and French (1987 and 1988) note the importance of allowing for time-varying risk premia as negative correlations between spot prices and risk premia can generate mean reversion in spot prices. Bessembinder (1992) shows that market prices of risk in financial and commodity markets are related to the covariance of the market portfolio and the futures returns. Routledge et al. (2001) and Bessembinder and Lemmon (2002) relate market prices of risk to several measures of uncertainty, such as price volatility, spikes and uncertainty in demand. Moosa and Al-Loughani (1994), Sardosky (2002) and Jalali-Naini and Kazemi-Manesh (2006) find evidence of variable risk premia in oil markets using GARCH models.

More recently, Kolos and Ronnn (2008) estimate the market prices of risk for energy commodities, finding positive long-term and negative short-term market prices of risk. Lucia and Torro (2008) find that risk premia in the Nordic Power Exchange (Nord Pool) vary seasonally over the year and are related to unexpected low reservoir levels.

There have also been several papers that have analyzed the importance of allowing for time-varying risk premia from the point of view of asset valuation. Following the ideas in Fama (1984) and Fama and Bliss (1987), Duffee (2002) and Dai and Singleton (2002) propose interest rate models where risk premia are linear functions of the state variables. Casassus and Collin-Dufresne (2005) propose and estimate a three-factor model for commodity spot prices, convenience yields and interest rates where convenience yields depend on spot prices and interest rates, and time-varying (state depending) risk premia using a maximum likelihood method. They also test the importance of the dependence of convenience yields on spot prices and of interest rates on the valuation of a set of theoretical commodity European call options. However, they do not test the importance of time-varying risk premia on the valuation of commodity derivatives.

In this chapter, we extend these ideas by proposing and estimating a commodity derivative valuation model with time-varying risk premia. Time series of market prices of risk for energy commodities (crude oil, heating oil, gasoline and natural gas) are estimated under the most widely used model for commodity derivatives valuation, which is the Schwartz and Smith (2000) model, using the Kalman filter method on a moving windows basis. The results show that market prices of risk vary through time accordingly with several macroeconomic variables related to the business cycle, such as crude oil prices, NAPM (National Association of Purchasing Managers) and S&P 500 indices. These results constitute preliminary evidence that the risk compensation that investors want in a commodity derivative contract varies as market conditions change.

Based on these results, a factor model with market prices of risk depending on the business cycle (proxied by the underlying asset short- and long-term factors) using the Kalman filter method is proposed and estimated[20]. The proposed model with time-varying risk premia is also maximal, in accordance with Dai and Singleton (2000). The valuation results obtained with an extensive sample of commodity American options, traded on the NYMEX, show that the proposed model with time-varying risk premia outperforms standard models with constant risk premia. These results confirm the previous findings shown in the literature of non-constant market prices of risk. Moreover, in the present chapter, it is found that allowing for variable market prices of risk has an important effect in commodity derivative valuation. To the best of our

[20] Contrary to previous papers, such as Casassus and Collin-Dufresne (2005), who use a maximum likelihood method, in the present chapter, the estimation is carried out using the Kalman Filter method, which employs all the information available in the forward curve of commodity futures prices.

knowledge, this is the first time that a model with time-varying (state depending) risk premia is applied to the valuation of exchange-traded commodity derivatives.

The remainder of this chapter is organized as follows. Section 2 presents the data sets used in the chapter. Some preliminary findings regarding the market prices of risk estimation using the maximum-likelihood method proposed by Kolos and Ronn (2008) and the Kalman filter method, and their relation to the business cycle are presented in Section 3. The factor model with time-varying business cycle related market prices of risk is proposed and estimated in Section 4. Section 5 presents the option valuation results obtained with the models with time-varying and constant market prices of risk. Finally, Section 6 concludes with a summary and discussion.

2.2 DATA

In this section, we briefly describe the data that will be used in this and the following sections**.** The data set used in this chapter consists of weekly observations of WTI (light sweet) crude oil, heating oil, unleaded gasoline (RBOB) and natural gas (Henry Hub) futures prices traded on the NYMEX, as well as a set of exogenous variables related to the business cycle.

Currently, there are futures being traded on NYMEX for WTI crude oil with maturities of one month to seven years, for heating oil from one month to eighteen months, for gasoline from one month to twelve months and for Henry Hub natural gas from one month to six years. However, there is not enough liquidity for the futures with longer maturities, especially in the case of gasoline. Therefore, in the cases of WTI crude oil and heating oil, our data set is comprised of futures prices from one to eighteen months (1,338 weekly observations) between 1/1/1985 and 8/16/2010. In the case of RBOB gasoline, the data set is comprised of futures prices from one to nine months (1,338 weekly observations) between 1/1/1985 and 8/16/2010. Finally, in the case of Henry Hub natural gas, the data set is comprised of futures prices from one to eighteen months (1,064 weekly observations) between 4/2/1990 and 8/16/2010. The main descriptive statistics of these variables are contained in Table 1.

To asses the robustness of the results, two different data sets have been employed for each commodity. The first set contains more windows but fewer futures contracts, while the second set contains fewer windows but more futures contracts.

In the case of WTI crude oil, the first data set is comprised of contracts F1, F3, F5, F7 and F9 from 1/1/1985 to 8/16/2010, with 180 windows, yielding a time series of 180 market prices of risk. F1 is the contract for the month closest to maturity, F2 is the contract for the second-closest month to maturity, and so on. The second data set for WTI crude oil is comprised of contracts F1, F4, F7, F11, F15 and F18 from 9/9/1996 to 8/16/2010, with 82 windows, yielding a time series of 82 market prices of risk.

In the case of heating oil, the first data set is comprised of contracts F1, F3, F6, F8 and F10 from 10/14/1985 to 8/16/2010, with 177 windows, yielding a time series of 177 market prices of risk. The second data set for heating oil is comprised of contracts F1, F4, F8, F11, F15 and F18 from 9/9/1996 to 8/16/2010, with 82 windows, yielding a time series of 82 market prices of risk.

In the case of RBOB gasoline, the first data set is comprised of contracts F1, F3, F4, F5 and F7 from 4/29/1985 to 8/16/2010, with 181 windows, yielding a time series of 181 market prices of risk. The second data set for heating oil is comprised of contracts F1, F3, F5, F7 and F9 from 7/17/1995 to 8/16/2010, with 92 windows, yielding a time series of 92 market prices of risk.

Finally, in the case of Henry Hub natural gas, the first data set is comprised of contracts F1, F4, F6, F9 and F11 from 4/16/1990 to 8/16/2010, with 135 windows, yielding a time series of 135 market prices of risk. The second data set for Henry Hub natural gas prices is comprised of contracts F1, F4, F8, F12, F15, F18, F22, F26, F29, F31 and F35 from 5/28/1997 to 8/16/2010, with 76 windows, yielding a time series of 76 market prices of risk.

The set of business cycle-related variables is composed of weekly observations from 1/1/1985 to 8/16/2010 of WTI one month futures prices and S&P 500 index prices, as well as monthly observations of the NAPM (National Association of Purchasing Managers) index and the indicator of the expansion of the economy, which takes the value 1 (0) if the NAPM index is above (below) 50.

2.3 PRELIMINARY FINDINGS

In this section, we present some preliminary findings regarding the time series evolution of market prices of risk for crude oil, heating oil, gasoline and natural gas, as well as the market prices of risk relationship with the business cycle, using the maximum likelihood method proposed by Kolos and Ronn (2008) and the Kalman filter method.

Market prices of risk estimation using the maximum-likelihood method

Kolos and Ronn (2008) obtain short- and long-term estimates of the market price of risk for several energy commodities assuming the two-factor model by Schwartz and Smith (2000). In this model, the log-spot price (X_t) is assumed to be the sum of two stochastic factors, a short-term deviation (χ_t) and a long-term equilibrium price level (ξ_t). Thus,

$$X_t = \xi_t + \chi_t \tag{1}$$

The stochastic differential equations (SDEs) for these factors are as follows:

$$\begin{cases} d\xi_t = \mu_\xi dt + \sigma_\xi dW_{\xi t} \\ d\chi_t = -\kappa\chi_t dt + \sigma_\chi dW_{\chi t} \end{cases} \tag{2}$$

where $dW_{\xi t}$ and $dW_{\chi t}$ can be correlated ($dW_{\xi t}dW_{\chi t} = \rho_{\xi\chi}dt$) and with $\rho_{\xi\chi}$ representing the coefficient of correlation between long- and short-term factors.

To value derivative contracts, we must rely on the "risk-neutral" version of the model. The SDEs for the factors under the equivalent martingale measure can be expressed as:

$$\begin{cases} d\xi_t = (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dW^*_{\xi t} \\ d\chi_t = (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dW^*_{\chi t} \end{cases} \tag{3}$$

where λ_ξ and λ_χ are the market prices of risk for the long- and short-term factors, respectively, and $W_{\xi t}{}^*$ and $W_{\chi t}{}^*$ are the factor Brownian motions under the equivalent martingale measure.

Schwartz and Smith (2000) and Kolos and Ronn (2008) obtain the SDE for forward contracts (under the historical measure):

$$\frac{dF_t}{F_t} = \left(e^{-\kappa\tau}\lambda_\chi\sigma_\chi + \lambda_\xi\sigma_\xi\right)dt + e^{-\kappa\tau}\sigma_\chi dW_{\chi t} + \sigma_\xi dW_{\xi t} \tag{4}$$

Discretizing equation (4) and applying Ito's Lemma, it is possible to obtain the log-likelihood function, which is (after omitting unessential constants):[21]

$$\ln L = -n\ln\sigma_\chi - \sum_{i=1}^{n}\ln\left(\sqrt{e^{-2\kappa\tau_i} + \left(\sigma_\xi/\sigma_\chi\right)^2}\right) -$$
$$-\frac{1}{2\sigma^2\Delta t}\sum_{i=1}^{n}\left(\frac{\Delta\ln F_i - \left(\lambda_\chi e^{-\kappa\tau_i} + \lambda_\xi\left(\sigma_\xi/\sigma_\chi\right) - \frac{\sigma_\chi\left(e^{-2\kappa\tau_i} + \left(\sigma_\xi/\sigma_\chi\right)^2\right)}{2}\right)\sigma_\chi\Delta t}{\sqrt{e^{-2\kappa\tau_i} + \left(\sigma_\xi/\sigma_\chi\right)^2}}\right)^2 \tag{5}$$

Maximum likelihood estimates of short- and long-term market prices of risk (λ_χ and λ_ξ, respectively), together with the rest of the model parameters, can be obtained by maximizing this log-likelihood function.

[21] See Kolos and Ronn (2008) for the details.

In this chapter, the maximization of the log-likelihood function has been performed subsequently over moving windows of 240 weeks, using weekly observations of one month futures prices for WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas. In this way, we obtain time series of market prices of risk for the four commodity series (180 observations in the case of WTI crude oil, heating oil and RBOB gasoline, and 134 observations in the case of Henry Hub natural gas).

In Figure 1, we plot the time series evolution of the estimated market prices of risk in the case of WTI crude oil[22]. The estimated series show high volatility, which is consistent with the results found by Kolos and Ronn (2008).

The results regarding the coefficients of correlation among the estimated market prices of risk and the business cycle-related variables described above are shown in Table 2. The correlations between short- and long-term risk premia are negative in all cases, although significant only in the case of Henry Hub natural gas.

Positive and significant correlations are found among market prices of risk and WTI one month futures prices[23], except for the long-term risk premium for RBOB gasoline and long- and short-term risk premia for Henry Hub natural gas.

Moreover, positive and significant correlations among market prices of risk and S&P500 (and its one week lag) are found, except for the long-term one in the case of RBOB gasoline and long- and short-term ones in the case of Henry Hub natural gas. In the case of the NAPM index (and its one month lag), positive and significant correlations with short-term market prices of risk for all four commodities and with long-term one in the case of WTI are also found, although the magnitude of the correlation is lower than the magnitude in the S&P500 case (except for the Henry Hub

[22] For the sake of brevity, only the plot of market prices of risk estimated with WTI crude oil are presented here. The plots for the other three commodities show a very similar pattern.

[23] WTI one month futures prices are calculated as the mean of the futures price during the window used to estimate the market price of risk.

natural gas). Finally, evidence of correlation among market prices of risk and the expansion indicator of the economy has not been found. In fact, as can be evidenced in Figure 1, market prices of risk seem to show a "noise pattern" that is not clear and that is not directly associated with market conditions.

In summary, the preliminary analysis performed with the Kolos and Ronn (2008) maximum likelihood method shows evidence of some linear relationship mostly among short-term market risk premia and business cycle-related variables, such as S&P 500 and NAPM indices. As will be discussed herein, the maximum likelihood method used by Kolos and Ronn (2008) and Casassus and Collin-Dufresne (2005) presents some disadvantages when compared to the Kalman filter method used in the next section.

Market Prices of Risk Estimation using the Kalman Filter Method

The Kalman filter method is, theoretically, superior to the maximum likelihood method for several reasons. First, the Kalman filter method estimates all of the dynamic of the underlying asset, whereas the maximum likelihood method only uses market prices of futures contracts without taking into account the dynamics of the common underlying asset. Second, with the Kalman filter method, we are able to use more futures contracts (more maturities), which will result in more stable estimates of the parameters than those obtained with the maximum likelihood method, such as in Kolos and Ronn (2008) and Casassus and Collin-Dufresne (2005).

As stated in Section 3.1 and in the context of the Schwartz and Smith (2000) two-factor model, the log spot price (X_t) is assumed to be the sum of two stochastic factors, a short-term deviation (χ_t) and a long-term equilibrium price level (ξ_t). Moreover, in the cases of commodities, such as natural gas, heating oil and gasoline, a deterministic seasonal component is added, as suggested by Sorensen (2002)[24]. Therefore, the log spot price for heating oil, gasoline and natural gas (X_t) is assumed to be the sum of two stochastic factors (χ_t and ξ_t) and a deterministic seasonal trigonometric component (α_t), $X_t = \xi_t + \chi_t + \alpha_t$. The SDEs for ξ_t and χ_t are given by expressions (2) and:

$$\begin{aligned} d\alpha_t &= 2\pi\varphi\alpha_t^* dt \\ d\alpha_t^* &= -2\pi\varphi\alpha_t \, dt \end{aligned} \tag{6}$$

where α_t^* is the other seasonal factor, which complements α_t, and φ is the seasonal period.

The SDEs for the long- and short- term factors under the equivalent martingale measure are given by expressions (3).

As stated in previous studies, one of the main difficulties in estimating the parameters of the two-factor model is that the short- and long-term factors (or state variables) are not directly observable. Instead, they must be estimated from spot and/or futures prices[25].

The formal method to estimate the model is to use the Kalman filter methodology, which is briefly described in the Appendix[26]. The Kalman filter method has been subsequently performed over moving windows of 240 weeks, using weekly observations of futures prices for WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas[27]. Two different data sets (defined in Section 2) have been employed for each commodity. The first set contains more windows but fewer futures contracts, while the second set contains fewer windows but more futures contracts.

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[24] Sorensen (2002) suggests introducing into the model a deterministic seasonal component for agricultural commodities. Here, we use Sorensen's proposal for heating oil, gasoline and natural gas, which present a strong seasonal behavior (see, for example, Garcia et al., 2011a).

[25] The exact expression for the futures price under the Schwartz and Smith (2000) two-factor model with seasonal factors can be found in Garcia et al. (2011a).

[26] Detailed accounts for Kalman filtering are given in Harvey (1989) and Puthenpura et al. (1995).

[27] Details about implementing the Kalman filter in Matlab can be found in Date and Bang (2009).

In Figure 2, we plot the time series evolution of the estimated market prices of risk in the case of WTI crude oil with the first data set, together with several business cycle-related variables[28]. Looking at the time-evolution of the estimated risk premia, it is clear that we obtain more stable estimates with the Kalman filter method than those obtained with the maximum likelihood method. The results show a negative relationship between long- and short-term market risk premia. Moreover, a positive (negative) relationship between the long- (short-) term market price of risk and the average price of one month WTI futures is found, suggesting that the long- (short-) term risk compensation that investors want to enter in a commodity derivative is positively (negatively) related to crude oil prices[29]. This finding suggests that when crude oil prices are high, the risk associated with the long-term factor (which is the factor that does not disappear with time) tends to not be diversifiable. Moreover, the volatility of one month WTI futures prices is negatively (positively) related to the long- (short-) term market price of risk.

Concerning the estimated market prices of risk p-values, it is found that risk premia are significant (and therefore not diversifiable) during expansion periods or when crude oil prices rise, whereas they are not significant in contraction periods or when crude oil prices decrease, although the pattern is somewhat clearer in the case of the long-term market risk premium, which confirms that the crude oil risk is not diversifiable when crude oil price is high enough. If we consider the relationship between the average long- and short-term factors and the estimated market prices of risk, we find that long-term (short-term) market prices of risk are positively (negatively) related to both long- and short-term factors. Moreover, the estimated market price of risk seems to be positively related to its respective (long- or short-term) factor standard deviation.

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[28] As before, for the sake of brevity, only the plot of market prices of risk estimated with WTI crude oil are presented here. The plots for the other three commodities show a very similar pattern.

[29] As in the previous section, the futures prices average is the mean of the futures price during the window used to estimate the market price of risk.

Finally, a positive (negative) relationship is found between the estimated long-term (short-term) market price of risk and the average NAPM index, the average S&P 500 index and the indicator of expansion[30], suggesting that the risk associated with the long-term factor tends to not be diversifiable during expansion periods.

The results regarding the coefficients of correlation among the estimated market prices of risk and the business cycle related variables described above are shown in Tables 3 for WTI crude oil, 4 for heating oil, 5 for RBOB gasoline and 6 for Henry Hub natural gas. The results confirm the graphical analysis of Figure 2. The relationship between the long- and short-term market prices of risk is found to be negative and significant in the case of WTI crude oil (Table 3) and positive and significant in the cases of heating oil (Table 4), RBOB gasoline (Table 5) and Henry Hub natural gas (Table 6).

It is also interesting to observe the positive and significant relationship found between the long-term market price of risk and WTI futures prices for WTI crude oil, heating oil and RBOB gasoline (the relationship is less clear in the case of Henry Hub natural gas). This result suggests, once again, that the long-term compensation that investors require to enter into a commodity contract rises as WTI futures prices rise[31].

Rather ambiguous relationships are found among the market prices of risk and the volatility of one month WTI futures price, the model volatility and the maximum likelihood, and the NAPM and S&P500 (and their lags) indices.

However, the most obvious relationship is the one found among the estimated market prices of risk and the underlying long- and short-term factors, although the relationship is less clear in the case of Henry Hub natural gas prices. Less clear is the relationship

[30] The indicator of the expansion of the economy takes the value 1 (0) if the NAPM index is above (below) 50.

[31] Cortazar, Milla and Severino (2008) and García, Población and Serna (2011b) show that crude oil and its main refined products (heating oil and gasoline) share common long-term dynamics. Therefore, it is not surprising that the long-term compensations associated with crude oil, heating oil and gasoline are (positively) related to WTI futures prices.

among the market prices of risk and the volatility of the underlying long- and short-term factors.

These findings confirm our previous assumption that the risk compensation that investors want to enter into a commodity derivative contract varies as market conditions change. Specifically, it is quite interesting to observe how the market prices of risk vary according to the underlying long- and short-term factors. Therefore, it seems natural to propose a factor model with market prices of risk depending on the business cycle, proxied by the underlying long- and short-term factors, along the lines suggested by Casassus and Collin-Dufresne (2005), although here we use the Kalman filter method instead of the maximum likelihood method.

2.4 A FACTOR MODEL WITH TIME-VARYING MARKET PRICES OF RISK DEPENDING ON THE BUSINESS CYCLE

Based on the previous results, in this section, a factor model with time-varying market prices of risk depending on the business cycle is proposed and estimated. The proxy for the business cycle will be the Schwartz and Smith (2000) long- and short-term factors, ξ_t and χ_t, respectively. These two factors are found to be the business cycle related variables with higher coefficients of correlation with the estimated market prices of risk. The model with time-varying risk premia will be an extension of the two-factor model described in Section 3, where the log spot price for heating oil, gasoline and natural gas (X_t) is assumed to be the sum of two stochastic factors (χ_t and ξ_t) and a deterministic seasonal trigonometric component (α_t), $X_t = \xi_t + \chi_t + \alpha_t$ ($X_t = \xi_t + \chi_t$ for crude oil), where α_t is defined in expressions (6). The SDEs for the long- and short- term factors under the equivalent martingale measure, with time-varying risk premia, can be expressed as:

$$\begin{cases} d\xi_t = (\mu_\xi - \lambda_{\xi t})dt + \sigma_\xi dW_{\xi t}^* \\ d\chi_t = (-\kappa\chi_t - \lambda_{\chi t})dt + \sigma_\chi dW_{\chi t}^* \end{cases} \tag{7}$$

where, as before, $W_{\xi t}^*$ and $W_{\chi t}^*$ are the factor Brownian motions under the equivalent martingale measure, and $\lambda_{\xi t}$ and $\lambda_{\chi t}$ are time-varying market prices of risk for the long- and short-term factors, respectively.

Following Duffee (2002), Dai and Singleton (2002) and Casassus and Collin-Dufresne (2005), the market prices of risk are expressed as linear functions of the underlying long- and short-term factors:

$$\begin{aligned} \lambda_{\xi t} &= \lambda_{\xi 0} + \lambda_{\xi 1} \cdot \xi_t + \lambda_{\xi 2} \cdot \chi_t \\ \lambda_{\chi t} &= \lambda_{\chi 0} + \lambda_{\chi 1} \cdot \xi_t + \lambda_{\chi 2} \cdot \chi_t \end{aligned} \tag{8}$$

The parameters of the model are estimated, as in Section 3.2, using the Kalman filter method rather than the maximum likelihood procedure used by Casassus and Collin-Dufresne (2005). The results of the estimation of this factor model with time-varying market risk premia, together with the results of the standard two-factor Schwartz and Smith (2000) model with constant risk premia for the four commodity series using both the first and the second data sets described in Section 2 are shown in Table 7 (WTI crude oil), Table 8 (heating oil), Table 9 (RBOB gasoline) and Table 10 (Henry Hub natural gas).

The results in Tables 7, 8, 9 and 10 confirm the presence of the mean reversion effect, typically observed in commodity markets (parameter κ is significant in all cases). Moreover, as expected, both long- and short-term factors are found to be stochastic (their corresponding standard deviations, σ_ξ and σ_χ, respectively, are significant), although the short-term standard deviation is found to be higher than the corresponding long-term standard deviation, suggesting that short-term effects have a higher impact on spot prices than long-term effects[32]. However, as explained above, it must be kept in mind that short-term effects tend to disappear with time (the short-term process is stationary), whereas long-term effects do not disappear with time (the long-term process is integrated).

However, the most important issue in Tables 7, 8, 9 and 10 from the point of view of the goal of this chapter, is that the parameters associated with the market prices of risk ($\lambda_{\xi 0}$, $\lambda_{\xi 1}$, $\lambda_{\xi 2}$, $\lambda_{\chi 0}$, $\lambda_{\chi 1}$ and $\lambda_{\chi 2}$) are significant in most of the cases, confirming that risk premia vary through time depending on the economic conditions (proxied in this chapter by the model long- and short-term factors).

If we define the Schwartz information criterion (SIC) as $\ln(L_{ML}) - q\ln(T)$, where q is the number of estimated parameters, T is the number of observations and L_{ML} is the value of the likelihood function using the q estimated parameters, then the fit is better when the SIC is higher. The same conclusions are obtained with the Akaike information criterion (AIC), which is defined as $\ln(L_{ML}) - 2q$. It is worth noting that in Tables 7, 8, 9 and 10, the values of the SIC and the AIC are higher in the model with time-varying risk premia. This finding confirms the results obtained by Casassus and Collin-Dufresne (2005), in that allowing for time-varying market risk premia improves the estimation results. However, in this chapter, the estimation is carried out using the Kalman filter method, which is theoretically superior to the maximum likelihood method used by Casassus and Collin-Dufresne (2005).

In the next section, we use these results for commodity option valuation purposes. Specifically, we show the importance of allowing for time-varying market risk premia in valuing a set of market traded commodity options. It should be noted that Casassus and Collin-Dufresne (2005) also propose a model with time-varying risk premia, but

[32] This fact is also found in Schwartz and Smith (2000) and Garcia et al. (2011b), among others.

they do not test the importance of time-varying market prices of risk on the valuation of commodity derivatives.

2.5 OPTION VALUATION WITH TIME-VARYING MARKET PRICES OF RISK DEPENDING ON THE BUSINESS CYCLE

As stated above, in this section, we apply our model with time-varying risk premia to the valuation of an extensive set of commodity market traded options.

Option Data

The data set used in the estimation procedure consists of daily observations of WTI, heating oil, RBOB gasoline and Henry Hub natural gas American call and put options quoted at the NYMEX and corresponding to the years from 2006 until 2010. The number of series is 1,293 call and 2,153 put (223,272 and 118,316 observations, respectively) in the case of WTI crude oil; 1,567 call and 302 put (177,927 and 45,725 observations, respectively) in the case of heating oil; 1,633 call and 938 put (145,354 and 59,576 observations, respectively) in the case of RBOB gasoline; and 681 call and 758 put (79,957 and 99,828 observations, respectively) in the case of Henry Hub natural gas.

In the NYMEX, WTI option contracts mature each month for the current year and for the next five years. Additionally, the June and December months are listed beyond the sixth year. Strike prices are the one at-the-money strike price, twenty strike prices in increments of $0.50 per barrel above and below the at-the-money strike price, and the next 10 strike prices in increments of $2.50 above the highest and below the lowest existing strike prices for a total of at least 61 strike prices.

In the case of heating oil and RBOB gasoline options, there are listed contracts for the next 36 consecutive months, and available strike prices are the at-the-money, twenty strike prices in \$0.01 per gallon increments above and below the at-the-money strike price, and the next 10 strike prices in \$0.05 increments above the highest and below the lowest existing strike prices for a total of at least 61 strike prices.

Finally, in the case of Henry Hub natural gas options, there are listed contracts for the consecutive months for the balance of the current year plus 5 additional years. Strike prices are the one at-the-money strike prices, twenty strike prices in increments of \$0.05 per mmBtu above and below the at-the-money strike price in all months, plus an additional 20 strike prices in increments of \$0.05 per mmBtu above the at-the-money price will be offered in the first three nearby months, and the next 10 strike prices in increments of \$0.25 per mmBtu above the highest and below the lowest existing strike prices in all months, for a total of at least 81 strike prices in the first three nearby months and a total of at least 61 strike prices for four months and beyond[33].

In all cases, the underlying asset is the corresponding WTI, heating oil, RBOB gasoline or Henry Hub natural gas futures contract.

Option Valuation Methodology

The computation of American option prices is a challenging problem which implies solving an optimal stopping problem. The problem can be simplified employing Monte Carlo techniques. The starting point of these methods is to replace the time interval of exercise dates by a finite subset. The solution of the corresponding discrete optimal stopping problem reduces to an effective implementation of the dynamic programming principle. However, the conditional expectations involve in the iterations of the dynamic programming cause the main difficulty for the development of the Monte Carlo techniques. One way of treating this problem is the method presented in Longstaff and

[33] Additional details about the contracts can be found on the CME Group web page.

Swchartz (2001), which is one of the most popular American option valuation methods and will be the method used in this section to value commodity American options.

Specifically, the method proposed by Longstaff and Schwartz (2001) consists of estimating the conditional expected pay-off to the holder of the option from continuation using least squares regression techniques.

For the purpose of option valuation, we need a full description of the model. In matrix form, the state dynamics can be described as follows:

$$dZ_t = (\mu + AZ_t)dt + dW_t . \tag{9}$$

To clarify, let us take U_t to be a unit of Brownian motion (i.e., $dU_t dU_t^T = I\,dt$) and rewrite (9) as:

$$dZ_t = (\mu + AZ_t)dt + RdU_t . \tag{10}$$

For parameter estimation purposes, we use Kalman filter equations to estimate $Z_{t|t-1} = E[Z_t / Z_1,...,Z_{t-1}]$, and as an intermediate result, $Z_{t-1|t-1} = E[Z_{t-1} / Z_1,...,Z_{t-1}]$. This process (estimating using current or even future information) is termed "aliasing" in the Kalman filter literature. The series $Z_{t|t}$ is used as initial states for option valuation.

Option Valuation Results

Table 11 presents several metrics to analyze the predictive power ability of the models for the data set of WTI, heating oil, RBOB gasoline and Henry Hub natural gas American options. The models considered are the time-varying risk premia and the standard constant (two-factor) risk premia. Moreover, the results shown in the table are based on the estimation results obtained from both the first and the second data sets described in Section 2.

The statistics presented in Table 11 are the root mean squared error (RMSE), the percentage root mean squared error (PRMSE) and the mean absolute error (MAE), which are defined as:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(f_{i,m} - f_{i,t}\right)^{2}}$$

$$PRMSE = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(f_{i,m} - f_{i,t}\right)^{2}}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} f_{i,m}^{2}}}$$

$$MAE = \frac{1}{n}\sum_{i=1}^{n}\left|f_{i,m} - f_{i,t}\right|$$

where $f_{i,m}$ and $f_{i,t}$ are the market and the theoretical prices, respectively, of option i.

The values shown in the table are the median of the different means for each option series. It can be observed that we achieve better results with the time-varying risk premia model for all commodities under study with all three statistics (except in the case of heating oil using the RMSE with the second data set). It is also worth noting that, in general, we achieve better results using the first data set (at least in the case of WTI crude oil, heating oil and RBOB gasoline). Furthermore, it can be appreciated that the best results of the time-varying model are achieved with RBOB gasoline, followed by heating oil.

These results confirm that the constant risk premia assumption in standard option valuation models has an important effect in terms of valuation errors. Therefore, the fact that market prices of risk vary over time according to the business cycle must be taken into account in option valuation models. Specifically, we have seen that the risk that investors face when they enter in a derivative contract cannot sometimes be diversified, depending on the market conditions, which has important implications in terms of derivative valuation. In particular, it is found that the risk associated with the long-term factor tends to not be diversifiable during expansion periods. Therefore, it seems natural that the risk associated with the model factors is sometimes not diversifiable, depending on the market conditions, and can somewhat affect option values. In this chapter, we have seen that, in fact, by allowing for time-varying (state-depending) market prices of risk option valuation, errors can be reduced compared to those obtained with standard (constant market prices of risk) models.

Finally, it should be noted that there have been several papers proposing factor models with time-varying (state depending) risk premia, such as Casassus and Collin-Dufresne (2005). However, these papers do not test the importance of time-varying risk premia on the valuation of commodity derivatives. To the best of our knowledge, this is the first time that a model with time-varying (state depending) risk premia is applied to the valuation of exchange-traded commodity derivatives.

2.6 CONCLUSIONS

In this chapter, we note the importance of allowing for time-varying market prices of risk in a commodity derivative model. Specifically, we show that the compensation that investors want in a commodity derivative contract varies through time according to several business related variables. More importantly, this business cycle dependence of market prices of risk has an important effect in terms of option valuation errors.

The chapter begins by estimating time series of market prices of risk for crude oil, heating oil, gasoline and natural gas under the two-factor model proposed by Schwartz and Smith (2000) and using the Kalman filter method. The results show that the risk compensation that investors want in a commodity derivative contract varies as market conditions change. Specifically, close relationships among market prices of risk and several variables related to the business cycle, such as NAPM and S&P 500 indices, crude oil prices, crude oil price volatility and long- and short-term price factors, among others, are found.

Based on these results, a factor model with market prices of risk depending on the business cycle and proxied by long- and short-term price factors is proposed and estimated. The valuation results obtained with a sample of futures contracts on crude oil, heating oil, gasoline and natural gas show that the proposed model with time-varying risk premia depending on the business cycle outperforms the standard two-factor model with constant risk premia. This finding confirms the results obtained by Casassus and Collin-Dufresne (2005) in that allowing for time-varying market risk premia improves the estimation results. Nonetheless, in this chapter, the estimation is carried out using the Kalman filter method, which is theoretically superior to the maximum likelihood method used by Casassus and Collin-Dufresne (2005).

However, the most important contribution of this chapter is the application of the model with time-varying risk premia to the valuation of an extensive sample of exchange-traded commodity derivatives. Specifically, the data base is comprised of American options on WTI, heating oil, RBOB gasoline and Henry Hub natural gas futures contracts, traded at NYMEX and yielding better results than those obtained with standard (constant market prices of risk) models. Specifically, we have seen that the risk that investors face when they enter in a derivative contract cannot always be diversified, depending on the market conditions. In particular, it is found that the risk associated with the long-term factor tends to not be diversifiable in expansion periods. Consequently, it is important to take into account the dependence of risk premia on the economic conditions in valuing derivative contracts.

To the best of our knowledge, this is the first time that a model with time-varying (state depending) risk premia is applied to the valuation of exchange-traded commodity derivatives.

APPENDIX

The Kalman filter technique is a recursive methodology that estimates the unobservable time series and the state variables or factors (Z_t) based on an observable time series (Y_t), which depends on these state variables. The *measurement equation* accounts for the relationship between the observable time series and the state variables such that:

$$Y_t = d_t + M_t Z_t + \eta_t \qquad t = 1, \ldots, N_t \tag{A1}$$

where $Y_t, d_t \in \Re^n, M_t \in \Re^{n\,x\,h}$, $Z_t \in \Re^h$, h is the number of state variables, or factors, in the model, and $\eta_t \in \Re^n$ is a vector of serially uncorrelated Gaussian disturbances with zero mean and covariance matrix H_t. To avoid dealing with a large number of parameters, we assume that H_t is diagonal with main diagonal entries equal to σ_η.

The *transition equation* accounts for the evolution of the state variables:

$$Z_t = c_t + T_t Z_{t-1} + \psi_t \qquad t = 1, \ldots, N_t \tag{A2}$$

where $c_t \in \Re^h, T_t \in \Re^{h\,x\,h} \, and \;\, \psi_t \in \Re^h$ are vectors of serially uncorrelated Gaussian disturbances with zero mean and covariance matrix Q_t.

Let $Y_{t|t-1}$ be the conditional expectation of Y_t and let Ξ_t be the covariance matrix of Y_t conditional on all information available at time t – 1. Then, after omitting unessential constants, the log-likelihood function can be expressed as:

$$l = -\sum_t \ln|\Xi_t| - \sum_t (Y_t - Y_{t|t-1})' \Xi_t^{-1} (Y_t - Y_{t|t-1}) \tag{A3}$$

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TABLES AND FIGURES

TABLE 1

DESCRIPTIVE STATISTICS

The table shows the mean and standard deviation (S.D.) of the four commodity series prices. F1 is the futures contract closest to maturity, F2 is the contract second-closest to maturity and so on. In the cases of WTI crude oil and heating oil the data set is comprised of futures prices from one to eighteen months (1338 weekly observations) from 1/1/1985 to 8/16/2010. In the case of RBOB gasoline, the data set is comprised of futures prices from one to nine months (1338 weekly observations) from 1/1/1985 to 8/16/2010. In the case of Henry Hub natural gas, the data set is comprised of futures prices from one to eighteen months (1064 weekly observations) from 4/2/1990 to 8/16/2010.

	WTI Crude Oil		**Heating Oil**		**RBOB Gasoline**		**Henry Hub Natural Gas**	
	Mean	S. D.	Mean	S. D.	Mean	S. D.	Mean	S. D.
F1	33.39	23.56	38.91	27.65	39.76	26.01	4.04	2.60
F2	33.4	23.79	38.94	27.94	39.53	26.01	4.13	2.66
F3	33.37	23.97	38.98	28.22	39.34	25.97	4.19	2.71
F4	33.32	24.11	38.99	28.46	39.16	25.91	4.22	2.73
F5	33.26	24.23	38.97	28.65	39.03	25.93	4.25	2.75
F6	33.2	24.33	38.94	28.81	38.92	25.97	4.27	2.76
F7	33.14	24.41	39	29.06	38.95	26.17	4.29	2.77
F8	33.08	24.48	38.98	29.17	39.56	26.71	4.3	2.78
F9	33.03	24.54	38.94	29.21	40.52	27.28	4.3	2.78
F10	33.15	24.67	38.94	29.27			4.29	2.76
F11	33.6	24.96	39.39	29.59			4.29	2.74
F12	34.24	25.27	40.44	30.17			4.33	2.73
F13	35.1	25.63	42.17	30.92			4.53	2.73
F14	35.35	25.75	42.78	31.25			4.52	2.73
F15	35.6	25.96	43.79	31.77			4.52	2.73
F16	35.7	26.09	44.62	32.29			4.52	2.73
F17	35.8	26.17	46.5	33.03			4.53	2.73
F18	36.2	26.43	49.27	33.84			4.56	2.73

TABLE 2

CORRELATION AMONG MAXIMUM LIKELIHOOD ESTIMATES OF MARKET PRICES OF RISK AND BUSINESS CYCLE RELATED VARIABLES

The table shows the coefficients or correlation among the estimated market prices of risk with the Kolos and Ronn (2008) maximum likelihood method and several business cycle variables. Coefficients of correlation are reported with their standard errors in parenthesis. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	λ_χ	λ_ξ	WTI F1	S&P 500	S&P 500 (-5)	NAPM	NAPM (-1)	Expans. Indicator
WTI CRUDE OIL								
λ_χ	1	-0.0022 (0.0749)	0.3957^{***} (0.06883)	0.5196^{***} (0.0640)	0.5095^{***} (0.0645)	0.1736^{**} (0.0738)	0.1894^{**} (0.0735)	0.0285 (0.0749)
λ_ξ		1	0.3265^{***} (0.0708)	0.3204^{***} (0.0710)	0.2936^{***} (0.0716)	0.1908^{***} (0.0736)	0.2140^{***} (0.0732)	0.0436 (0.0749)
HEATING OIL								
	λ_χ	λ_ξ	WTI F1	S&P 500	S&P 500 (-5)	NAPM	NAPM (-1)	Expans. Indicator
λ_χ	1	-0.0824 (0.0749)	0.3651^{***} (0.0699)	0.4672^{***} (0.0665)	0.4773^{***} (0.0660)	0.1265^{*} (0.0746)	0.1492^{**} (0.0743)	0.0158 (0.0751)
λ_ξ		1	0.2794^{***} (0.0722)	0.3047^{***} (0.0716)	0.3199^{***} (0.0712)	-0.01615 (0.0751)	0.0045 (0.0752)	0.0421 (0.0751)
RBOB GASOLINE								
	λ_χ	λ_ξ	WTI F1	S&P 500	S&P 500 (-5)	NAPM	NAPM (-1)	Expans. Indicator
λ_χ	1	-0.3030 (0.7016)	0.3902^{***} (0.0692)	0.4746^{***} (0.0662)	0.4666^{***} (0.0665)	0.1862^{**} (0.0738)	0.1949^{***} (0.0737)	0.1129 (0.0747)
λ_ξ		1	-0.0512 (0.0751)	0.0821 (0.0749)	0.0713 (0.0750)	0.1042 (0.0747)	0.1087 (0.0747)	-0.0293 (0.0751)
HENRY HUB NATURAL GAS								
	λ_χ	λ_ξ	WTI F1	S&P 500	S&P 500 (-5)	NAPM	NAPM (-1)	Expans. Indicator
λ_χ	1	-0.4930^{***} (0.0749)	-0.0623 (0.0859)	0.2131^{**} (0.0841)	0.1832^{**} (0.0846)	0.2052^{**} (0.0842)	0.2059^{**} (0.0842)	0.1098 (0.0855)
λ_ξ		1	-0.0679 (0.0859)	-0.1399 (0.0852)	-0.1134 (0.0855)	0.0306 (0.0860)	0.0224 (0.0860)	0.0918 (0.0857)

TABLE 3

CORRELATION AMONG WTI KALMAN FILER ESTIMATES OF MARKET PRICES OF RISK AND BUSINESS CYCLE RELATED VARIABLES

The table shows the coefficients or correlation among the estimated WTI crude oil market prices of risk with the Kalman filter method and several business cycle related variables. Coefficients of correlation are reported with their standard errors in parenthesis. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	WTI FRIST DATA SET		WTI SECOND DATA SET	
	λ_ξ	λ_χ	λ_ξ	λ_χ
λ_ξ	1	-0.5813*** (0.0610)	1	-0.3351*** (0.1053)
λ_χ	-0.5813*** (0.0610)	1	-0.3351*** (0.1053)	1
WTI F1	0.5492*** (0.0626)	-0.8503*** (0.0394)	0.3829*** (0.1033)	-0.8691*** (0.0553)
VOLAT. F1	0.1228** (0.0744)	-0.0324 (0.0749)	-0.8765*** (0.0538)	0.1132 (0.1111)
MODEL VOLAT.	0.0666 (0.0748)	0.0581 (0.0748)	-0.7846*** (0.0693)	0.3445*** (0.1050)
LIKELIHOOD	-0.2528*** (0.0725)	0.4568*** (0.0667)	-0.8910*** (0.0508)	0.2716** (0.1076)
ξ	0.6574*** (0.0565)	-0.8424*** (0.0404)	0.4614*** (0.0992)	-0.7809*** (0.0698)
χ	0.1981*** (0.0735)	-0.5239*** (0.0638)	0.4989*** (0.0969)	-0.7385*** (0.0754)
σ_ξ	0.8234*** (0.0425)	-0.6554*** (0.0566)	0.8642*** (0.0562)	-0.4755*** (0.0984)
σ_χ	0.0370 (0.0449)	-0.0046 (0.0750)	-0.6753*** (0.0825)	0.3362*** (0.1053)
NAPM	0.1823** (0.0737)	-0.0318 (0.0749)	0.8479*** (0.0593)	-0.2518** (0.1082)
NAPM(-1)	0.1896** (0.0736)	-0.0608 (0.0748)	0.8408*** (0.0605)	-0.3158*** (0.1061)
S&P 500	0.6945*** (0.0539)	-0.6880*** (0.0544)	-0.1899** (0.1098)	-0.4351*** (0.1007)
S&P 500 (-5)	0.6897*** (0.0543)	-0.6570*** (0.0565)	0.0507 (0.1112)	-0.5552*** (0.0930)

TABLE 4

CORRELATION AMONG HEATING OIL KALMAN FILER ESTIMATES OF MARKET PRICES OF RISK AND BUSINESS CYCLE RELATED VARIABLES

The table shows the coefficients or correlation among the estimated heating oil market prices of risk with the Kalman filter method and several business cycle related variables. Coefficients of correlation are reported with their standard errors in parenthesis. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	HEATING OIL FRIST DATA SET		HEATING OIL SECOND DATA SET	
	λ_ξ	λ_χ	λ_ξ	λ_χ
λ_ξ	1	0.7896*** (0.0464)	1	0.5207*** (0.0955)
λ_χ	0.7896*** (0.0464)	1	0.5207*** (0.0955)	1
WTI F1	0.3711*** (0.0702)	0.0309 (0.0756)	0.4068*** (0.1021)	-0.1922* (0.1097)
VOLAT. F1	0.0904 (0.0753)	0.0608 (0.0755)	-0.3687*** (0.1039)	0.0824 (0.1114)
MODEL VOLAT.	0.1560** (0.0747)	0.1383* (0.0749)	0.2519** (0.1082)	0.5875*** (0.0905)
LIKELIHOOD	-0.0225 (0.0756)	0.1857** (0.0743)	-0.3440*** (0.1050)	0.0812 (0.1114)
ξ	0.6106*** (0.0599)	0.4365*** (0.0680)	0.6884*** (0.0811)	0.4267*** (0.1011)
χ	-0.6041*** (0.0602)	-0.6356*** (0.0584)	-0.5586*** (0.0927)	-0.8958*** (0.0497)
σ_ξ	0.2922*** (0.0726)	-0.0458 (0.0755)	0.4454*** (0.1001)	-0.0069 (0.1118)
σ_χ	-0.0022 (0.0756)	-0.0960 (0.0752)	-0.3736*** (0.1037)	-0.0765 (0.1115)
NAPM	-0.0569 (0.0755)	-0.0969 (0.0752)	0.3371*** (0.1053)	0.0748 (0.1115)
NAPM(-1)	-0.0559 (0.0755)	-0.1016 (0.0752)	0.3419*** (0.1051)	0.0496 (0.1117)
S&P 500	0.1657** (0.0745)	-0.0542 (0.0755)	0.0388 (0.1117)	-0.1803 (0.1098)
S&P 500 (-5)	0.1152 (0.0751)	-0.0878 (0.0753)	0.1025 (0.1112)	-0.2097* (0.1093)

TABLE 5

CORRELATION AMONG RBOB GASOLINE KALMAN FILER ESTIMATES OF MARKET PRICES OF RISK AND BUSINESS CYCLE RELATED VARIABLES

The table shows the coefficients or correlation among the estimated RBOB gasoline market prices of risk with the Kalman filter method and several business cycle related variables. Coefficients of correlation are reported with their standard errors in parenthesis. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	RBOB GASOLINE FRIST DATA SET		RBOB GASOLINE SECOND DATA SET	
	λ_ξ	λ_χ	λ_ξ	λ_χ
λ_ξ	1	0.8618*** (0.0379)	1	0.8520*** (0.0552)
λ_χ	0.8618*** (0.0379)	1	0.8520*** (0.0552)	1
WTI F1	0.1992*** (0.0732)	0.0234 (0.0747)	0.2644*** (0.1017)	0.0608 (0.1052)
VOLAT. F1	0.1942*** (0.0733)	0.0267 (0.0747)	0.1230 (0.1046)	-0.0222 (0.1054)
MODEL VOLAT.	-0.0442 (0.0747)	-0.1434** (0.0740)	-0.1613 (0.1040)	-0.1499 (0.1042)
LIKELIHOOD	-0.1191 (0.0742)	-0.0845 (0.0745)	-0.0105 (0.1054)	-0.1235 (0.1046)
ξ	0.6041*** (0.0596)	0.4824*** (0.0655)	0.6415*** (0.0809)	0.5540*** (0.0878)
χ	-0.6850*** (0.0545)	-0.6929*** (0.0539)	-0.7215*** (0.0730)	-0.8156*** (0.0610)
σ_ξ	0.0660 (0.0746)	-0.0744 (0.0745)	0.2166** (0.1029)	-0.0656 (0.1052)
σ_χ	-0.1613** (0.0738)	-0.1109 (0.0743)	-0.0919 (0.1050)	0.0044 (0.1054)
NAPM	-0.0491 (0.0747)	-0.0617 (0.0746)	0.2012* (0.1033)	0.0189 (0.1054)
NAPM(-1)	-0.0375 (0.0747)	-0.0615 (0.0746)	0.2263** (0.1027)	0.0365 (0.1053)
S&P 500	0.1478** (0.0739)	-0.0358 (0.0747)	0.0226 (0.1054)	-0.0410 (0.1053)
S&P 500 (-5)	0.1357* (0.0741)	-0.0450 (0.0747)	0.0400 (0.1053)	0.0002 (0.1054)

TABLE 6

CORRELATION AMONG HENRY HUB NATURAL GAS KALMAN FILER ESTIMATES OF MARKET PRICES OF RISK AND BUSINESS CYCLE RELATED VARIABLES

The table shows the coefficients or correlation among the estimated Henry Hub natural gas market prices of risk with the Kalman filter method and several business cycle related variables. Coefficients of correlation are reported with their standard errors in parenthesis. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	HENRY HUB NATURAL GAS FRIST DATA SET		HENRY HUB NATURAL GAS SECOND DATA SET	
	λ_{ξ}	λ_{χ}	λ_{ξ}	λ_{χ}
λ_{ξ}	1	0.3024^{***} (0.0827)	1	0.2313^{**} (0.1131)
λ_{χ}	0.3024^{***} (0.0827)	1	0.2313^{**} (0.1131)	1
WTI F1	0.4506^{***} (0.0774)	-0.3204^{***} (0.0774)	-0.0284 (0.1162)	-0.6381^{***} (0.0895)
VOLAT. F1	0.3790^{***} (0.0802)	0.0022 (0.0802)	0.3384^{***} (0.1094)	0.4662^{***} (0.1028)
MODEL VOLAT.	0.2315^{***} (0.0844)	0.3538^{***} (0.0844)	0.3085^{***} (0.1106)	0.3269^{***} (0.1099)
LIKELIHOOD	0.0005 (0.0867)	-0.0779 (0.0867)	-0.1138 (0.1155)	-0.6353^{***} (0.0898)
ξ	0.6706^{***} (0.0643)	-0.0153 (0.0643)	0.0125 (0.1162)	0.0209 (0.1162)
χ	-0.6737^{***} (0.0641)	-0.5463^{***} (0.0641)	0.0095 (0.1162)	-0.8082^{***} (0.0685)
σ_{ξ}	0.4327^{***} (0.0782)	-0.1896^{**} (0.0782)	0.7007^{***} (0.0829)	0.1682 (0.1147)
σ_{χ}	-0.0157 (0.0867)	-0.0634 (0.0867)	-0.3838^{***} (0.1073)	0.3283^{***} (0.1098)
NAPM	0.0155 (0.0867)	-0.1080 (0.0867)	0.6385^{***} (0.0895)	-0.2403^{**} (0.1128)
NAPM(-1)	0.0200 (0.0867)	-0.1235 (0.0867)	0.6058^{***} (0.0925)	-0.2710^{**} (0.1119)
S&P 500	0.4897^{***} (0.0756)	-0.1823^{**} (0.0756)	-0.4528^{***} (0.1036)	-0.2309^{**} (0.1131)
S&P 500 (-5)	0.4911^{***} (0.0755)	-0.1787^{**} (0.0755)	-0.2605^{**} (0.1122)	-0.3593^{***} (0.1085)

TABLE 7

ESTIMATION RESULTS OF THE FACTOR MODELS WITH TIME-VARYING BUSINESS CYCLE RELATED AND CONSTANT MARKET PRICES OF RISK

WTI CRUDE OIL

The table shows the estimation results of the model with time-varying market prices of risk (MPR), depending on the model long- and short-term factors, together with those obtained with the standard Schwartz and Smith (2000) two-factor model with constant risk premia. The table shows the results obtained with both the first and the second data sets described in section 3. Standard errors are in parentheses. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	First Data Set		Second Data Set	
	WTI Constant MPR	WTI Variable MPR	WTI Constant MPR	WTI Variable MPR
μ_ξ	0.0452* (0.0270)	0.0404 (0.0268)	0.1128*** (0.0334)	0.0868** (0.0410)
κ	1.9748*** (0.0234)	1.1859*** (0.2318)	1.1254*** (0.0103)	1.3257*** (0.2478)
σ_ξ	0.1936*** (0.0030)	0.1919*** (0.0170)	0.1761*** (0.0037)	0.2160*** (0.0283)
σ_χ	0.2467*** (0.0043)	0.1799*** (0.0266)	0.2763*** (0.0065)	0.1393*** (0.0384)
$\lambda_{\xi 0}$	0.0907*** (0.0271)	0.3821* (0.1991)	0.1669*** (0.0334)	0.4945*** (0.1505)
$\lambda_{\xi 1}$	-	-0.0856 (0.0569)	-	-0.1009*** (0.0387)
$\lambda_{\xi 2}$	-	0.8030* (0.4184)	-	1.2291* (0.6864)
$\lambda_{\chi 0}$	0.0453 (0.0346)	1.1308*** (0.3736)	-0.0333 (0.0524)	0.3670** (0.1758)
$\lambda_{\chi 1}$	-	-0.3251*** (0.1084)	-	-0.1022** (0.0473)
$\lambda_{\chi 2}$	-	0.8121*** (0.2647)	-	-0.1048 (0.2702)
$\rho_{\xi\chi}$	0.1494*** (0.0239)	0.5775*** (0.0949)	0.0445 (0.0320)	0.7349*** (0.0791)
σ_η	0.0079*** (0.0001)	0.0078*** (0.0001)	0.0093*** (0.0001)	0.0093*** (0.0001)
Log-L	50007.70	50160.36	32146.72	32163.08
AIC	49991.70	50136.36	32130.72	32139.08
SIC	49950.11	50073.97	32093.99	32084.00

TABLE 8

ESTIMATION RESULTS FOR HEATING OIL

The table shows the estimation results of the model with time-varying market prices of risk (MPR), depending on the model long- and short-term factors, together with those obtained with the standard Schwartz and Smith (2000) two-factor model with constant risk premia. The table shows the results obtained with both the first and the second data sets described in section 3. Standard errors are in parentheses. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	First Data Set		Second Data Set	
	Heating Oil Constant MPR	Heating Oil Variable MPR	Heating Oil Cosntant MPR	Heating Oil Variable MPR
μ_ξ	0.1659*** (0.0310)	0.3994*** (0.1237)	0.2248*** (0.0327)	0.0635 (0.0283)
κ	1.8522*** (0.0911)	1.1312*** (0.0205)	1.7080*** (0.0158)	0.4026*** (0.0000)
σ_ξ	0.1770*** (0.0035)	0.3691*** (0.0031)	0.1696*** (0.0037)	0.1905*** (0.0000)
σ_χ	0.3343*** (0.0158)	0.6105*** (0.0206)	0.2441*** (0.0072)	0.3758*** (0.0000)
ϕ	0.9976*** (0.0000)	0.9974*** (0.0000)	0.9972*** (0.0002)	0.9974*** (0.0002)
$\lambda_{\xi 0}$	0.2239*** (0.0327)	0.3642 (0.2214)	0.2322*** (0.0329)	0.2883 (0.0000)
$\lambda_{\xi 1}$	-	-0.2846*** (0.0000)	-	0.0216*** (0.0000)
$\lambda_{\xi 2}$	-	-0.8459*** (0.0225)	-	-0.5506*** (0.0198)
$\lambda_{\chi 0}$	0.6999*** (0.1079)	0.5173** (0.2067)	0.3546*** (0.0496)	-0.9568*** (0.0000)
$\lambda_{\chi 1}$	-	0.5407*** (0.0198)	-	0.0462*** (0.0000)
$\lambda_{\chi 2}$	-	0.6409*** (0.0378)	-	1.1536*** (0.0000)
$\rho_{\xi\chi}$	-0.1229*** (0.0431)	-0.5889*** (0.0340)	0.4488*** (0.0307)	-0.4248*** (0.0000)
σ_η	0.0209*** (0.0003)	0.0143*** (0.0001)	0.0190*** (0.0001)	0.0187*** (0.0001)
Log-L	47709.23	51511.79	42417.57	42573.40
AIC	47691.23	51485.79	42399.57	42547.40
SIC	47644.72	51418.61	42358.26	42487.73

TABLE 9

ESTIMATION RESULTS FOR RBOB GASOLINE

The table shows the estimation results of the model with time-varying market prices of risk (MPR), depending on the model long- and short-term factors, together with those obtained with the standard Schwartz and Smith (2000) two-factor model with constant risk premia. The table shows the results obtained with both the first and the second data sets described in section 3. Standard errors are in parentheses. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	First Data Set		Second Data Set	
	RBOB Constant MPR	RBOB Variable MPR	RBOB Cosntant MPR	RBOB Variable MPR
μ_ξ	-0.4000*** (0000)	0.2855*** (0.0173)	0.2621*** (0.0315)	0.0909 (0.0200)
κ	3.1144*** (0.0916)	0.4002*** (0.0206)	2.0500*** (0.0558)	2.1285*** (0.1590)
σ_ξ	0.2093*** (0.0034)	0.3023*** (0.0000)	0.1877*** (0.0045)	0.2458*** (0.0000)
σ_χ	0.3770*** (0.0088)	0.3212*** (0.0000)	0.3084*** (0.0084)	0.5067*** (0.0000)
ϕ	0.9947*** (0.0001)	0.9940*** (0.0000)	1.0028*** (0.0004)	1.0002*** (0.0003)
$\lambda_{\xi 0}$	-0.3919*** (0.0041)	0.6893*** (0.0000)	0.3439*** (0.0323)	0.0298 (0.1130)
$\lambda_{\xi 1}$	-	0.0576*** (0.0140)	-	-0.0045 (0.0396)
$\lambda_{\xi 2}$	-	0.3876*** (0.0000)	-	-0.8974*** (0.0000)
$\lambda_{\chi 0}$	-0.3849*** (0.0288)	0.6412* (0.0000)	0.3791*** (0.0548)	0.9855*** (0.2980)
$\lambda_{\chi 1}$	-	0.4053*** (0.0000)	-	-0.1642* (0.0901)
$\lambda_{\chi 2}$	-	1.1309*** (0.0482)	-	-0.0410 (0.1778)
$\rho_{\xi\chi}$	0.0764** (0.0322)	-0.2500*** (0.0000)	0.1072*** (0.0404)	-0.7064*** (0.0000)
σ_η	0.0162*** (0.0001)	0.0151*** (0.0001)	0.0162*** (0.0002)	0.0159*** (0.0002)
Log-L	42227.28	42409.56	25273.54	25374.57
AIC	42209.28	42383.56	25255.54	25348.57
SIC	42162.60	42316.14	25213.51	25287.86

TABLE 10

ESTIMATION RESULTS FOR HENRY HUB NATURAL GAS

The table shows the estimation results of the model with time-varying market prices of risk (MPR), depending on the model long- and short-term factors, together with those obtained with the standard Schwartz and Smith (2000) two-factor model with constant risk premia. The table shows the results obtained with both the first and the second data sets described in section 3. Standard errors are in parentheses. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	First Data Set		Second Data Set	
	Henry Hub Constant MPR	Henry Hub Variable MPR	Henry Hub Cosntant MPR	Henry Hub Variable MPR
μ_ξ	-0.3996*** (0.0307)	-0.2996*** (0.0621)	0.0719*** (0.0252)	0.0284*** (0.0000)
κ	1.8158*** (0.0000)	0.8416*** (0.0607)	1.1163*** (0.0138)	1.0458*** (0.0000)
σ_ξ	0.2515*** (0.0000)	0.2334*** (0.0295)	0.1297*** (0.0040)	0.3325*** (0.0000)
σ_χ	0.5547*** (0.0087)	0.5475*** (0.0285)	0.4779*** (0.0155)	0.1714*** (0.0000)
ϕ	0.9957*** (0.0001)	0.9997*** (0.0002)	0.9999*** (0.0001)	0.9992*** (0.0001)
$\lambda_{\xi 0}$	-0.1397*** (0.0488)	0.4892*** (0.0284)	0.1236*** (0.0253)	-0.0587*** (0.0000)
$\lambda_{\xi 1}$	-	-0.3987*** (0.0539)	-	-0.0029 (0.0506)
$\lambda_{\xi 2}$	-	-0.3539*** (0.0768)	-	1.9929*** (0.0000)
$\lambda_{\chi 0}$	0.0008 (0.0994)	0.4166* (0.1922)	-0.2177** (0.0928)	-0.0252*** (0.0000)
$\lambda_{\chi 1}$	-	0.1618 (0.1466)	-	-0.0358 (0.0212)
$\lambda_{\chi 2}$	-	0.8608*** (0.0000)	-	0.0452*** (0.0000)
$\rho_{\xi\chi}$	-0.5678*** (0.0000)	-0.7205*** (0.0677)	-0.0222 (0.0471)	0.9166*** (0.0000)
σ_η	0.0916*** (0.0006)	0.0914*** (0.0006)	0.0399*** (0.0002)	0.0383*** (0.0002)
Log-L	22625.01	22758.45	39438.12	40032.74
AIC	22607.01	22732.45	39420.12	40006.74
SIC	22562.30	22667.87	39379.28	39947.75

TABLE 11

AMERICAN OPTION VALUATION RESULTS

ERROR DESCRIPTIVE STATISTICS

The table presents several metrics, root mean squared error (RMSE), percentage root mean squared error (PRMSE) and mean absolute error (MAE), to analyze the predictive power ability of the models under study: the time-varying risk premia model and the standard (two-factor) model with constant risk premia. The data set is comprised of daily observations of WTI American call and put options quoted at NYMEX during the years 2006 to 2010. For each series, we have calculated the corresponding statistic. These results correspond to the median value of these multiple means. The total number of observations is 341588, 223652, 204930 and 179785 for WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas respectively.

PANEL A: WTI AMERICAN OPTIONS						
	CONSTANT RISK PREMIA			TIME-VARYING RISK PREMIA		
	RMSE	PRMSE	MAE	RMSE	PRMSE	MAE
FIRST DATA SET	0.9727	27.4746	0.6853	0.8974	25.20429	0.6416
SECOND DATA SET	0.9675	28.49709	0.6835	0.9313	26.39755	0.6672

PANEL B: HEATING OIL AMERICAN OPTIONS						
	CONSTANT RISK PREMIA			TIME-VARYING RISK PREMIA		
	RMSE	PRMSE	MAE	RMSE	PRMSE	MAE
FIRST DATA SET	3.1407	43.2489	2.7943	1.3379	14.0521	1.0127
SECOND DATA SET	1.3484	16.3243	1.1052	1.3807	17.0274	1.0706

TABLE 11 (CONT.)

AMERICAN OPTION VALUATION RESULTS

ERROR DESCRIPTIVE STATISTICS

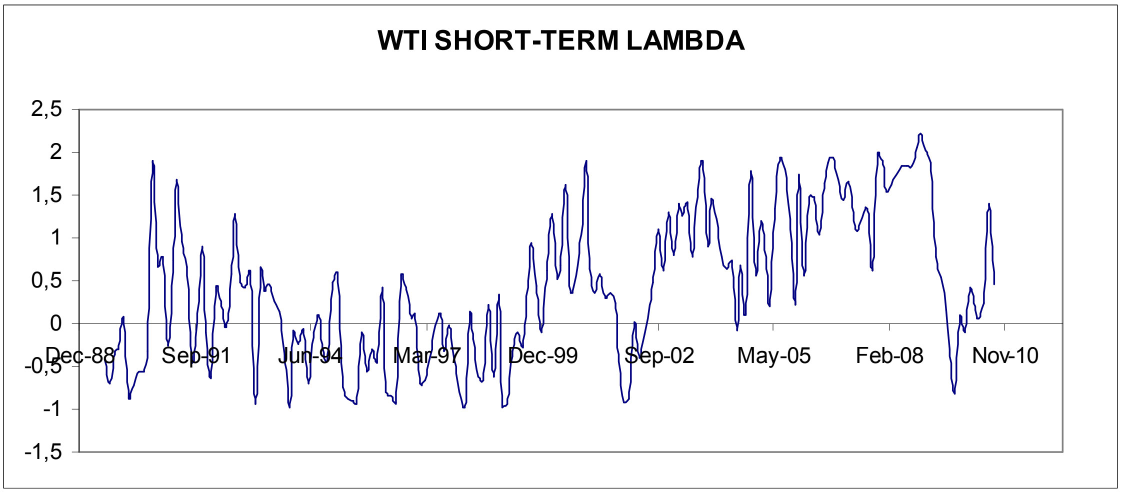
The table presents several metrics, root mean squared error (RMSE), percentage root mean squared error (PRMSE) and mean absolute error (MAE), to analyze the predictive power ability of the models under study: the time-varying risk premia model and the standard (two-factor) model with constant risk premia. The data set is comprised of daily observations of WTI American call and put options quoted at NYMEX during the years 2006 to 2010. For each series, we have calculated the corresponding statistic. These results correspond to the median value of these multiple means. The total number of observations is 341588, 223652, 204930 and 179785 for WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas respectively.

PANEL C: RBOB GASOLINE AMERICAN OPTIONS						
	CONSTANT RISK PREMIA			TIME-VARYING RISK PREMIA		
	RMSE	PRMSE	MAE	RMSE	PRMSE	MAE
FIRST DATA SET	5.5027	164.1529	4.6894	1.4065	39.7270	1.0744
SECOND DATA SET	1.4952	37.8901	1.1821	0.9294	28.1595	0.7342

PANEL D: HENRY HUB NATURAL GAS AMERICAN OPTIONS						
	CONSTANT RISK PREMIA			TIME-VARYING RISK PREMIA		
	RMSE	PRMSE	MAE	RMSE	PRMSE	MAE
FIRST DATA SET	0.1192	64.2655	0.0913	0.1124	54.8146	0.0878
SECOND DATA SET	0.1055	71.8211	0.0846	0.0864	14.0521	0.0678

FIGURE 1

TIME-SERIES EVOLUTION OF MAXIMUM LIKELIHOOD MARKET PRICES OF RISK FOR WTI CRUDE OIL



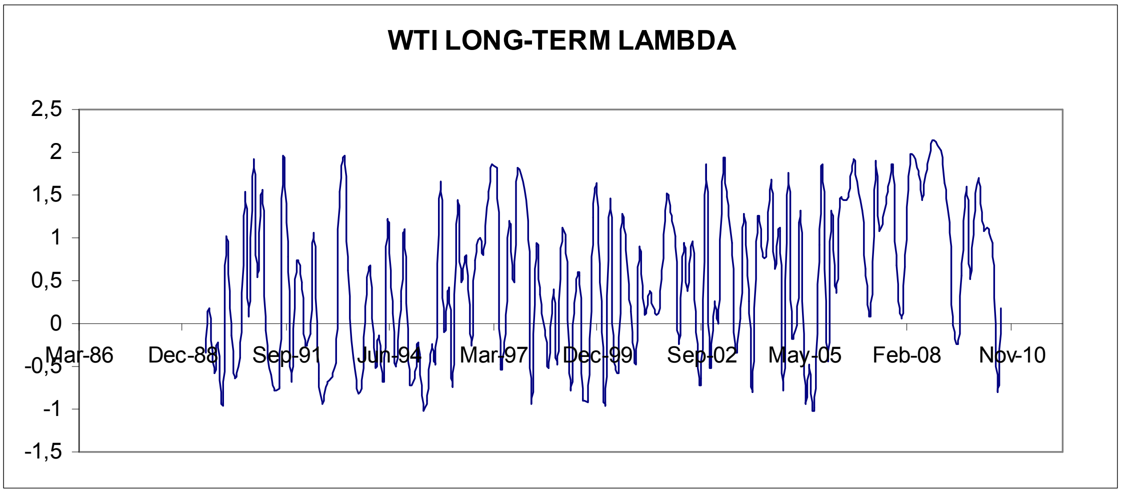
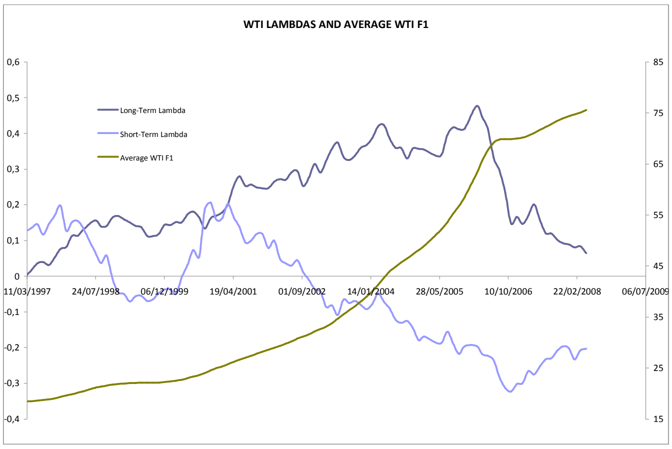
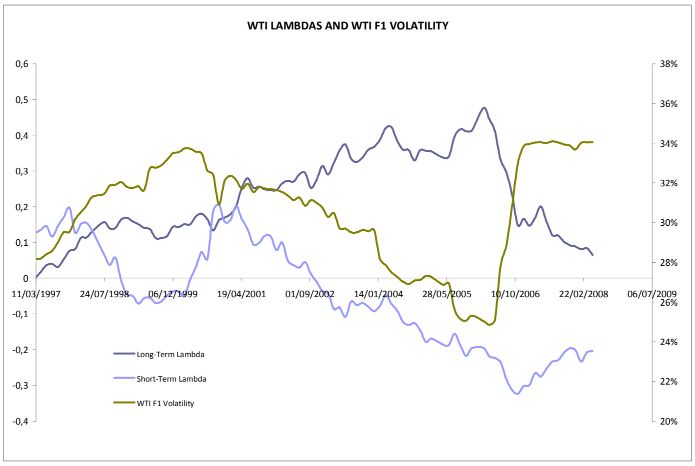


FIGURE 2

TIME-SERIES EVOLUTION OF KALMAN FILTER MARKET PRICES OF RISK FOR WTI CRUDE OIL AND BUSINESS CYCLE RELATED VARIABLES





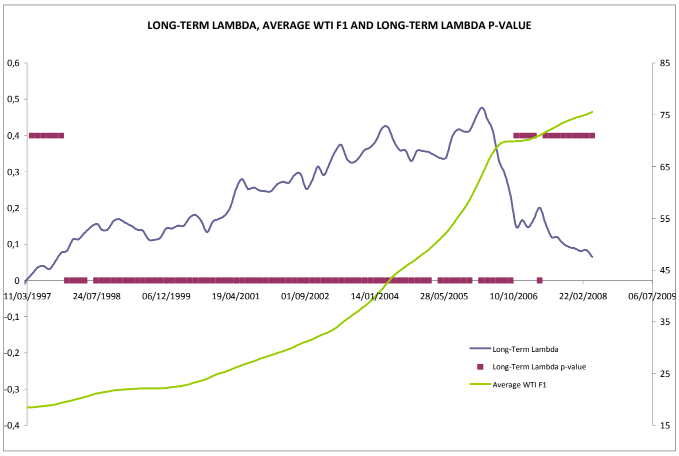
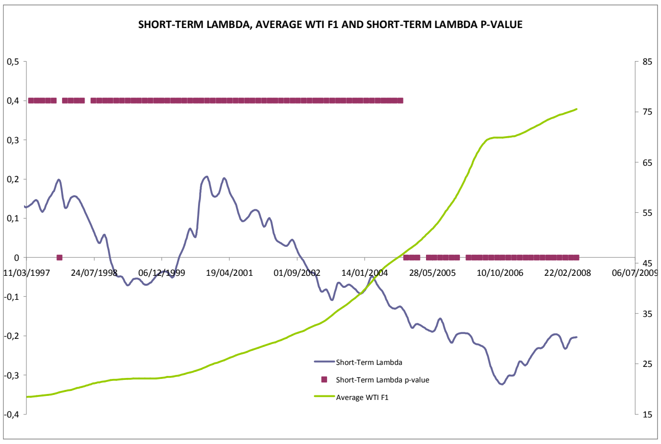
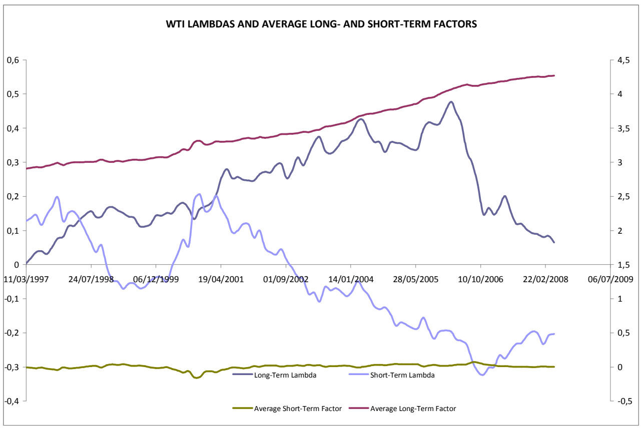


FIGURE 2

TIME-SERIES EVOLUTION OF KALMAN FILTER MARKET PRICES OF RISK FOR WTI CRUDE OIL AND BUSINESS CYCLE RELATED VARIABLES (CONT.)





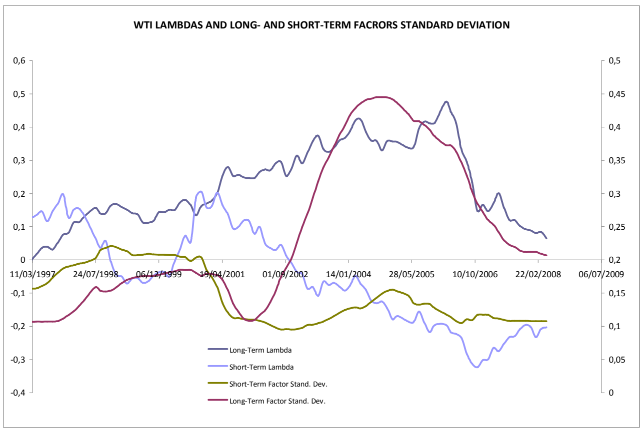
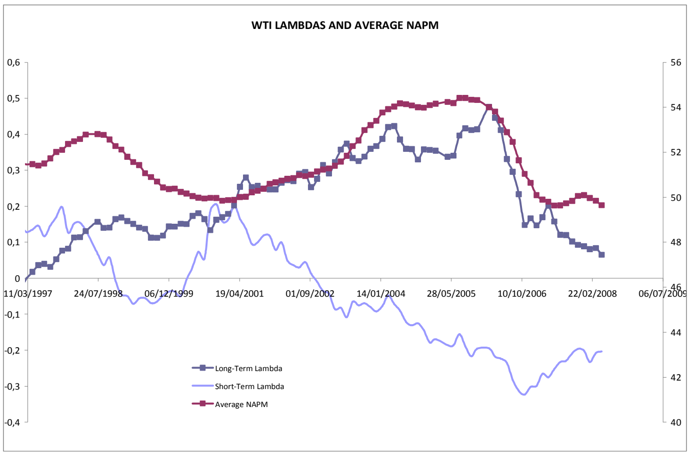
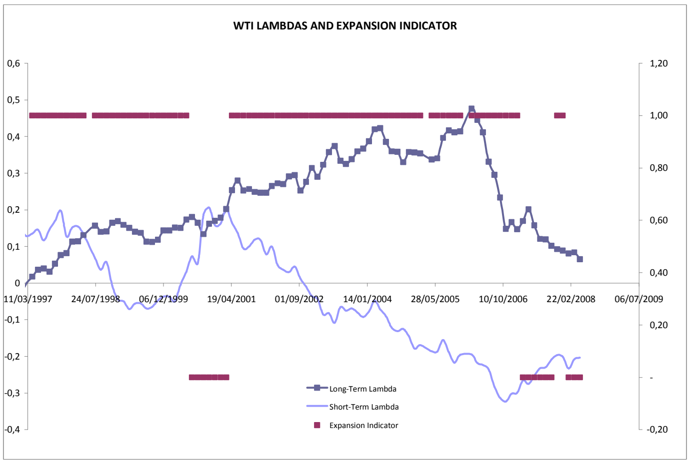
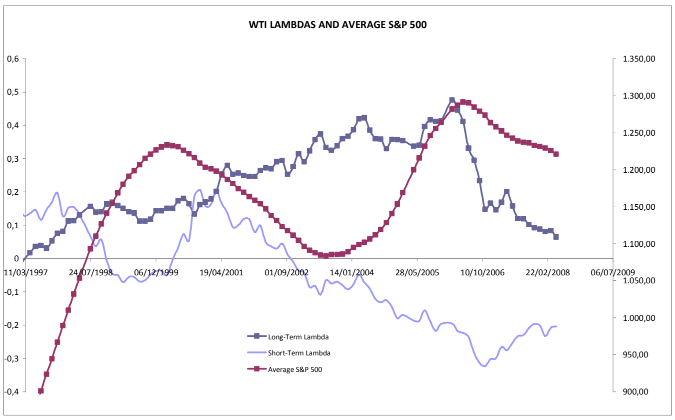


FIGURE 2

TIME-SERIES EVOLUTION OF KALMAN FILTER MARKET PRICES OF RISK FOR WTI CRUDE OIL AND BUSINESS CYCLE RELATED VARIABLES (CONT.)







CHAPTER 3: THE STOCHASTIC SEASONAL BEHAVIOR OF ENERGY COMMODITY CONVENIENCE YIELS

3.1 INTRODUCTION

In consumption commodities (commodities that are consumption assets rather than investment assets) the benefit from holding the physical asset net of storage cost is sometimes referred to as the "convenience yield" provided by the commodity (see for example Hull, 2003).

In other words, if we denote by F and S the futures and spot prices respectively, in the case of consumption commodities we do not necessarily have equality in $F \leq S \cdot e^{(r+u) \cdot T}$ (where r and u represent the risk free rate and storage costs respectively and T is the time to maturity), because users of a consumption commodity may feel that ownership of the physical commodity provides benefits that are not obtained by holders of futures contracts. For example, an oil refiner is unlikely to regard a futures contract on crude oil as equivalent to crude oil held in inventory. The crude oil in inventory can be an input to the refining process whereas a futures contract cannot be used for this purpose. In general, ownership of the physical asset enables a manufacturer to keep a production process running and perhaps profit from temporary local shortages. A futures contract does not do the same (see for example Brennan and Schwartz, 1985). Therefore the convenience yield net of storage costs, denoted by δ, is defined so that: $F \cdot e^{\delta \cdot T} = S \cdot e^{r \cdot T}$.

Previous studies have considered the convenience yield as a deterministic function of time, such as Brennan and Schwartz (1985), or as a stochastic process, such as Gibson and Schwartz (1990) and Schwartz (1987). Specifically, Gibson and Schwartz (1990) allow for stochastic convenience yield of crude oil in order to develop a two-factor oil contingent claims price model. Moreover, Gibson and Schwartz (1990) show that convenience yields exhibit mean reversion, which is consistent with the theory of storage (see, for example, Brennan, 1985) in which it is established an inverse relationship between the net convenience yield and the level of inventories. Schwartz (1997) presents and empirically compares several factor models in which the convenience yield is assumed to be a stochastic factor. Hilliard and Reis (1998) and Miltersen and Schwartz (1998) use models with stochastic convenience yield to value commodity derivatives (futures and options). More recently, Casassus and Collin-Dufresne (2005) characterize a three-factor model, "maximal" in a sense of Dai and Singleton (2000), of commodity spot prices, convenience yields and interest rates, which nests many existing specifications.

Wei and Zhu (2006) investigate the empirical properties of convenience yields in the US natural gas market, finding that convenience yields are highly variable and economically significant, with their variability depending on spot price level, spot price variability and the variability of lagged convenience yields.

In spite of there have been many papers analyzing the seasonal behavior of some commodity prices (Lucia and Schwartz, 2002, Sorensen, 2002, Manoliu and Tompaidis, 2002, Garcia et al., 2012, among others), considerably less attention has been paid to the seasonal behavior of convenience yields. Based on the finding of seasonality in the convenience yield made by Fama and French (1987), Amin et al. (1994) propose a one-factor model for the spot price with deterministic seasonal convenience yield. More recently, Borovkova and Geman (2006) present a two-factor model in which the first factor is the average forward price, instead of the commodity spot price, and the second factor is the stochastic convenience yield. These authors allow for a deterministic seasonal premium within the convenience yield.

In this chapter, we go further by presenting a factor model in which the (stochastic) convenience yield exhibits stochastic seasonality. Specifically, we show that the four-factor model presented by Garcia et al. (2012), with two long- and short-term factors and two additional trigonometric seasonal factors, can generate stochastic seasonal convenience yields. An expression for the instantaneous convenience yield within this model is obtained, showing that the instantaneous convenience yield exhibits mean reversion and stochastic seasonality. Moreover, it is found a $\pi/2$ lag in the convenience yield seasonality with respect to spot price seasonality.

Based on this evidence, the next step is to present a theoretical model to characterize the commodity convenience yield dynamics which is coherent with the previous findings. Specifically, the model takes into account mean reversion and stochastic seasonal effects in the convenience yield. The model is estimated using data from a variety of energy commodity futures prices: crude oil, heating oil, gasoline and natural gas. We also show that commodity price seasonality can be better estimated through convenience yields rather than through futures prices. The reason is that futures prices are driven for many things, such as supply, demand, political aspects, speculation, weather conditions, etc. Therefore, sometimes it may be difficult to extract the seasonal component from futures prices. However, as we will show in Section 2, the convenience yield is estimated though a ratio of two futures prices, so many of these non-seasonal factors tend to disappear, facilitating the estimation of the seasonal component.

The remainder of this chapter is organized as follows. Section 2 presents the data and some preliminary findings regarding seasonality in convenience yields. We show that convenience yields show mean reversion and stochastic seasonality, using data from heating oil, gasoline and natural gas futures markets. In section 3 we present the four-factor model accounting for stochastic seasonality in commodities and the expression for the instantaneous convenience yield derived from this four-factor model. In section 3 we also discuss the properties of the model estimated convenience yields for the four commodities under study, showing that in fact they exhibit mean reversion, stochastic seasonality and a $\pi/2$ lag with respect to spot price seasonality. Based on this empirical evidence, in section 4 it is proposed and estimated a factor model characterizing the commodity convenience yield dynamics, taking into account mean reversion and stochastic seasonal effects in the convenience yield. Finally, Section 5 concludes with a summary and discussion.

3.2 DATA AND PRELIMINARY FINDINGS

In this section, we present a data description of the futures prices for the four commodities used in the chapter, i.e. WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas. Moreover, it is described the procedure presented by Gibson and Schwartz (1990) in order to obtain the convenience yield data. The section concludes analyzing the main empirically observed characteristics of the convenience yield data.

Data description

Futures Prices

The data set used in this chapter consists of weekly observations of WTI (light sweet) crude oil, heating oil, unleaded gasoline (RBOB) and natural gas futures prices traded at NYMEX, during the period 9/27/1999 to 7/4/2011 (615 weekly observations).

Actually, there are futures being traded on NYMEX with maturities from one month up to seven years for WTI crude oil, from one to eighteen months for heating oil, from one to twelve months for RBOB gasoline and from one month to six years in the case of

Henry Hub natural gas. However, liquidity is scarce for the futures with longer maturities, mostly in the case of gasoline.

In the estimation of the models presented below a representative set of maturities has been used for each commodity. Thus, in the case of WTI crude oil, the data set is comprised of contracts F1, F4, F7, F11, F14, F17, F20, F24 and F27, where F1 is the contract for the month closest to maturity, F2 is the contract for the second-closest month to maturity, and so on. In the case of heating oil, the data set is comprised of contracts F1, F3, F5, F7, F10, F12, F14, F16 and F18. In the case of RBOB gasoline, the data set contains contracts F1, F3, F5, F7, F9 and F12. Finally, in the case of Henry Hub natural gas, the data set contains contracts F1, F5, F9, F14, F18, F22, F27, F31 and F35. The main descriptive statistics of these variables are contained in Table 1.

Convenience Yield

The estimation of the convenience yield series is carried out using the procedure defined in Gibson and Schwartz (1990). Based on the convenience yield definition, $F \cdot e^{\delta \cdot T} = S \cdot e^{r \cdot T}$, we have:

$$F(S, x\_months) = S \cdot \exp\{(r_{x\_months} - \delta_{x\_months}) \cdot (x/12)\}$$

where r_{x_months} is the interest rate of a zero coupon bond with x months to maturity and δ_{x_months} is the convenience yield in x months for this commodity. Analogously:

$$F(S, x+1\_month) = S \cdot \exp\{(r_{x+1\_month} - \delta_{x+1\_month}) \cdot ((x+1)/12)\}$$

where r_{x+1_month} is the interest rate of a zero coupon bond with x+1 months to maturity and δ_{x+1_month} is the convenience yield in x+1 months for this commodity.

From these expressions we have:

$$\frac{F(S, x+1\_months)}{F(S, x\_month)} = \exp\left\{\left[\left((x+1) \cdot r_{x+1\_months} - x \cdot r_{x\_month}\right) - \left((x+1) \cdot \delta_{x+1\_months} - x \cdot \delta_{x\_month}\right)\right](1/12)\right\} \quad (1)$$

On the other hand, by definition:

$$\exp\left\{\left((x+1)\cdot r_{x+1\_months} - x\cdot r_{x\_month}\right)(1/12)\right\} = \exp\{r_{implicit\_x\_to\_x+1\_months}\cdot(1/12)\}$$

where $r_{implicit\_x\_to\_x+1\_months}$ is the implicit interest rate from x months to x+1 months, and

$$\exp\left\{\left((x+1)\cdot \delta_{x+1\_months} - x\cdot \delta_{x\_month}\right)(1/12)\right\} = \exp\{\delta_{implicit\_x\_to\_x+1\_months}\cdot(1/12)\}$$

where $\delta_{implicit\_x\_to\_x+1\_months}$ is the implicit convenience yield from x months to x+1 months.

Taking into account these definitions, expression (1) can be written as:

$$\frac{F(S, x+1\_months)}{F(S, x\_month)} = \exp\{(r_{implicit\_x\_to\_x+1\_months} - \delta_{implicit\_x\_to\_x+1\_months})\cdot(1/12)\}$$

or equivalently:

$$\delta_{implicit\_x\_to\_x+1\_months} = r_{implicit\_x\_to\_x+1\_months} - 12\cdot\ln\left\{\frac{F(S, x+1\_months)}{F(S, x\_month)}\right\}$$

$\delta_{implicit\_x\_to\_x+1\_months}$ can be used as a proxy for the instantaneous convenience yield δ_t.

Following this procedure we have estimated the convenience yield series for the four commodity futures prices series described above. The main descriptive statistics of these convenience yield series are summarized in Table 2. In Figure 1 we plot the time series evolution of some of the estimated convenience yields for the four commodities under study. It can be appreciated in the figures the mean-reverting and seasonality effects, although the pattern is less clear in the case of WTI crude oil. These issues are further discussed below.

Preliminary Findings

Previous studies found evidence of mean reversion in the convenience yield dynamics. From convenience yield data obtained as in the previous sub-section, Gibson and Schwartz (1990) show a strong mean reverting tendency in the convenience yield, which is consistent with the theory of storage (see, for example, Brennan,1985) in which it is established an inverse relationship between the level of inventories and the relative net convenience yield.

Fama and French (1987) pointed out that seasonals in production or demand can generate seasonals in inventories. Under the theory of storage, inventory seasonals generate seasonals in the marginal convenience yield. Following this reasoning, Borovkova and Geman (2006) present a model allowing for a deterministic seasonal premium within the convenience yield.

Here, using the estimated convenience yield series from the previous sub-section for the four commodities under study, we will investigate the existence of mean reverting and seasonal effects in the convenience yield.

Table 3 presents the results of the unit root tests for WTI, heating oil, gasoline and Henry Hub natural gas convenience yield series. The empirical evidence from previous studies of mean reversion is confirmed in the present work using the standard Augmented Dickey-Fuller test. Specifically, we are able to reject the null hypothesis of a unit root in all the cases, with the only exception of WTI crude oil (mostly as we go further in time). These results are coherent with the time evolution of the series shown in Figure 1.

The presence of seasonality in the estimated convenience yield series is assessed through the Kurskal-Wallis test. To perform the test we have computed monthly averages from the weekly estimated convenience yield series. The null hypothesis of the test is that there are no monthly seasonal effects. The results of the test are shown in Table 4. The results indicate the rejection of the null hypothesis of no seasonal effects in all cases, except for WTI crude oil. The seasonal effects are even clearer in the cases of RBOB gasoline and Henry Hub natural gas convenience yield series. These seasonal effects are evident in Figure 1.

As explained above, Borovkova and Geman (2006) allow for a deterministic seasonal premium within the convenience yield. However, it may be possible that seasonal effects in the convenience yield are stochastic rather than deterministic. Garcia et al. (2012) present a model for the stochastic behavior of commodity prices allowing for stochastic seasonality in commodity prices. Following this idea, we will check for the existence of stochastic seasonal effects in the convenience yield series.

The RBOB gasoline[34] convenience yield spectrum and its first differences are depicted in Figure 2, assuming that the series follows an AR(1) process with yearly seasonality, following the procedure described in Garcia et al. (2012). As explained by Garcia et al. (2012), sharp spikes in the spectrum are likely to indicate a deterministic cyclical component, while broad peaks often indicate a nondeterministic seasonal component. The asterics (*) shown in the Figure denote harmonic points, calculated as $2\pi k/12$ (peaks) and $\pi(2k\text{-}1)/12$ (troughs), where k = 1, 2, 3, 4, 5 and 6.

Looking at Figure 2, it seems that, more or less, the spectrum exhibits broad peaks and thoughts, suggesting that seasonality in convenience yields is stochastic rather than deterministic. However, these results must be taken with care, as aliasing effects and estimation errors can confuse deterministic and stochastic patterns.

In Figure 3 we plot the forward curves for the estimated convenience yield series on a representative date (July 4, 2011) in the case of Henry Hub natural gas prices[35]. Looking at the figure it can be appreciated that both futures and convenience yield series present an evident seasonal pattern. Moreover, it is interesting to observe how the seasonal picks in the convenience yield series are delayed three months compared to those observed in the futures series.

[34] The patter for the rest of commodities is very similar.

[35] For short only the figure for Henry Hub natural gas is presented. The pattern is similar in the rest of the cases.

3.3 THE PRICE MODEL

In this section, we show that a four-factor model for the stochastic behavior of commodity prices, with two long- and short-term factors and two additional seasonal factors, can accommodate some of the most important empirically observed characteristics of commodity convenience yields described in Section 2, such as mean reversion, stochastic seasonality and a three months delay in the convenience yield seasonality with respect to the spot price seasonality.

General Considerations

Based on the convenience yield definition, $F \cdot e^{\delta \cdot T} = S \cdot e^{r \cdot T}$,taking into account that the spot price (S_t) and the convenience yield (δ_t) are stochastic if $T > 0$, the previous equation can be expressed as an SDE in the following way:

$$dS_t = S_t(r - \delta_t)dt + \sigma dW_t^* \tag{2}$$

which is the classical definition of the convenience yield under the Q-measure (see, for example, Schwartz, 1997, or Casassus and Collin-Dufresne, 2005). Under the P-measure the SDE can be expressed in the following way:

$$dS_t = S_t(\mu - \delta_t)dt + \sigma dW_t \tag{3}$$

To characterize the convenience yield dynamics, let $X_t = \log(S_t)$ be the log of the spot price. If we assume a linear model, like in the studies listed above, its general dynamics is given by:

$$\begin{cases} dX_t = (m + AX_t)dt + RdW_t \\ S_t = \exp(\phi_0 + CX_t) \end{cases} \tag{4}$$

As it shall be proven in appendix B, the model above has an explicit (unique) solution (note that it is enough to solve for X_t):

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As} m ds + \int_0^t e^{-As} R dW_s \right].$$

Note that $S_t = \exp(\phi_0 + CX_t)$ and we would like to establish a stochastic differential equation for S_t. Taking differentials and using Ito's lemma:

$$dS_t = \exp(\phi_0 + CX_t) CdX_t + \frac{1}{2}\exp(\phi_0 + CX_t) C(dX_t)(dX_t)' C = S_t\left[CdX_t + \frac{1}{2} C(dX_t)(dX_t)' C' \right]$$

Using the fact that $dtdt = dtdW_t = 0$ and $(dW_t)(dW_t)' = Idt$ we obtain:

$$dS_t = S_t\left[C(m + AX_t)dt + CRdW_t + \frac{1}{2} CRR'C'dt \right]$$

and finally:

$$dS_t = S_t\left[C\left(m + \frac{1}{2} RR'C' + AX_t \right) dt + CRdW_t \right] \tag{5}$$

If m is defined as $m = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, which is necessary to the model be maximal (or globally identifiable), we get that $Cm = \mu$ and from (4):

$$\delta_t = -C\left(\frac{1}{2} RR'C' + AX_t \right) \tag{6}$$

Therefore, with (6) we can obtain the convenience yield dynamics from the model factors dynamics.

Theoretical Model

Here we are going to present a model to characterize the commodity prices dynamics which takes into account the seasonal effects and which is coherent with the previous findings.

In the four-factor model in Garcia et al. (2012), the log spot price (X_t) is the sum of three stochastic factors, a long-term component (ξ_t), a short-term component (χ_t) and a seasonal component (α_t).

$$X_t = \xi_t + \chi_t + \alpha_t \tag{7}$$

The fourth stochastic factor is the other seasonal factor (α_t^*), which complements α_t. The SDEs of these factors are:

$$d\xi_t = \mu_\xi dt + \sigma_\xi dW_{\xi t} \tag{8}$$

$$d\chi_t = -\kappa \chi_t dt + \sigma_\chi dW_{\chi t} \tag{9}$$

$$d\alpha_t = 2\pi\varphi\alpha_t^* dt + \sigma_\alpha dW_{\alpha t} \tag{10}$$

$$d\alpha_t^* = -2\pi\varphi\alpha_t dt + \sigma_\alpha dW_{\alpha^* t} \tag{11}$$

Equations (8) and (9) are identical to equations (2) and (1), respectively, in Schwartz and Smith (2000).

This model is "maximal" in a sense of Dai and Singleton (2000). Even more this model is Dai-Singleton $A_0(4)$ as can be seen in Appendix C. To see this, note that in the canonical form given by expressions (4):

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & k & 2\pi\varphi \\ 0 & 0 & -2\pi\varphi & k \end{pmatrix}$$

and the model is globally identifiable. The García et al. (2012) model imposes the restriction $a = k = 0$ and $\alpha > 0$. And, as a restriction of a globally identifiable model imposing concrete values and intervals to the parameters, it is also globally identifiable and maximal.

As stated above, in Garcia et al. (2012) model we have[36]:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 2\pi\varphi \\ 0 & 0 & -2\pi\varphi & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix},\ m = \begin{pmatrix} \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And:

$$RR' = \begin{pmatrix} \sigma_\xi^2 & - & - & - \\ \sigma_\xi \sigma_\chi \rho_{\xi\chi} & \sigma_\chi^2 & - & - \\ \sigma_\xi \sigma_\alpha \rho_{\chi\alpha} & \sigma_\chi \sigma_\alpha \rho_{\chi\alpha} & \sigma_\alpha^2 & - \\ \sigma_\xi \sigma_\alpha \rho_{\chi\alpha*} & \sigma_\chi \sigma_\alpha \rho_{\chi\alpha*} & 0 & \sigma_\alpha^2 \end{pmatrix}$$

Under this model, using expression (6), the convenience yield can be written in the following way:

$$\delta_t = -\frac{1}{2}(\sigma_\xi^2 + \sigma_\chi^2 + 2\sigma_\alpha^2 + 2\sigma_\xi\sigma_\chi\rho_{\xi\chi} + 2\sigma_\xi\sigma_\alpha\rho_{\chi\alpha} + 2\sigma_\xi\sigma_\alpha\rho_{\chi\alpha*} + 2\sigma_\chi\sigma_\alpha\rho_{\chi\alpha} + 2\sigma_\chi\sigma_\alpha\rho_{\chi\alpha*}) + k\chi_t - 2\pi\varphi\alpha_t^* \quad (12)$$

As can be appreciated in the previous expression, δ_t does not depend on the long-term factor, ξ_t, neither the seasonal factor, α_t. However, it depends on the sum of factor variances, the short-term factor, χ_t, (times the speed of mean reversion) and the seasonal factor that complements the one defined in the spot price, α_t^*, (times the seasonal frequency). In other words, the convenience yield is the sum of a constant term plus a short-term factor plus a seasonal factor.

The fact that δ_t is stationary (does not depend on the long-term factor and depends on the short-term one) in the previous expression is coherent with the fact that the two factor model defined in Schwartz-Smith (2000) is equivalent to the one defined in Schwartz (1997) in which δ_t follows an Ornstein-Uhlenbeck process, which is a mean-reverting one. It is clear, therefore, that δ_t should depends on χ_t instead of ξ_t. It is also clear that the dependency should be modulated by k because the higher the mean-

[36] As can be seen in García et al. (2012), $\rho_{\alpha\alpha*} = 0$ and $\sigma_\alpha = \sigma_{\alpha*}$.

reverting speed, the higher the benefit of holding the physical asset. Think, for example, in a shortage, if the price come back to its equilibrium level in a short-term period (high mean-reverting speed) the owner of the physical asset can sell the commodity and buy it again in a short-period (consequently with a low cost) getting the benefit. In the other hand, if the price delay in coming back to their equilibrium level (low mean-reverting speed), the owner of the physical asset has to buy the commodity again at a higher price or he is not going to be able to keep the production process running.

Taking into account expression (2), and getting around the stochastic part of it, it is clear that: $\frac{dS_t}{S_t dt} \propto -\delta_t$. As $\frac{d\alpha_t}{dt} = 2\pi\varphi\alpha_t^*$, it is not suppressive that δ_t depends on α_t^* instead of α_t, that implies a $\pi/2$ lag in the convenience yield seasonality with respect spot price seasonality. As in the previous case, the dependency should be modulated by φ because the higher the seasonal frequency, the higher the benefit of holding the physical asset.

The same can be said about the sum of factor variances, the higher the variance the higher is the convenience yield (in absolute value) because the benefit of holding the physical asset is higher. It is interesting to note that the convenience yield depends on the sum of the factor variances instead of the spot price variance, that is, depends on the whole system variance and not only the variance of the factors which compose the spot price.

Finally, it is worth noting that expression (12) for the convenience yield is coherent with the empirical facts observed for the convenience yield in Section 2.2: mean reversion, (stochastic) seasonality and a three months ($\pi/2$) lag in the convenience yield seasonality with respect to the spot price one.

Estimation Results

Here, we present the results of the estimation of the four-factor model for the four commodities presented above. The models presented in Section 3.1 were estimated using the Kalman filter methodology, which is briefly described in Appendix A. The results are shown in Table 5.

It is found that in all cases the seasonal factor volatility (σ_α) is significantly different from zero and the seasonal period (φ) is more or less one year, implying that seasonality in all four commodity prices is stochastic with a period of one year, which is consistent with the results obtained by Garcia et al. (2012). Moreover, the speed of adjustment (k) is highly significant, implying, mean reversion in commodity prices, which is coherent with the results obtained by Schwartz (1997). It is also found that the long-term trend (μ_ξ) is positive and significantly different from zero in all cases, implying long-term growth in commodity prices, specially in the cases of RBOB gasoline, heating oil and WTI crude oil.

It is also interesting to note that short-term volatility (σ_χ) is higher than long-term volatility (σ_ξ) in all cases, which is coherent with the results found by Schwartz (1997) and Garcia et al. (2012).

Concerning the market prices if risk, it is found that the risk premium associate with the long-term factor (λ_ξ) is significantly different from zero in all cases, whereas the risk premium associated with the short-term one (λ_χ) is not, suggesting that the risk associated with the long-term factor is more difficult to diversify than the risk associated with the short-term one. Moreover, the market prices of risk associated with the real and complex parts of the seasonal component (λ_α and λ_{α^*} respectively) are not significantly different from zero in most of the cases, suggesting that the risk associated to the seasonal component can be diversified in most of the cases.

However, from the point of view of the goal of this chapter it is interesting to analyze the influence of the estimated parameters for each commodity on its convenience yield. As stated above, the speed of adjustment (k) is relatively high and significantly different from zero in all cases, implying high convenience yield, especially in the case of RBOB gasoline, followed by Henry Hub natural gas. It is also found that the highest value of the seasonal period (φ) is found in the case of Henry Hub natural gas, followed by RBOB gasoline , heating oil and WTI crude oil, implying higher convenience yield for Henry Hub natural gas and lower for WTI crude oil (in absolute value). Finally, from the estimated vales shown in Table 5 it is easy to compute the term in parenthesis in expression (12), involving the standard deviations and the correlations among the model factors. It is found that the highest value for this term, and therefore the highest absolute value for the convenience yield, corresponds to Henry Hub natural gas (with a value of 0.2336), followed by RBOB gasoline (0.1296), WTI (0.1128) and heating oil (0.0993). Therefore, we can conclude that the highest estimated values of the convenience yield are found in the cases of Henry Hub natural gas and RBOB gasoline.

Finally, Figure 4 shows the time series evolution of the estimated seasonal components and the estimated convenience yield, both obtained with the four-factor model. It can be appreciated the three months delay of convenience yields (green line) seasonality with respect to the commodity price seasonality (blue line), although the pattern is less clear in the case of WTI crude oil. The seasonal pattern is less clear in the case of WTI, which is coherent with the results found in Section 2.

3.4 THE CONVENIENCE YIELD MODEL

Here we present a model for the stochastic behavior of convenience yields. This model will account for stochastic seasonality. Moreover, it could be the case that in certain commodities like crude oil, in which there were not observe seasonality, it is possible that there is a weak seasonal component, which is hidden by other factors, and this seasonal component can be estimated through the convenience yield.

Specifically, the proposed model for the convenience yield is the three-factor model by Garcia et al. (2012). This model will allow us to estimate crude oil seasonality through its convenience yield and to compare spot price and convenience yield seasonality.

Theoretical Model

Here we present a model to characterize the commodity convenience yield dynamics which takes into account the seasonal effects and which is coherent with the previous findings.

The proposed model for the stochastic behavior of convenience yields is the three-factor model in Garcia et al. (2012)[37]. In this three-factor model the spot convenience yield (X_t) is the sum of a deterministic long-term factor (ξ_t) and two stochastic factors[38], a short-term component (χ_t) and a seasonal component (α_t):

$$X_t = \xi_t + \chi_t + \alpha_t \tag{13}$$

The third stochastic factor is the other seasonal factor (α_t^*), which complements α_t. The SDEs of these factors are:

$$d\xi_t = \mu_\xi dt \tag{14}$$

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dW_{\chi t} \tag{15}$$

$$d\alpha_t = 2\pi\varphi\alpha_t^* dt + \sigma_\alpha dW_{\alpha t} \tag{16}$$

[37] A four factor model like the one presented in section 3 has been estimated for the convenience yield, however the stochastic parameters related with the long-term factor were no significant, which confirms previous evidence regarding the strong mean-reverting behavior of convenience yield series.

[38] It should be noted that in the original three-factor model by Garcia et al. (2012) the log-spot price is the sum of three stochastic factors. However, here we model directly the convenience yield price instead of its log, given that the convenience yield can take negative values.

$$d\alpha_t^* = -2\pi\varphi\alpha_t dt + \sigma_\alpha dW_{\alpha^* t} \quad (17)$$

As shown in the case of the four-factor model, this model is “maximal” in the sense of Dai and Singleton (2000). Even more this model is Dai-Singleton $A_0(3)$, as can be seen in Appendix C.

Estimation Results

The three factor model presented above has been estimated though the Kalman filter methodology, using the convenience yield data estimated in Section 2. The results of the model estimation are shown in Table 6. The results indicate a high degree of mean reversion (high value of κ), mostly in the case of Henry Hub natural gas, which is coherent with the preliminary results obtained in Section 2.

However, the most important issue in Table 6, from the point of view of this chapter goal, is the fact that the standard deviation of the seasonal factor (σ_α) is significantly different from zero for all four commodities. This result is suggesting that convenience yields not only show seasonality, but this seasonality is stochastic rather than deterministic. Moreover, the values of the standard deviation of the seasonal factor obtained in Table 6 for the convenience yield series are considerable higher than those obtained in Table 5 for the commodity price series. This result is suggesting that seasonality is even clearer in the convenience yield series than in the commodity price ones. It is interesting to observe the high values of σ_α obtained in the cases of RBOB gasoline and Henry Hub natural gas convenience yield series, which is coherent with results shown in Figure 1. It is also very interesting to observe that the WTI convenience yield series (and the WTI futures prices series in Table 5) also shows evidence of stochastic seasonality, although the tests in Section 2 did not detected evidence of seasonality in the case of WTI crude oil convenience yield series.

Looking at expression (12) it is clear that the short-term component in the convenience yield is equal to the short-term component in the spot price multiplied by the speed of adjustment in the four-factor model (κ). Given that the estimated values of κ in the four-factor model (Table 5) are not very far from one, the standard deviations of the short-term components in the convenience yield and the spot price series should be similar. This is the result found in the cases of RBOB gasoline and heating oil. The values of the standard deviations of the short-term component in the WTI and Henry Hub natural gas convenience yield series (Table 6) are higher than the corresponding values in the spot price series (Table 5) due to the high variability found in these convenience yield series, as can be appreciated in Figure 1.

Moreover, from expression (12) we can conclude that the seasonal component in the convenience yield is equal (in absolute value) to the complementary seasonal component in the spot price multiplied by $2\pi\varphi$. Given that the estimated values of the seasonal period (φ) in Table 5 are very close to one, the standard deviation of the spot price complementary factor[39] should be similar to the standard deviation of the convenience yield divided by 2π. In the case of WTI crude oil the standard deviation of the complementary seasonal factor in the spot price model is 0.0106, whereas the standard deviation of the seasonal factor in the convenience yield model (divided by 2π) is 0.00844. The figures in the case of heating oil are 0.0118 and 0.0115 respectively. In the case of RBOB gasoline these figures are 0.0425 and 0.0760 respectively. Finally, the figures in the case of Henry Hub natural gas are 0.0385 and 0.0600 respectively.

[39] Remember that in the four-factor model $\sigma_\alpha = \sigma_{\alpha^*}$.

This result can be corroborated looking at Figure 4. In this Figure the estimated convenience yield (green line) shows a very similar patter to the complementary seasonal factor (α^*) in the four-factor model (red line), although as before the pattern is less clear in the case of WTI crude oil.

Table 7 presents a summary of the influence of the seasonal components on the commodity price (four-factor model for commodity spot prices) and on the convenience yield (three-factor model for convenience yields). Specifically the table shows the average weights of the seasonal factors (α and α^*) in the log-price of the commodity (Panel A) and in the convenience yield (Panel B)[40]. It is quite striking to observe how the weights of the seasonal components are considerable higher in the model for the convenience yield (Panel B). In both panels the highest weights are achieved in the cases of RBOB, heating oil and Henry Hub natural gas. Finally, it is also interesting to observe the relative high weight of the seasonal pattern on the convenience yield in the case of WTI crude oil, suggesting that in commodities like crude oil, in which there were not observe seasonality, that there is a weak seasonal component and this seasonal component can be estimated through the convenience yield.

In summary, we can conclude that the estimated convenience yield series show evidence of stochastic seasonality and that this seasonality is even clearer than in the case of commodity spot prices series. This result is suggesting that commodity price seasonality can be better estimated through convenience yields rather than through futures prices. The reason is that futures prices are driven for many things, such as supply, demand, political aspects, speculation, weather conditions, etc. Therefore, sometimes it may be difficult to extract the seasonal component from futures prices.

[40] The weight of the sum of the two seasonal factors (α and α^*) over the convenience yield price in Panel B of Table 7 is greater than 100%. This is due to the fact that in the three-factor model the convenience yield is the sum of a long-term (ξ, deterministic) component, a short-term (χ, stochastic) component and a seasonal (α, stochastic) component. The other seasonal component, α^*, does not influence the convenience yield price.

However, as shown in Section 2, the convenience yield is estimated though a ratio of two futures prices, so many of these non-seasonal factors tend to disappear, facilitating the estimation of the seasonal component.

3.5 CONCLUSIONS

This chapter focuses on commodity convenience yields. Convenience yields for four energy commodities (WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas) are estimated using the procedure defined in Gibson and Schwartz (1990), finding, as in previous studies, that convenience yields exhibit seasonality and mean reversion. Based on this empirical evidence, we present a factor model in which the convenience yield exhibits mean reversion and stochastic seasonality. Specifically, we show that the four-factor model presented by Garcia et al. (2012), with two long- and short-term factors and two additional trigonometric seasonal factors, can generate stochastic seasonal mean-reverting convenience yields. Moreover, it is found a $\pi/2$ lag in the convenience yield seasonality with respect to spot price seasonality.

Based on this evidence, the next step is to present a theoretical model to characterize the commodity convenience yield dynamics which is coherent with the previous findings. Specifically, the model takes into account mean reversion and stochastic seasonal effects in the convenience yield. We also show that commodity price seasonality can be better estimated through convenience yields rather than through futures prices. The reason is that futures prices are driven for many things, such as supply, demand, political aspects, speculation, weather conditions, etc. Therefore, sometimes it may be difficult to extract the seasonal component from futures prices. However, the convenience yield is estimated though a ratio of two futures prices, so many of these non-seasonal factors tend to disappear, facilitating the estimation of the seasonal component.

APPENDIX A. ESTIMATION METHODOLOGY

The Kalman filter technique is a recursive methodology that estimates the unobservable time series, the state variables or the factors (Z_t) based on an observable time series (Y_t) that depends on these state variables. The *measurement equation* accounts for the relationship between the observable time series and the state variables:

$$Y_t = d_t + M_t Z_t + \eta_t \qquad t = 1, \ldots, N_t, \qquad \text{(A1)}$$

where $Y_t, d_t \in \Re^n, M_t \in \Re^{n\,x\,n}$, $Z_t \in \Re^h$, h is the number of state variables, or factors, in the model, and $\eta_t \in \Re^n$ is a vector of serially uncorrelated Gaussian disturbances with zero mean and covariance matrix H_t.

In the estimation procedure, a discrete time version of this equation is necessary; in the case of the joint model with a common long-term trend for the three commodities, this equation is given by the following expressions:

$$Y_t = \begin{pmatrix} \ln F_{T1}^1 \\ \vdots \\ \ln F_{Tn}^1 \\ \ln F_{T1}^2 \\ \vdots \\ \ln F_{Tn}^2 \\ \ln F_{T1}^3 \\ \vdots \\ \ln F_{Tn}^3 \end{pmatrix}, \quad d_t = \begin{pmatrix} A^1(T_1) \\ \vdots \\ A^1(T_n) \\ A^2(T_1) \\ \vdots \\ A^2(T_n) \\ A^3(T_1) \\ \vdots \\ A^3(T_n) \end{pmatrix}, \quad M_t = \begin{pmatrix} 1 & e^{-k_1 T_1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-k_1 T_n} & 0 & 0 \\ 1 & 0 & e^{-k_2 T_1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & e^{-k_2 T_n} & 0 \\ 1 & 0 & 0 & e^{-k_3 T_1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & e^{-k_3 T_n} \end{pmatrix}$$

and F_{T1}^i is the price of a futures contract for the commodity "*i*" (*i*=1,2,3) with maturity at time "T_1+*t*" traded at time *t*. In principle, it would be possible to use a different number of futures contracts for each commodity; however, in this work, we consider it more suitable to use the same number ("*n*") of futures contracts for all commodities.

The *transition equation* accounts for the evolution of the state variables:

$$Z_t = c_t + T_t Z_{t-1} + \psi_t \qquad t = 1, \ldots, N_t, \qquad \text{(A2)}$$

where $c_t \in \Re^h, T_t \in \Re^{h\,x\,h}\, and\ \psi_t \in \Re^h$ is a vector of serially uncorrelated Gaussian disturbances with zero mean and covariance matrix Q_t.

In the case of the joint model with a common long-term trend for the three commodities, the discrete time version of this equation, which is needed in the estimation procedure, is given by the following expressions:

$$Z_t = \begin{pmatrix} \xi_{1t} \\ \chi_{1t} \\ \chi_{2t} \\ \chi_{3t} \end{pmatrix}, \; c_t = \begin{pmatrix} \mu_{\xi 1}\Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; T_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-k_1\Delta t} & 0 & 0 \\ 0 & 0 & e^{-k_2\Delta t} & 0 \\ 0 & 0 & 0 & e^{-k_3\Delta t} \end{pmatrix} \text{ and}$$

$$Var(\psi_t) = \begin{pmatrix} \sigma_{\xi_1}^2\Delta t & \sigma_{\xi_1}\sigma_{\chi_1}\rho_{\xi_1\chi_1}(1-e^{-k_1\Delta t})/k_1 & \sigma_{\xi_1}\sigma_{\xi_2}\rho_{\xi_1\xi_2}\Delta t & \sigma_{\xi_1}\sigma_{\chi_2}\rho_{\xi_1\chi_2}(1-e^{-k_1\Delta t})/k_2 \\ \sigma_{\xi_1}\sigma_{\chi_1}\rho_{\xi_1\chi_1}(1-e^{-k_1\Delta t})/k_1 & \sigma_{\chi_1}^2(1-e^{-2k_1\Delta t})/(2k_1) & \sigma_{\chi_1}\sigma_{\chi_2}\rho_{\chi_1\chi_2}(1-e^{-(k_1\Delta t+k_2\Delta t)})/(k_1+k_2) & \sigma_{\chi_1}\sigma_{\chi_3}\rho_{\chi_1\chi_3}(1-e^{-(k_1\Delta t+k_3\Delta t)})/(k_1+k_3) \\ \sigma_{\xi_1}\sigma_{\chi_2}\rho_{\xi_1\chi_2}(1-e^{-k_1\Delta t})/k_2 & \sigma_{\chi_1}\sigma_{\chi_2}\rho_{\chi_1\chi_2}(1-e^{-(k_1\Delta t+k_2\Delta t)})/(k_1+k_2) & \sigma_{\chi_2}^2(1-e^{-2k_2\Delta t})/(2k_2) & \sigma_{\chi_2}\sigma_{\chi_3}\rho_{\chi_2\chi_3}(1-e^{-(k_2\Delta t+k_3\Delta t)})/(k_2+k_3) \\ \sigma_{\xi_1}\sigma_{\chi_3}\rho_{\xi_1\chi_3}(1-e^{-k_3\Delta t})/k_3 & \sigma_{\chi_1}\sigma_{\chi_3}\rho_{\chi_1\chi_3}(1-e^{-(k_1\Delta t+k_3\Delta t)})/(k_1+k_3) & \sigma_{\chi_2}\sigma_{\chi_3}\rho_{\chi_2\chi_3}(1-e^{-(k_2\Delta t+k_3\Delta t)})/(k_2+k_3) & \sigma_{\chi_3}^2(1-e^{-2k_3\Delta t})/(2k_3) \end{pmatrix}$$

Here, $Y_{t|t-1}$ is the conditional expectation of Y_t, and Ξ_t is the covariance matrix of Y_t conditional on all information available at time t – 1. After omitting unessential constants, the log-likelihood function can be expressed as

$$l = -\sum_t \ln|\Xi_t| - \sum_t (Y_t - Y_{t|t-1})'\Xi_t^{-1}(Y_t - Y_{t|t-1}) \,. \tag{A3}$$

APPENDIX B. STOCHASTIC DIFERENTIAL EQUATIONS (SDE) INTEGRATION

Most of the models proposed in the literature assume that the risk-neutral dynamics of a commodity price (or its log) is given by a linear stochastic differential system:

$$\begin{cases} dX_t = (b + AX_t)dt + RdW_t \\ Y_t \quad = cX_t \end{cases}$$

where Y_t is the commodity price (or its log), b, A, R and c are deterministic parameters[41] independent of t ($b \in \Re^n$, $A, R \in \Re^{n x n}$, $c \in \Re^n$) and W_t is a n-dimensional canonical Brownian motion (i.e. all components uncorrelated and its variance equal to unity) under the risk-neutral measure.

Let us see that the solution of that problem is[42]:

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] \tag{B1}$$

In order to proof it, we shall apply the general rule for the derivation of the product of stochastic components (Oksendal, 1992):

$$dX_t = \left(de^{At}\right)\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] + e^{At} d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] + \\ + \left(de^{At}\right) d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right]$$

It is easy to show that:

$$d\left[X_0 + \int_0^t e^{-As} bds + \int_0^t e^{-As} RdW_s\right] = e^{-At} bdt + e^{-At} RdW_t$$

[41] Again note that R does not need to be computed.

[42] Even in the case that b, A and R were function of t, if A_t and $\int_0^t A_s ds$ commute, the solution of that problem is (B1).

The first differential only has elements of type *dt*, hence the product of the first differential times the second differential is zero.

Thus:

$$dX_t = Ae^{At}dt\left[X_0 + \int_0^t e^{-As}bds + \int_0^t e^{-As}RdW_s\right] + e^{At}\left[e^{-At}bdt + e^{-At}RdW_t\right] = A_t X_t dt + bdt + RdW_t$$

Consequently we obtain expression (B1):

$$X_t = e^{At}\left[X_0 + \int_0^t e^{-As}bds + \int_0^t e^{-As}RdW_s\right].$$

It is easy to prove that the solution is unique (Oksendal, 1992).

APPENDIX C. CANONICAL REPRESENTATION

Introduction

In this appendix, we shall see how our models can be related to Dai-Singleton $A_0(n)$ class, with the important distinction of allowing complex eigenvalues. Afterwards, we shall show global identification properties.

General setup

Let $Z_t = \log(S_t)$ be the log of the spot price. If we assume a linear model, its real dynamics is given by:

$$\begin{cases} dX_t = (m + AX_t)dt + RdW_t \\ S_t = \exp(\phi_0 + CX_t) \\ X_0 \text{ given} \end{cases} \tag{F}$$

whereas its risk neutral dynamics is given by:

$$\begin{cases} dX_t = (m - \lambda + AX_t)dt + RdW_t \\ S_t = \exp(\phi_0 + CX_t) \\ X_0 \text{ given} \end{cases} \tag{FN}$$

where R is full rank lower triangular (we shall examine this assumption later). We would like to know how this general setup can be reduced to a model which is maximal, i.e. cannot be reduced to an equivalent model with less states and parameters (another way to see this is saying that has the maximum number of identificable parameters). We shall concentrate first in (F).

First of all, (see for example Sontag 1990), a model has the minimal number of states if and only if is observable and controlable, i.e. $rank\begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} = n$ (observability condition) and $rank\left(R \; AR \; A^2R \ldots A^{n-1}R\right) = n$ (controlability condition). As the latter is always satisfied if R is full rank, we just impose the former. Moreover, in the context of stochastic systems, controlability plays a small role as it means that some states are unaffected by noise so whether they are observationally equivalent to other system depends only on initial states.

Invariant transformations

Following Dai and Singleton, we allow for the following transformations

1. Affine transformations of states: $\widetilde{X}_t = v + GX_t$ where G is nonsingular and v is an arbitrary vector. Note the important role of constants ϕ_0 and ϕ_1. If they where not present and output equation were CX_t, then v could not be arbitrary but instead would have to accomplish $Cv = 0$.
2. Rotations of brownian motions. $\widetilde{W}_t = UW_t$ where $UU^T = I$ as Brownian motion is unobserved.

Note that these transformations preserve observability and rank of *R*.

Relationship with $A_0(n)$

We shall first show now how to relate our model to Dai-Singleton $A_0(n)$ class, i.e. a system like: (DS) $\begin{cases} dY_t = -KY_t dt + \Sigma d\widetilde{W}_t \\ S_t = \exp\left(\delta_0 + \widetilde{C}Y_t\right) \end{cases}$ where $R = I$, $C = (1 \ldots 1)$ and K is lower triangular with all their diagonal elements strictly positive, i.e. $K_{ii} > 0$.

This means several restrictions within the system:

1. The dynamics matrix -*K* is full rank and all their eigenvalues are real and negative.
2. Noise matrix is also full rank.

All these properties are preserved through invariant transformations, so we would have to impose them on our system. But we have complex eigenvalues, so we have to use a different, although similar, canonical form. To sum up, we replace Dai-Singleton restrictions with others, so our approaches are similar but not directly comparable.

First canonical form

If all eigenvalues are different then the pair (F) can be reduced to:

(F1) $\begin{cases} d\widetilde{X}_t = \left(\widetilde{m} + \widetilde{A}\widetilde{X}_t\right)dt + \widetilde{R}d\widetilde{W}_t \\ S_t = \exp\left(\widetilde{C}\widetilde{X}_t\right) \end{cases}$, where

1. $\widetilde{A}$ is diagonal (real only if there are no complex eigenvalues)
2. $C = (1\ 1\ ...1)$.
3. $\widetilde{R}$ is lower triangular and all its diagonal elements are strictly possitive.
4. $\widetilde{m} = \begin{pmatrix} m_0 \\ 0 \end{pmatrix}$ with $m_0 \in \Re$

Moreover, if we start with a canonical form (F1) the system is observable and controlable (therefore has the minimal number possible of states).

Proof

If all the eingenvalues are different, then A is diagonalizable. Therefore, changing the base, we have a representation where $\widetilde{A}$ is diagonal. We shall see now that all elements in C are not null.

Let $\widetilde{A} = diag(d_1...d_n)$. By the observability condition, the matrix $\begin{pmatrix} \widetilde{C} \\ \widetilde{C}\widetilde{A} \\ ... \\ \widetilde{C}\widetilde{A}^{n-1} \end{pmatrix}$ is full rank.

But this matrix equals $\begin{pmatrix} c_1 & c_2 & ... & c_n \\ c_1 d_1 & c_2 d_2 & ... & c_n d_n \\ ... & ... & ... & ... \\ c_1 d_1^{n-1} & c_2 d_2^{n-1} & ... & c_n d_n^{n-1} \end{pmatrix}$. Should any of the c_i be null, then its full column would be null and therefore the system would not be observable. This also proves that, starting from canonical form (F1), the system is observable.

As a result, we can define the transformation $L_0 = diag\left(\frac{1}{c_1},...,\frac{1}{c_n}\right)$. Under this change of variable, $\widetilde{C} = (1...1)$ and $\widetilde{A}$ is diagonal. Using a suitable ortogonal transformation of the noise, we can also impose the conditions on $\widetilde{R}$ via a Choleski decomposition (thus proving also that the system is controlable, due to the fact that noise matrix is full rank).

Now for the form of $\widetilde{m}$. We define the new state as $\widetilde{\widetilde{X}}_t = \widetilde{X}_t - \begin{pmatrix} \phi_0 + \mu_2/d_2 + ... + \mu_n/d_n \\ -\mu_2/d_2 \\ ... \\ -\mu_n/d_n \end{pmatrix}$. Clearly it verifies the conditions.

Complex eigenvalues

It is now time to consider complex eigenvalues. The results are essentially the same, but the canonical form is slightly different. Both are, however, perfectly equivalent. We need a few previous lemmas.

Lemma

If A is a 2x2 real matrix with complex eigenvalues $k \pm i\varphi$ and C ia a 2x1 real matrix such that the pair (A, C) is observable then

1. A is diagonalizable and, if $\Lambda = \begin{pmatrix} k + i\varphi & 0 \\ 0 & k - i\varphi \end{pmatrix}$ and $H = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ then $H^{-1}\Lambda H = \begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix}$

2. There exist a real matrix T such that $T^{-1}AT = \begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix}$ and $CT = (1\ \ 0)$

Proof

A has two all eigenvalues distinct therefore is diagonalizable. As it is real, its eigenvalues are conjugate.

We just have to do the product. $H^{-1}\Lambda H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} k+i\varphi & 0 \\ 0 & k-i\varphi \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. It equals

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} k+i\varphi & -ik+\varphi \\ k-i\varphi & ik+\varphi \end{pmatrix} = \begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix}$$

In order to proof part 2, let us get back to the original A. It has two eigenvectors, but is a real matrix. Therefore, if v is an eingenvector associated to an eigenvalue λ, then $Av = \lambda v$. Taking conjugates, $\overline{A}\,\overline{v} = \overline{\lambda}\,\overline{v}$. But A is real, therefore $A = \overline{A}$ so $A\overline{v} = \overline{\lambda}\,\overline{v}$. It means that $\overline{v}$ is the eingenvector associated to the other eigenvalue.

Let $T_0 = \begin{pmatrix} v_1 & \overline{v}_1 \\ v_2 & \overline{v}_2 \end{pmatrix}$ be the matrix of eigenvectors. Then $\begin{pmatrix} k+i\varphi & 0 \\ 0 & k-i\varphi \end{pmatrix} = T_0^{-1} A T_0$.

Let $T_1 = T_0 H$. We know then, $\begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix} = T_1^{-1} A T_1$. We shall proof know that T_1 is real.

$$\begin{pmatrix} v_1 & \overline{v}_1 \\ v_2 & \overline{v}_2 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} v_1+\overline{v}_1 & -iv_1+i\overline{v}_1 \\ v_2+\overline{v}_2 & -iv_2+\overline{v}_2 \end{pmatrix} = 2\begin{pmatrix} \mathrm{Re}[v_1] & \mathrm{Im}[v_1] \\ \mathrm{Re}[v_2] & \mathrm{Im}[v_2] \end{pmatrix} \in \Re^{2\mathrm{x}2}$$

Finally, let $C = (c_1, c_2)$. As (A, C) is observable, $c_1^2 + c_2^2 > 0$.We define

$H_0 = \frac{1}{c_1^2+c_2^2}\begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}$ We know $CH_0 = (1\ \ 0)$

$$H_0^{-1}\begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix} H_0 = \frac{1}{c_1^2+c_2^2}\begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}\begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix} =$$

$$\frac{1}{c_1^2+c_2^2}\begin{pmatrix} c_1k+c_2\varphi & c_1\varphi-c_2k \\ c_2k-c_1\varphi & c_1k+c_2\varphi \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix} = \begin{pmatrix} k & \varphi \\ \varphi & k \end{pmatrix}$$

So, defining $T = T_1 H_0$ we get the result.

Lemma

Let $C \in \Re^{k\,x\,n}, A \in \Re^{n\,x\,n}$ be real matrices where all eigenvalues of A are different.

There exist a real matrix T such that $T^{-1}AT = \begin{pmatrix} A_1 & 0 & ... & 0 \\ 0 & A_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & A_r \end{pmatrix}$ $CT^{-1} = (C_1 ... C_r)$

where $eig(A_j) = \lambda_j \in \Re$ or $eig(A_j) = \{k_j + i\varphi_j, k_j + i\varphi_j\}$

<u>Proof</u>

Let $\lambda_1,...,\lambda_p$ be the real eigenvalues and $\mu_1, \overline{\mu}_1, ..., \mu_q, \overline{\mu}_q$ be the complex ones. Let $v_1,...,v_p$ and $w_1, \overline{w}_1, ..., w_q, \overline{w}_q$ be the corresponding eigenvectors. We define the subspaces $V_i = Sp(v_i)$ and $W_i = Sp(w_i, \overline{w}_i)$. V_i is defined by a real vector (and thus has a real basis) and $W_i = Sp\left\{w_i + \overline{w}_i, \frac{w_i - \overline{w}_i}{i}\right\}$ therefore has also a real basis. Let T be the basis of all the subspaces together, which is a real matrix.

Clearly $\Re^{p+2q} = V_1 \oplus ... \oplus V_p \oplus W_1 \oplus ... \oplus W_q$ and all subspaces are $A-$invariant. Using the above real basis, we can partition $T^{-1}AT = \begin{pmatrix} A_1 & 0 & ... & 0 \\ 0 & A_2 & ... & 0 \\ .. & ... & ... & ... \\ 0 & 0 & ... & A_r \end{pmatrix}$ where $A_i = A|_{V_i}$ or $A_i = A|_{W_i}$ thus verifying the thesis.

We are now ready to state the complex canonical form.

Second canonical form

If all eigenvalues are different then (F) can be reduced to (F2) $\begin{cases} d\widetilde{X}_t = (\widetilde{m} + \widetilde{A}X_t)dt + \widetilde{R}d\widetilde{W}_t \\ S_t = \exp(\widetilde{C}X_t) \end{cases}$, where all matrices are real and:

1. $\widetilde{A} = \begin{pmatrix} A_1 & 0 & ... & 0 \\ 0 & A_2 & ... & 0 \\ .. & ... & ... & ... \\ 0 & 0 & ... & A_r \end{pmatrix}$ and either $A_i = \lambda_i \in \Re$ or $A_i = \begin{pmatrix} k & \varphi \\ -\varphi & k \end{pmatrix}$

2. $C = (C_1 ... C_r)$ each corresponding to A_i and $C_i = 1$ if $A_i = \lambda_i \in \Re$ or $C_i = (1\ 0)$ otherwise.

3. $\widetilde{R}$ is lower triangular and all its diagonal elements are strictly possitive.

4. $\widetilde{m} = \begin{pmatrix} m_0 \\ 0 \end{pmatrix}$ with $m_0 \in \Re$

<u>Proof</u>

Combining the two previous lemmas, it is obvious that there is a real matrix that transforms A and C into the previous forms. By proceding as in the other third reduced form, we obtain the rest of the result.

Maximality

In order to show that the model set is maximal we see that the model is globally identificable, as in general the latter implies the former if all parameters are admisible. To see this, remember that in a globally identifiable model, different parameters give different realizations. Suppose that a model has n parameters and is not maximal but admits a representation with $k<n$ parameters. By redefining the parameter space (under some conditions) it means that the last parameters are functions of the first, formally $\theta = (\phi, \varphi(\phi))$.

But, for a value ϕ^*, we can take a differente value $(\phi^*, \varphi^*) \neq (\phi^*, \varphi(\phi^*))$ thus obtaining a different admisible value. The only way to avoid contradiction would be that

$(\phi^*,\varphi^*)\neq(\phi^*,\varphi(\phi^*))$ achieve the same realization, but this is imposible since the model is globally identificable. We thus have to conclude that the model is not maximal.

We shall first proof the version where spot prices are observable and then explain why risk premia can also be identified.

Proposition

If S_t is observable, model (F3) is globally identificable (incluiding the initial state X_0)

Proof

Let $Z_t=\log(S_t)$. We assume that we can observe the mean and variance of Z_t at any moment in time. If the model has complex eigenvalues, we perform the transformation $\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}^{-1}$ for each A_i thus converting C into $(1...1)$ and making A diagonal. If after this transformation the model is globally identificable, so is the original model.

We know that $\mathrm{var}(Z_t)=\_Cvec^{-1}\left[\int_0^t(\exp(Au)\otimes\exp(Au))vec(RR')dt\right]C'$ (see García et al., 2012). It is the sum of exponencials of eigenvalues of A and in all sums appears $\frac{e^{d_{ii}T}-1}{d_{ii}}(RR')_{ii}$. As $(RR')_{ii}$ is not null and d_{ii} is the double of an eingenvalue all eigenvalues are identified and so is *A*. Note that this argument os even valid if 0 is an engenvalue, as we would only be able to identify $n-1$ values, which means that the other is 0. Therefore, no restrictions exists in the eigenvalues of *A* so any maximal model needs all.

But, as $Cvec^{-1}\left[\int_0^t(\exp(Au)\otimes\exp(Au))vec(RR')dt\right]C'$, if A is identified, so is RR' (in the complex case is $HRR'H'$ where H is the change of variable, but we can get the original by multiplying by both inverses). We just have to extract from the integrals (as all integrals are positive). Therefore RR' and A are identified.

We have now two cases. Let us first assume A is NOT full rank. Then,

$$[Y_t] = Ce^{At}\left[X_0 + \int_0^t e^{-As}\begin{pmatrix} m_0 \\ 0 \end{pmatrix} ds\right] = (1 \; ...1)\begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & e^{d_1 t} & ... & 0 \\ ... & .... & ... & ... \\ 0 & 0 & ... & e^{d_n t} \end{pmatrix}\left[X_0 + \begin{pmatrix} tm_0 \\ 0 \end{pmatrix}\right]$$

So we have the equality $E[Y_t] = X_{01} + tm_0 + e^{d_2 t}X_{02} + ...e^{d_n t}X_{0n}$. As all this functions are linearly independent, it means that all their coefficients are univocaly defined.

Now, we shall assume that A is full rank. We define $\overline{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$, $\overline{X}_0 = \begin{pmatrix} \phi_0 \\ X_0 \end{pmatrix}$ and $\overline{C} = (1 \; C_1 ... C_r)$. The system is still observable, by construction and we are back to the previous case.

Risk premia

It is now time to consider whether risk premia can be identified. If we start with model

$$\text{(F2)}\begin{cases} d\widetilde{X}_t = (\widetilde{m} + \widetilde{A}X_t)dt + \widetilde{R}d\widetilde{W}_t \\ S_t = \exp(\widetilde{C}X_t) \end{cases}$$ its risk neutral version is given by:

$$\text{(F2N)}\begin{cases} d\widetilde{X}_t = (\widetilde{m} - \lambda + \widetilde{A}X_t)dt + \widetilde{R}d\widetilde{W}_t \\ S_t = \exp(\widetilde{C}X_t) \end{cases}$$

We shall now assume that all futures are observable and show that the system, with the risk neutral dynamics is also globally identificable.

Proposition

In the above conditions, if $F_{t,T} = E^Q[S_{t+T} / I_t]$ is observable, then model (F3N) is globally identificable.

Proof

First, if $F_{t,T}$ is observable, making $T=0$ it means that S_t is observable. So all parameters apart from (possibly) risk premia are identified.

However, $Z_{t+T}=\phi_1+Ce^{AT}\left[X_t+\int_0^T e^{-As}\lambda ds+\int_0^T e^{-As}RdW_{s+t}\right]$. If we take expectations with respect first to the first measure and after to the second $E\left[E^Q[\bullet/I_t]\right]$, the Ito integral disapears and $E^Q[X_t/I_t]=X_t$ only depends on identifiable parameters. Therefore we are left with $E\left[E^Q[Z_t/I_t]\right]=\phi_1+t(m_0-\lambda_1)-e^{d_2t}\lambda_2-\ldots-e^{d_nt}\lambda_n$ in the singular A case and without the t term in the nonsingular case. Anyway, independent functions which means identifiable parameters.

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TABLES AND FIGURES

TABLE 1

DESCRIPTIVE STATISTICS. FUTURES PRICES

The table shows the mean and volatility of the four commodity futures prices series. The sample period is 9/27/1999 to 7/4/2011 (615 weekly observations). F1 is the futures contract closest to maturity, F2 is the contract second-closest to maturity and so on.

	WTI Crude Oil			Heating Oil			Gasoline			Henry Hub	
	Mean	Volatility		Mean	Volatility		Mean	Volatility		Mean	Volatility
F1	55.06	31.30%	F1	64.46	31.73%	F1	64.59	36.81%	F1	5.68	46.80%
F4	55.59	26.49%	F3	64.96	28.08%	F3	64.19	30.13%	F5	6.04	32.53%
F7	55.57	23.83%	F5	65.17	26.04%	F5	63.73	26.26%	F9	6.17	26.91%
F11	55.36	21.69%	F7	65.27	24.08%	F7	63.37	24.53%	F14	6.15	22.48%
F14	55.17	20.57%	F10	65.23	21.59%	F9	63.24	24.31%	F18	6.13	20.80%
F17	54.98	19.72%	F12	65.13	20.61%	F12	63.00	23.77%	F22	6.06	21.55%
F20	54.80	19.05%	F14	65.07	20.10%	-	-	-	F27	5.99	19.57%
F24	54.60	18.44%	F16	65.04	20.04%	-	-	-	F31	5.96	20.05%
F27	54.48	18.13%	F18	65.02	19.95%	-	-	-	F35	5.89	19.17%

TABLE 2

DESCRIPTIVE STATISTICS. CONVENIENCE YIELD

The table shows the mean and volatility of the commodity convenience yield estimated prices series for the four commodities under study. The sample period is 9/27/1999 to 7/4/2011 (615 weekly observations). δ_{x_x+1} denotes the implicit convenience yield from "x" month to "x+1" months.

	WTI Crude Oil			Heating Oil			Gasoline			Henry Hub	
	Mean	Stand. Dev.		Mean	Stand. Dev.		Mean	Stand.Dev.		Mean	Stand. Dev.
δ_{1_2}	-0.01	0.29	δ_{1_2}	0.01	0.28	δ_{1_2}	0.08	0.41	δ_{1_2}	-0.30	0.58
δ_{4_5}	0.06	0.15	δ_{3_4}	0.04	0.24	δ_{3_4}	0.07	0.40	δ_{5_6}	-0.06	0.52
δ_{7_8}	0.08	0.11	δ_{5_6}	0.06	0.21	δ_{5_6}	0.09	0.34	δ_{9_10}	0.03	0.52
δ_{11_12}	0.08	0.09	δ_{7_8}	0.06	0.19	δ_{7_8}	0.08	0.33	δ_{14_15}	0.06	0.48
δ_{14_15}	0.07	0.08	δ_{10_11}	0.07	0.18	δ_{9_10}	0.07	0.35	δ_{18_19}	0.05	0.48
δ_{17_18}	0.07	0.07	δ_{12_13}	0.06	0.17	δ_{12_13}	-1.76	3.20	δ_{22_23}	0.07	0.50
δ_{20_21}	0.06	0.06	δ_{14_15}	0.06	0.16	-	-	-	δ_{27_28}	0.08	0.47
δ_{24_25}	0.06	0.05	δ_{16_17}	0.06	0.15	-	-	-	δ_{31_32}	0.06	0.51
δ_{27_28}	0.06	0.04	-	-	-	-	-	-	-	-	-

TABLE 3

UNIT ROOT TEST

The table shows the statistic of the Augmented Dickey-Fuller (ADF) test. The MacKinnon critical values for the rejection of the null hypothesis of a unit root tests are -3.4408 (1%), -2.8661 (5%) and -2.5692 (10%).

	δ_{1_2}	δ_{2_3}	δ_{3_4}	δ_{5_6}	δ_{8_9}	δ_{9_10}	δ_{13_14}	δ_{15_16}	δ_{19_20}
WTI	-3.9166	-2.7213	-2.8424	-2.8563	-2.4789	-2.5284	-2.3123	-2.0427	-2.2670
Heating Oil	-4.6782	-3.5201	-5.3899	-5.1484	-5.4680	-6.0645	-5.4280	-5.5776	-
RBOB	-7.7077	-6.7132	-6.4348	-6.8391	-5.7703	-5.4273	-5.8969	-	-
Henry Hub	-5.8121	-5.8404	-6.2003	-6.8154	-6.7486	-7.5143	-6.8356	-7.4454	-7.7367

TABLE 4

SEASONALITY TEST

The table shows the statistic of the Kruskal-Wallis test for the presence of seasonal effects in the estimated convenience yield series. The test statistic is distributed, under the null hypothesis of no seasonal effects, as a χ^2 with 11 degrees of freedom. The critical value for the rejection of the null hypothesis at 99% is 24.725.

	δ_{1_2}	δ_{2_3}	δ_{3_4}	δ_{4_5}	δ_{5_6}	δ_{6_7}	δ_{7_8}	δ_{8_9}	δ_{9_10}
WTI	3.1077	1.9255	1.3683	1.2326	1.1293	0.8077	1.1751	1.2313	1.5310
H. Oil	44.1184	43.2305	49.3397	55.8876	65.7434	67.7825	72.0962	79.1446	82.5958
RBOB	74.5228	85.3857	91.9193	92.2936	94.8757	99.9759	99.1085	99.1205	101.5181
H. Hub	80.7324	82.6555	88.6157	101.8594	106.3644	107.5883	112.3180	113.4766	115.0153

TABLE 5

ESTIMATION RESULTS. FOUR-FACTOR MODEL

The table presents the results for the four-factor model applied to the four commodities under study: WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas. Standard errors are in parentheses. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	WTI	Heating Oil	RBOB	Henry Hub
μ_ξ	0.1132***	0.1158***	0.1084**	0.0655**
	(0.0409)	(0.0395)	(0.0474)	(0.0302)
κ	1.0225***	1.0301***	1.9649***	1.1323***
	(0.0101)	(0.0143)	(0.1691)	(0.0206)
φ	0.9566***	0.9978***	1.0029***	1.0088***
	(0.0051)	(0.0002)	(0.0009)	(0.0002)
σ_ξ	0.1626***	0.1573***	0.1885***	0.1201***
	(0.0045)	(0.0044)	(0.0058)	(0.0049)
σ_χ	0.2752***	0.2458***	0.3051***	0.4367***
	(0.0090)	(0.0072)	(0.0119)	(0.0165)
σ_α	0.0106***	0.0118***	0.0425***	0.0385***
	(0.0005)	(0.0006)	(0.0020)	(0.0022)
$\rho_{\xi\chi}$	0.0518	0.1311***	0.0573	0.0117
	(0.0429)	(0.0409)	(0.0722)	(0.0603)
$\rho_{\xi\alpha}$	-0.2794***	-0.1600**	-0.1050*	-0.0892
	(0.0719)	(0.0650)	(0.0556)	(0.0845)
$\rho_{\xi\alpha*}$	-0.2488***	-0.1357*	0.2353***	-0.0067
	(0.0695)	(0.0693)	(0.0620)	(0.0797)
$\rho_{\chi\alpha}$	0.3073***	0.0994	0.1760***	0.2518***
	(0.0759)	(0.0685)	(0.0655)	(0.0812)
$\rho_{\chi\alpha*}$	0.3166***	0.2957***	-0.3956***	0.3145***
	(0.0727)	(0.0722)	(0.0549)	(0.0740)
λ_ξ	0.1372***	0.1532***	0.1515***	0.1025***
	(0.0409)	(0.0396)	(0.0492)	(0.0303)
λ_χ	0.0503	-0.0011	-0.0651	-0.0869
	(0.0692)	(0.0619)	(0.0825)	(0.1101)
λ_α	-0.0017	-0.0014	-0.0062	0.0111
	(0.0029)	(0.0032)	(0.0112)	(0.0105)
$\lambda_{\alpha*}$	-0.0050*	-0.0077**	0.0027	-0.0138
	(0.0029)	(0.0031)	(0.0130)	(0.0106)
σ_η	0.0112***	0.0094***	0.0117***	0.0376***
	(0.0001)	(0.0001)	(0.0002)	(0.0003)
Log-likelihood	27057.14	28139.46	16835.78	19318.02
AIC	27025.14	28107.46	16803.78	19286.02
SIC	26949.69	28032.02	16728.33	19210.58

TABLE 6

ESTIMATION RESULTS. THREE-FACTOR MODEL FOR THE CONVENIENCE YIELD

The table presents the results for the three-factor model applied to the four commodity convenience yield series under study: WTI crude oil, heating oil, RBOB gasoline and Henry Hub natural gas. Standard errors are in parentheses. The estimated values are reported with * denoting significance at 10%, ** denoting significance at 5%, and *** denoting significance at 1%.

	WTI	Heating Oil	RBOB	Henry Hub
μ_ξ	0.7882***	-0.0069	0.0304	-2.5086***
	(0.1823)	(0.0800)	(0.0990)	(0.7047)
κ	1.2705***	0.9639***	0.8112***	5.8064***
	(0.0002)	(0.0294)	(0.2255)	(0.0003)
φ	0.7906***	1.0114***	1.0080***	1.0012***
	(0.0000)	(0.0031)	(0.0103)	(0.0000)
σ_χ	0.6205***	0.2727***	0.3000***	2.2847***
	(0.0002)	(0.0168)	(0.0483)	(0.0003)
σ_α	0.0530***	0.0725***	0.4773***	0.3772***
	(0.0002)	(0.0054)	(0.0407)	(0.0003)
$\rho_{\chi\alpha}$	0.7771***	0.6338***	0.4382***	-0.3040***
	(0.0002)	(0.0806)	(0.1263)	(0.0003)
$\rho_{\chi\alpha^*}$	0.3725***	0.1087	0.4922***	0.1639***
	(0.0002)	(0.01055)	(0.1160)	(0.0003)
λ_χ	0.8087***	-0.0615	0.0941	-1.0521
	(0.1809)	(0.0798)	(0.0983)	(0.6875)
λ_α	0.0602***	0.0660***	-0.0660	-0.0038
	(0.0163)	(0.0240)	(0.1510)	(0.1186)
λ_{α^*}	-0.0129	-0.0427**	0.1003	0.1375
	(0.0164)	(0.0235)	(0.1437)	(0.1219)
σ_η	0.0517***	0.0779***	0.2170***	0.3772***
	(0.0002)	(0.0008)	(0.0031)	(0.0003)
Log-likelihood	13008.48	9719.43	3386.01	2501.60
AIC	12986.48	9697.43	3364.01	2479.60
SIC	12937.85	9648.79	3315.37	2430.97

TABLE 7

WEIGHTS OF SEASONAL COMPONENTS

The table presents the average weights of the estimated seasonal factors in the spot price (four-factor model for spot commodity prices) and in the convenience yield (three-factor for convenience yields), for the four commodities under study: WTI crude oil, heating oil, RBOB gasoline and Henry Hub.

PANEL A: FOUR FACTOR MODEL, COMMODITY SPOT PRICES		
	$\lvert\alpha\rvert/\log(S)$	$(\lvert\alpha\rvert + \lvert\alpha^*\rvert)/\log(S)$
Henry Hub	2.8866%	5.7299%
Heating Oil	0.5270%	1.0625%
RBOB	0.8295%	1.6809%
WTI	0.0826%	0.1439%

PANEL B: THREE-FACTOR MODEL, CONVENIENCE YIELDS		
	$\lvert\alpha\rvert/\log(S)$	$(\lvert\alpha\rvert + \lvert\alpha^*\rvert)/\log(S)$
Henry Hub	34.5652%	94.0058%
Heating Oil	46.6621%	128.1067%
RBOB	63.6727%	180.6800%
WTI	9.8590%	21.0083%

FIGURE 1

TIME SERIES EVOLUTION OF ESTIMATED CONVENIENCE YIELDS

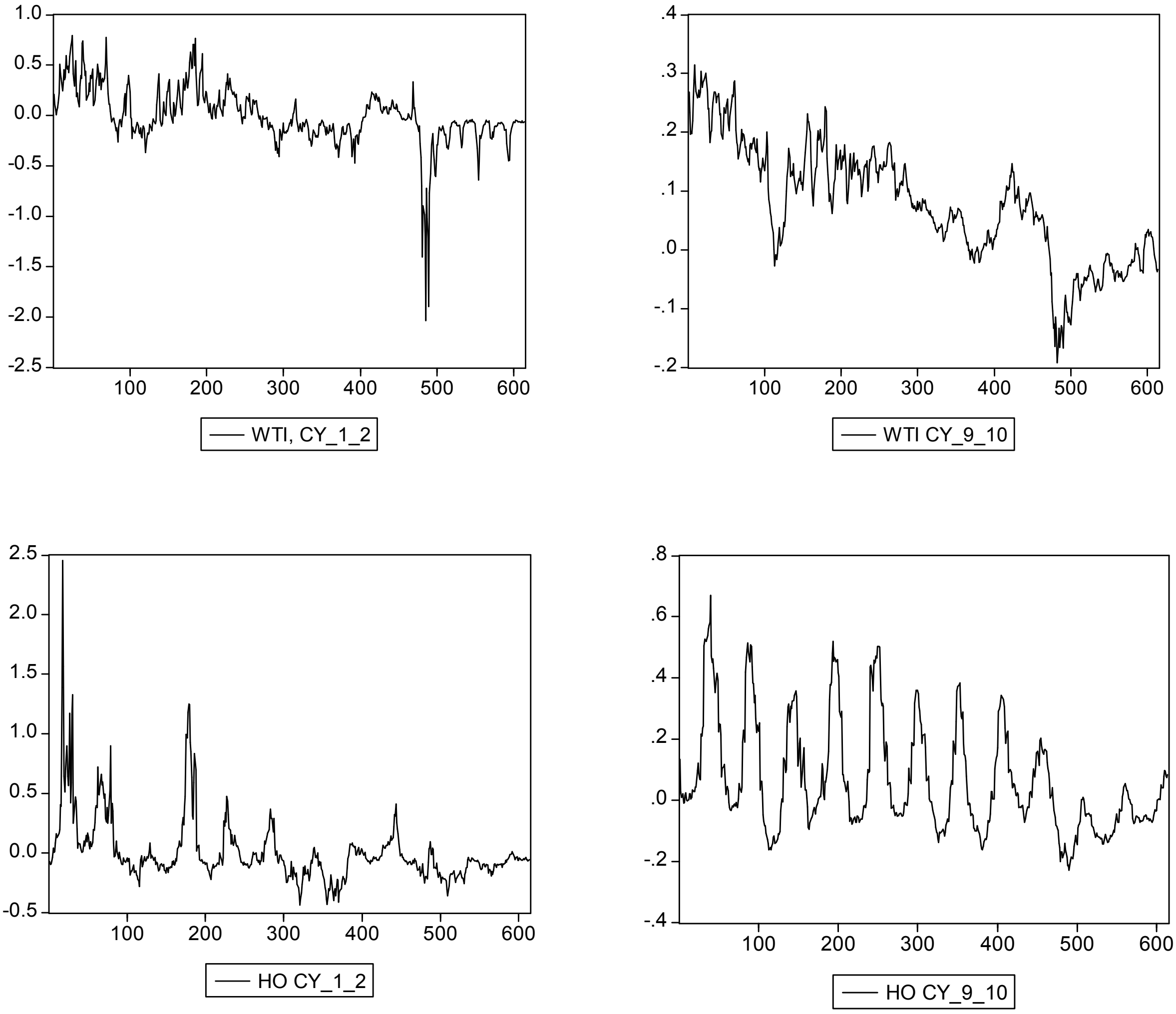


FIGURE 1

TIME SERIES EVOLUTION OF ESTMATED CONVENIENCE YIELDS (CONT.)

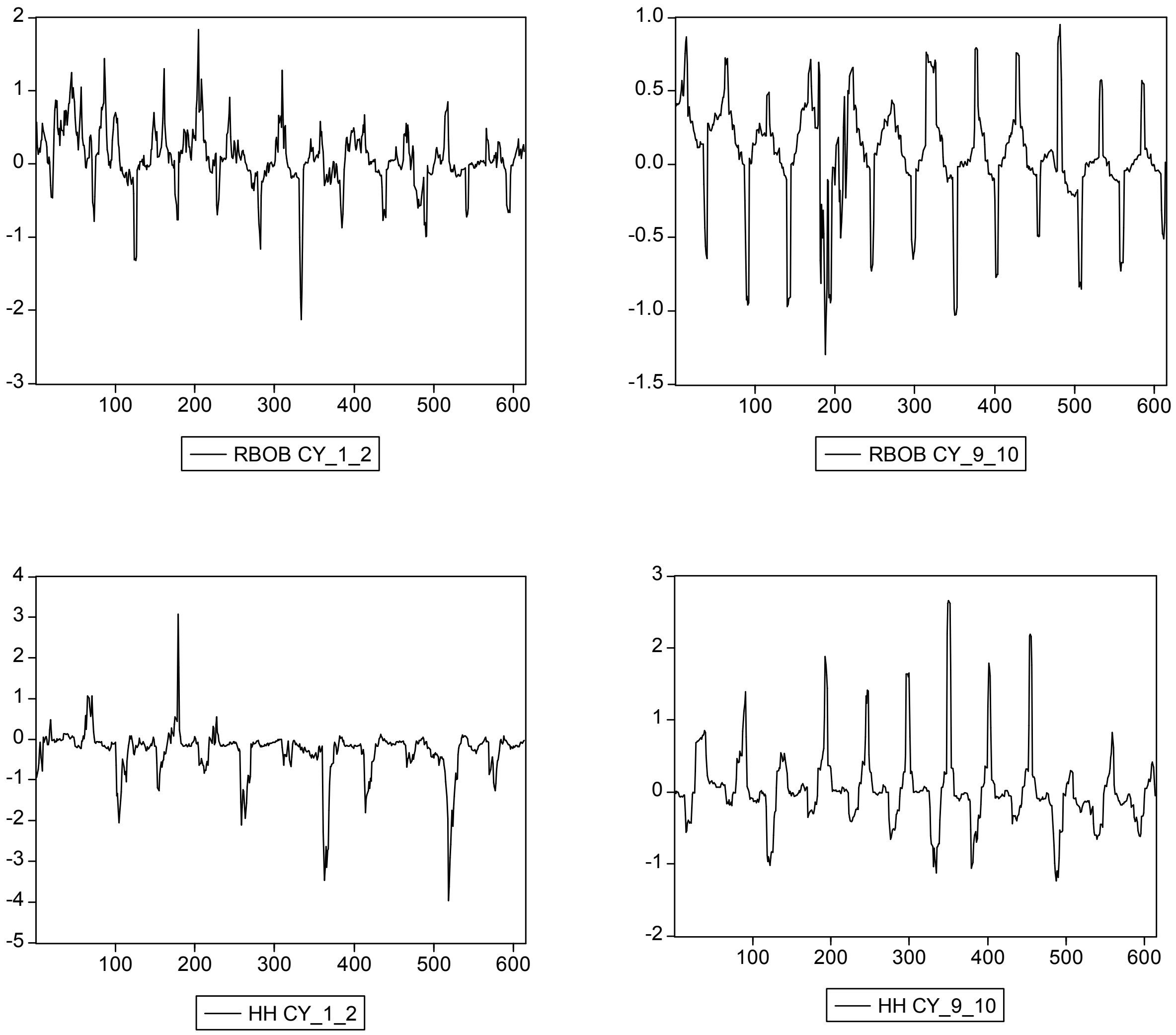
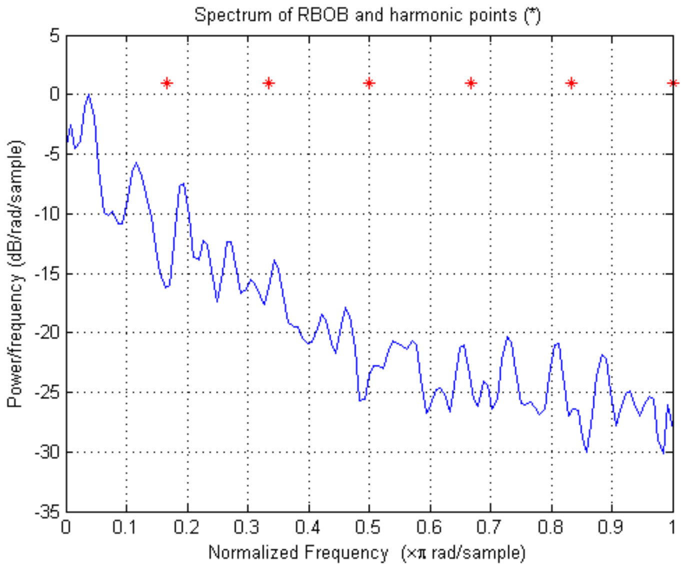


FIGURE 2

RBOBO GASOLINE CONVENIENCE YIELD SPECTRUM



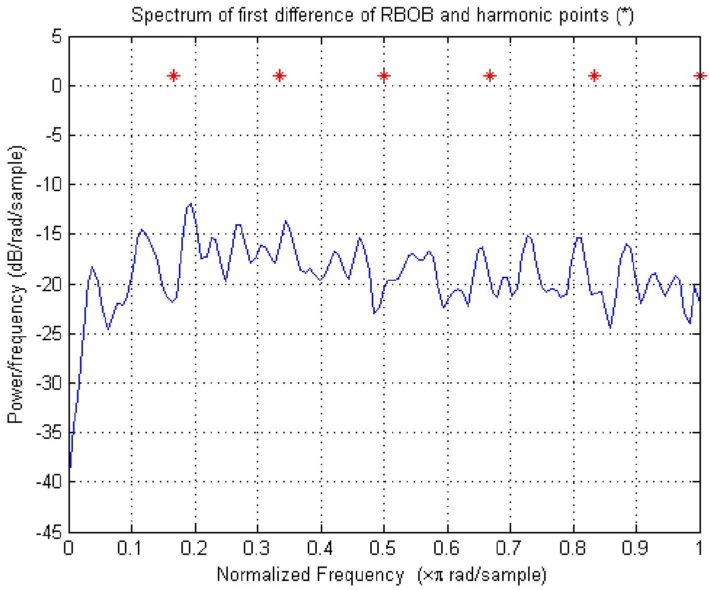


FIGURE 3

FORWARD CURVES FUTURES AND CONVENIENCE YIELD

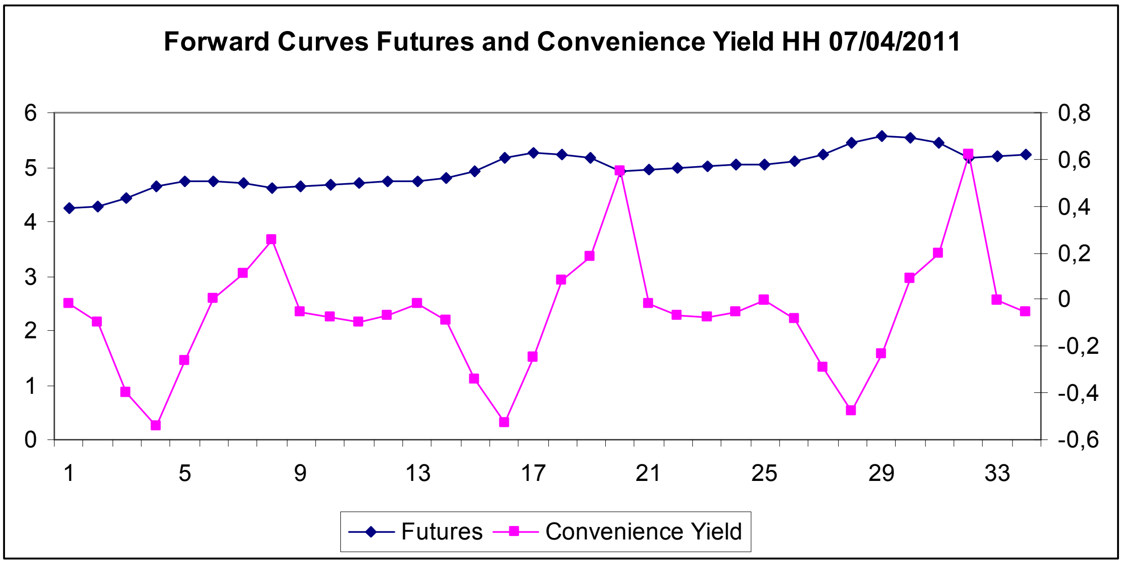


FIGURE 4

COMMODITY SEASONAL COMPONENTS AND CONVENIENCE YIELD