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Is there a need for more than three models?

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An important open issue in Functional Measurement is whether the three most important models of cognitive algebra are sufficient to describe the great majority of possible response behaviors. Generally speaking, the individual response R is a function of the subjective scale values s_k and can be imagined as a continuous manifold. First and second order terms of a Taylor series are often used to locally approximate the shape of such a generic function. In this work we suggest that almost any response surface can be approximated by an additive, multiplicative or averaging model, considering that the Taylor expansion is cut at the second order at most. In particular, additive and multiplicative models appear to hold as global approximations, while the averaging model appears to be a connection of local approximations.

INTRODUCTION

Although, from a general point of view, Functional Measurement allows for various and different algebraic rules to describe cognition, only three of them, additive, multiplicative and averaging, appear to be the most important and are successfully applied to cover a wide range of experimental results and findings (Anderson, 1981).

After a brief review of cognitive algebra, generalizing its rules to a continuous framework, the concept of a Taylor series is introduced with the aim of emphasizing how additive, multiplicative and averaging models can be seen as simple approximations of a generic Response surface.

The surface representing all the possible responses R to a set of n stimuli can indeed be seen as a manifold in an $(n+1)$-dimensional space (Box & Wilson, 1951). Since Taylor series is used to approximate the value of a function (or in this case a surface or a manifold), an interesting insight on the need of just three models to describe many psychological findings arises when one notices that additive, multiplicative, and averaging models can be used to describe, within a reasonable degree of approximation, a

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wide set of response surfaces. This also suggests that human responses to different stimuli can be characterized by smooth behavior.

COGNITIVE ALGEBRA

The most important models of cognitive algebra are the additive, the multiplicative and the averaging model (Anderson, 1981; 1982). Given for instance two different stimuli with associated subjective scale values s_1, s_2 and an integration function $r(s_1, s_2)$ they can be represented as:

Additive: $$r(s_1,s_2) = s_1 + s_2 \quad (1.1)$$

Multiplicative: $$r(s_1,s_2) = s_1 s_2 \quad (1.2)$$

Averaging: $$r(s_1,s_2) = \frac{w_1}{w_1 + w_2} s_1 + \frac{w_2}{w_1 + w_2} s_2 \quad (1.3)$$

In what follows the subjective scale values will be considered as continuous variables. This generalizes the treatise to any number of levels in any factor. In such a perspective, w_k represents the weight of a generic factor k that corresponds to the particular subjective value s_k. Using continuous variables and functions instead of discrete structures can be seen both as a generalization and as a practical interpolation of the discrete values usually attained by each subjective scale value or by the integration function in the different cells of the factorial design.

Assuming a linear implicit response scale (Anderson, 1981) and using a continuous representation, every response can be seen as a function $R(s_1, s_2)$ over the subjective scale values s_1, s_2; that is, a generic real valued function of two continuous variables:

$$R(s_1,s_2) = C_0 + C_1 r(s_1,s_2) \quad (2)$$

Additive, multiplicative, and averaging models of cognitive algebra, are just three out of an infinite possible number of shapes that a response function could attain. For instance the category of ratio models:

$$R(s_1,s_2) = \frac{s_1}{s_1 + s_2}$$

has appeared several times in decision theory, equity theory, speech perception and psycholinguistics (Anderson, 1981).

Furthermore, let us consider now a 3-dimensional space, where the dimensions are labeled by s_1, s_2 and R, so that the response is plotted against the subjective scale values. As in Response surface methodology (Box and Wilson, 1951), any equation like (2) describes a Response surface, namely a manifold representing the value of all the responses R for each pair (s_1, s_2).

See for an example figure 1, below.

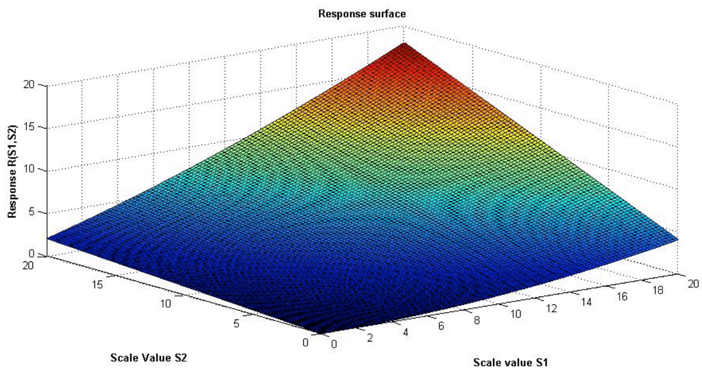

Figure 1: Response surface given by equation $R(s_1, s_2) = 0.01s_1^2+0.1s_2^2+0.03\, s_1s_2$.

Generally, then, a Response surface could be any given (and rather complex) shape of such a manifold. Cognitive algebra indeed accounts for the existence of other models than the three fundamental ones.

In addition, several models that have been found in psychological research are just generalizations involving the three fundamental operations of adding, multiplying, and averaging (Anderson, 1981), so that their interpretation as response surfaces is straightforward.

Finally, considering the generalization to multi-linear models, that extend to n the number of stimulus variables, from an analytical point of view it is represented by an $(n+1)$-dimensional manifold $R(s_1,.., s_n)$ that is the most general case of Response surface.

TAYLOR SERIES

The Taylor series is a very important and useful representation of a function in the neighborhood of a point of its dominion, by means of an infinite sum of elements. In particular, it is commonly used to approximate the value of a function using a partial sum of terms (often called the Taylor polynomial) and to compute its value numerically.

Given a function *f(x)*, of independent variable x, in a neighborhood of a particular point a of its dominion, it can be written as:

$$f(x) = \sum_{k=1}^{\infty} c_k (x-a)^k$$

that is a linear combination of increasing powers of the differences between x and a. The coefficients c_k are related to the k-derivatives of the function, evaluated in the point $x=a$ (see the Appendix for further details).

Often, a good approximation is to cut away the terms that are greater than the second order, thus obtaining the Taylor polynomial:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \varepsilon \tag{3}$$

that is a quadratic (or parabolic) approximation to the real value of the function. In the language of the Response surface methodology, this is a second order model (Box and Wilson, 1951). In such a case, the Taylor's theorem ensures that the remainder term ε, that is the approximation error given by the difference between the real value attained by the function and its Taylor polynomial, is negligible if compared to the size of $(x-a)^3$ when x approaches the value a. From a statistical point of view, rearranging and regrouping the terms, equation (3) can be seen as the linear regression:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon \tag{4}$$

In the case of a Response surface $R(s_1, s_2)$, it can be shown (see the Appendix) that the Taylor series can be closely related to a linear regression:

$$R(s_1, s_2) = \beta_0 + \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_1 s_2 + \varepsilon \tag{5}$$

Equation (5) is a second order approximation of a generic response function in the neighborhood of a particular choice of the scale values s_1 and s_2, when the quadratic terms (but not the interaction) are negligible. More in general, the Taylor series could be read as a linear regression model for which the regression coefficients indicates whenever a term is negligible.

Interesting results arise when considering, under this perspective, the relation of a generic Response surface to the models of cognitive algebra.

ADDITIVE MODEL

The additive model is simply a plane:

$$R(s_1, s_2) = \beta_0 + \beta_1 s_1 + \beta_2 s_2 + \varepsilon$$

Or from a graphical point of view, as in Figure 2.

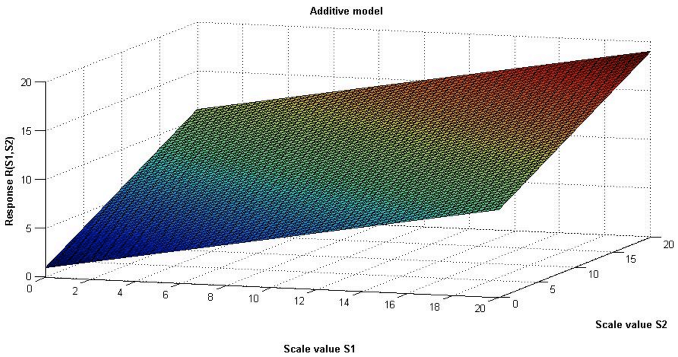

Figure 2: Response surface of an additive model.

Hence any Response surface *R(s1, s2)* that is quietly close to a plane, or that can be approximated in the entire dominion of the response function with a plane, can be described by means of an additive model, which in the language of response surface methodology corresponds to a first order model (Box and Wilson, 1951). Notice that it does not matter how much the surface is different from a plane, as long as its fluctuations can be contained within the approximation error ε. Moreover, the additive model is a linear global approximation, that is a first order approximation of a generic response function in the whole dominion.

MULTIPLICATIVE MODEL

Whenever a surface cannot be approximated by a plane since the quadratic terms or the interaction term are too strong, yet the surface is smooth enough to be globally approximated by means of a parabolic surface, we could expect a multiplicative model like:

$$R(s_1, s_2) = \beta_0 + \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_1 s_2 + \varepsilon$$

That accounts for both interactions and main effects, being a mixture between the additive (1.1) and the multiplicative (1.2) models. See Figure 3.

The classical shape of a multiplicative model (1.2) is the sub-case when $\beta_1 = \beta_2 = 0$, defined as (see Figure 4):

$$R(s_1, s_2) = \beta_0 + \beta_3 s_1 s_2 + \varepsilon$$

The multiplicative model is a global approximation like the additive one, and like the additive model holds only for response surfaces that do not fluctuate too much around this particular shape.

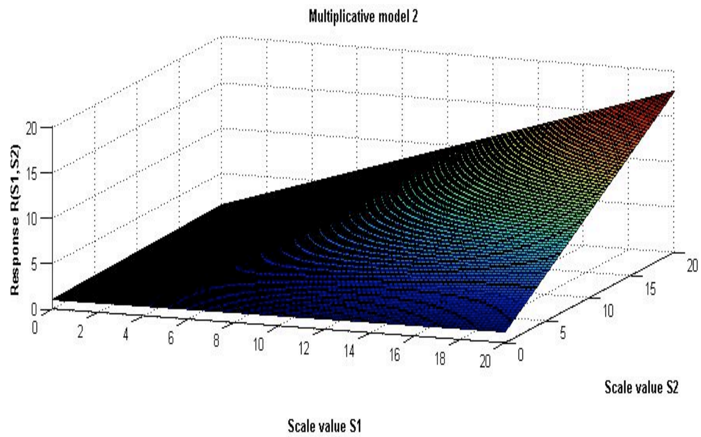

Figure 3: Response surface of a multiplicative model.

But, what happens if we consider a generic Response surface $R(s_1, s_2)$, that cannot be globally approximated with linear or quadratic terms?

Surely, we lose the concept of a global approximation, yet we could still describe the surface as a collection of local approximations given by Taylor polynomials. Like a patchworks of different functions approximating the real Response surface in the neighborhood of every pair of subjective scale values (s_1, s_2). In particular, since usually experimental factors possess a discrete number of levels, using different continuous functions like Taylor polynomials gives a local approximation in each cell of the model.

Interestingly the Averaging model behaves exactly like one of these collection of local approximations.

AVERAGING MODEL

The averaging model is a linear model like the additive one, but its slopes are bounded by the additional requirements that their sum has to be equal to one. This implies that an increase in the relative importance of a factor leads to a decrease in the relative importance of the other factor:

$$\begin{cases} R(s_1, s_2) = \beta_0 + \beta_1 s_1 + \beta_2 s_2 + \varepsilon \\ \beta_0 + \beta_1 + \beta_2 = 1 \end{cases}$$

Furthermore, the regression coefficients can be rewritten as:

$$\begin{cases} \beta_1 = \dfrac{w_1}{w_1 + w_2} \\ \beta_2 = \dfrac{w_2}{w_1 + w_2} \end{cases}$$

giving equation (1.3), where (w_1, w_2) are the absolute importance of each factor. Notice that β_0 has been omitted for simplicity of notation.

Since the previous approximation holds in each cell of the factorial design, the averaging model possesses different slopes in different cells and behaves like a local first order approximation. In particular, it approximates the response to every pair of stimuli that elicit the subjective scale values (s_1, s_2) with a plane. The Averaging model then is a collection of planes, with each plane being in the neighborhood of a different cell of the factorial design.

Hence, from an analytical point of view, the Averaging model is a local approximation of the Response surface, with the additional requirement that the slopes of the planes must sum to one. The latter from an analytical point of view is a very strong requirement that narrows the possible values of the slopes, but adds to the system an important psychological insight since it allows to define the absolute importance parameters (w_1, w_2), related to each subjective scale value, while keeping different magnitudes of the slopes in the cells (that is, different inclination of the planes to better approximate the real response surface). This introduces a connection between different cells that compensate on an interpretational level the absence of interaction due to the truncation of the Taylor series at the first order.

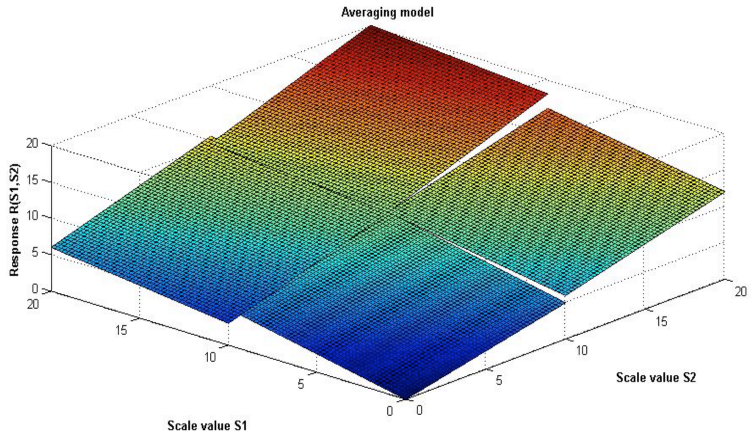

Figure 4: Example of Response surface for a 2x2 Averaging model.

Furthermore, the application of the methods of sub-design (Norman, 1976; Anderson, 1982), can be used to identify the regression coefficients β_k, representing the relative importance of the factors, and thus the weights w_k, representing the absolute importance, by means of a multivariate linear regression when the one-way sub designs are treated as linear models to evaluate the subjective scale values s_k (Noventa et al., 2010).

CONCLUSIONS AND DISCUSSION

Additive, multiplicative, and averaging are considered to be the most important and successfully applied models in the cognitive algebra of Functional Measurement, since they appear to cover a very wide range of experimental results and psychological findings.

In our opinion, this happens because human responses over a rating scale show a certain smooth behavior, so that their Response surface can be approximated, with maximally a second order Taylor polynomial.

In particular, additive and multiplicative models seem to describe all those Response surfaces that show a long range order (their fluctuations around a linear or a quadratic approximation are not excessive) so that they can be approximated in the whole dominion by a first or a second order model (corresponding to the presence of main effects and interactions).

Averaging model, instead, appears to be a collection of local linear approximations in every cell of the design (that is, in every neighborhood of every pair of scale values) that replaces a generic Response surface with a collection of planes. In spite of the loss of terms, that implies a loss in the mathematical descriptive precision and details of the model, it provides full psychological insight by defining absolute importance weights and requiring that the relative importance of the subjective scale values sum to one, thus connecting the different planes of approximation, each one to the other.

Obviously, all these considerations imply the existence of infinite other possible models out of the three fundamental ones. Yet, considering how much psychological raw data can be coarse grained, and how smooth human response behavior seems to be, first and second order approximation may suffice to describe the great part of experimental findings. Hence, the idea of approximating a generic Response surface with just a collection of adjacent planes sounds sensible if we are willing to give up a perfect analytical description in behalf of a more simple and good psychological insight.

APPENDIX

To be defined a Taylor series:

$$f(x) = \sum_{k=1}^{\infty} c_k (x-a)^k$$

requires a function to be C^{∞} over its dominion: that is a function must possesses all of its derivatives so that the c_k coefficients are defined as:

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{k!} \left. \frac{d^k f(x)}{dx^k} \right|_{x=a}$$

If the series is truncated now at a term k, the Taylor's theorem ensure that the remainder, that is the approximation error given by the difference between the real value attained by the function and its Taylor polynomial is a little-o of the third order, namely:

$$\varepsilon_k = o(|x-a|^k), \qquad x \to a$$

that means:

$$\lim_{x \to a} \frac{\varepsilon_k}{(x-a)^k} = 0$$

In the case of more than two variables (or factors), the Taylor series of a function *f(x)* becomes:

$$f(x_1,...,x_n) = \sum_{k_1=0}^{\infty} ... \sum_{k_n=0}^{\infty} \frac{(x_1 - a_1)^{k_1} ... (x_n - a_n)^{k_n}}{k_1!...k_n!} \left(\frac{\partial^{k_1+...+k_n} f(a_1,...,a_n)}{\partial^{k_1} x_1 ... \partial^{k_n} x_n} \right)$$

involving also the partial derivatives of the function respect to any variable. In the case of multi-linear model, describing an *(n+1)*-dimensional manifold or Response Surface, then:

$$R(s_1,...,s_n) = \sum_{k_1=0}^{\infty} ... \sum_{k_n=0}^{\infty} \frac{(s_1 - a_1)^{k_1} ... (s_n - a_n)^{k_n}}{k_1!...k_n!} \left(\frac{\partial^{k_1+...+k_n} R(a_1,...,a_n)}{\partial^{k_1} s_1 ... \partial^{k_n} s_n} \right)$$

That, for two variables, becomes:

$$R(s_1, s_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(s_1 - a_1)^{k_1} (s_2 - a_2)^{k_2}}{k_1! k_2!} \left(\frac{\partial^{k_1+k_2} R(a_1, a_2)}{\partial^{k_1} s_1 \partial^{k_2} s_2} \right)$$

The previous equation, rewritten by considering the first and second order terms, but cutting the quadratic terms except for the interaction one, gives:

$$R(s_1, s_2) \approx R(a_1, a_2) + \frac{\partial R(a_1, a_2)}{\partial s_1}(s_1 - a_1) + \frac{\partial R(a_1, a_2)}{\partial s_2}(s_2 - a_2) + ...$$

$$... + \frac{\partial^2 R(a_1, a_2)}{\partial s_1 \partial s_2}(s_1 - a_1)(s_2 - a_2)$$

That rearranged is equation (5).

REFERENCES

Anderson, N. H. (1981). *Foundations of Information Integration Theory*. New York: Academic Press.

Anderson, N. H. (1982). *Methods of Information Integration Theory*. New York: Academic Press.

Box, G. E. P. and Wilson, K. B. (1951). On the Experimental Attainment of Optimum Conditions (with discussion). *Journal of the Royal Statistical Society Series B* 13(1):1–45

Norman, K. L. (1976). A solution for weights and scale values in functional measurement. *Psychological Review*, 83 (1), 80-84.

Noventa, S., Massidda, D., and Vidotto, G. (2010) Initial conditions in the averaging cognitive model. *Psicològica*, 31, 477-489.

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