Tachyon correlators in $c_M < 1$ non-rational Liouville gravity

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Two approaches to $c_M < 1$ non-critical string - compared -
world-sheet CFT and matrix model (**target space** CFT)

I. Continuum - conformal gauge $g_{ab} = e^{2b\phi}\,\hat{g}_{ab}\,, \quad \hat{g}_{ab} = \delta_{ab}$ locally

effective action - 2d CFT: $c_M < 1$ - **"matter"** , $c_L > 25$ - **Liouville**

$$\mathcal{A}^{\text{free}} = \frac{1}{4\pi}\int d^2x\left[(\partial_a\chi)^2 + (\partial_a\phi)^2 + (Q\phi + ie_0\chi)\sqrt{\hat{g}}\hat{R}\right] + \mathcal{A}^{\text{ghosts}}(\mathbf{b},\mathbf{c})$$

background charges $\quad Q = b + \frac{1}{b}\,, \quad e_0 = \frac{1}{b} - b, \quad b$ - **real**

vanishing conformal anomaly $\quad c^{\text{tot}} = c_M + c_L + c_{ghosts} = 0$

interaction - matter and Liouville **screening charges** -

$$\mathcal{A}_{\text{int}} = \int \left(\mu_{_L} e^{2b\phi} + \mu_{_M} e^{-2ib\chi}\right) = \lambda_{_L}\, T^{+}_{b} + \lambda_{_M}\, T^{+}_{0}\,,$$

or/and

$$\tilde{\mathcal{A}}_{\text{int}} = \int \left(\tilde{\mu}_{_L} e^{\frac{2}{b}\phi} + \tilde{\mu}_{_M} e^{\frac{2}{b}i\chi}\right) = \tilde{\lambda}_{_L}\, T^{-}_{1/b} + \tilde{\lambda}_{_M}\, T^{-}_{0}\,,$$

$$\lambda_{_L} = \pi\gamma(b^2)\,\mu_L,\;\; \lambda_{_M} = \pi\gamma(-b^2)\,\mu_{_M}\,, \quad \tilde{\lambda}_{_L} = \lambda_{_L}{}^{1/b^2}\,, \quad \tilde{\lambda}_{_M} = \lambda_{_M}{}^{-1/b^2}$$

- examples of "massless tachyons" - BRST -invariant operators of ghost number 1 associated with vertex operators of dim $(1,1)$,

$$V^{\epsilon}_{\alpha} \sim \; e^{2ie\chi}\, e^{2\alpha\phi}, \qquad \triangle_M(e) + \triangle_L(\alpha) = 1\,,$$

"mass-shell" condition parametrized by the target space **momentum** P

$$\alpha = \tfrac{1}{2}(Q - \varepsilon P) = \varepsilon\, e + b^{\varepsilon}\,,$$

and **chirality** $\varepsilon = \pm 1$,

- **integrated (1,1)-forms** $\qquad T^{\pm}_{\alpha} \equiv \int d^2x V^{\pm}_{\alpha}$

- Q_{BRST}**-closed (0,0)-forms**: $\qquad W^{\pm}_{\alpha} \equiv \mathbf{c}\bar{\mathbf{c}}\; V^{\pm}_{\alpha}$

Problem: tachyon correlation "functions" (numbers) on the sphere

$$G^{(\epsilon)}_n = \left\langle\, W^{\varepsilon_1}_{\alpha_1} W^{\varepsilon_2}_{\alpha_2} W^{\varepsilon_3}_{\alpha_3} T^{\varepsilon_4}_{\alpha_4} \cdots T^{\varepsilon_n}_{\alpha_n} \right\rangle$$

$$= \int \cdots \int \langle \quad \rangle_M \, \langle \quad \rangle_L$$

$$= \lambda_{_L}{}^{\frac{1}{b}(Q - \sum_i \alpha_i)}\, \lambda_{_M}{}^{\frac{1}{b}(\sum_i e_i - e_0)}\; \mathcal{G}^{(\epsilon)}_n(P_1, P_2, \ldots P_n)$$

• **Ground ring** – OPE of ghost number 0, dim 0 physical operators [*Witten '91*]

$$\mathcal{O}_i\,\mathcal{O}_j = n_{ij}^k\ \mathcal{O}_k$$

mod $\mathrm{Q_{BRST}}$ – exact terms

$$\mathcal{O}_i\,W_\alpha = c_{i\alpha}^\beta\ W_\beta$$

tachyons - modules of the ring,

• (free field) **generators of the ground ring** $a_\pm(x) = a_\pm(z)\,a_\pm(\bar z)\,,$

$$a_-(z) = \ :\Big(\mathbf{bc} - \tfrac{1}{b}\,\partial_z(\phi + i\chi)\Big)\,e^{-b(\phi - i\chi)}:$$

$$a_+(z) = \ :(\mathbf{bc} - b\,\partial_z(\phi - i\chi))\,e^{-\frac{1}{b}(\phi + i\chi)}:$$

both **matter and Liouville degenerate** vertex operators (unlike the tachyons)

• **Functional relations** for the tachyon correlators (on the sphere)

[*'91- Kutasov-Martinec-Seiberg,*]

[*Bershadsky- Kutasov*] , [*'03- Seiberg-Shih, Kostov*]

[*'91 Di Francesco-Kutasov*]

old work - **trivial matter** - now OPE deformed by Liouville and **matter** interactions

• 3-point tachyon correlator $G_3(\alpha_1\,,\,\alpha_2\,,\,\alpha_3) = N(\alpha_1,\alpha_2,\alpha_3)\lambda_{_L}{}^{\frac{1}{b}(Q-\sum_i \alpha_i)}\,\lambda_{_M}{}^{\frac{1}{b}(\sum_i e_i - e_0)}$

$$\sum_{\pm} N(\alpha_1 \pm \tfrac{b^\rho}{2},\alpha_2,\alpha_3) = \sum_{\pm} N(\alpha_1,\alpha_2 \pm \tfrac{b^\rho}{2},\alpha_3)\,, \qquad \rho = \pm 1$$

• simplest solution - for generic momenta - $N(\alpha_1,\alpha_2,\alpha_3) = 1$,
obtained also from 3-point matter -Liouville factorization

$$\left\langle\, W_{\alpha_1}^{\varepsilon_1}\, W_{\alpha_2}^{\varepsilon_2}\, W_{\alpha_3}^{\varepsilon_3} \,\right\rangle = \frac{C^{\mathrm{Liou}}(\alpha_1,\alpha_2,\alpha_3) C^{\mathrm{Matt}}(e_1,e_2,e_3)}{\prod_{j=1}^3 b^{-\epsilon_j}\, \gamma(\alpha_j^2 - e_j^2)}$$

with the generic matter constant $C^{\mathrm{Matt}}(e_1,e_2,e_3)$ computed similarly as the DOZZ Liouville constant

• **non-trivial solutions** for momenta corresponding to degenerate matter
$P_i = e_0 - 2e_i = n_i/b - m_i b$, $n_i, m_i \in \mathbb{N}$ (or Liouville) reps

$$N(P_1,P_2,P_3) = N_{m_1,m_2,m_3}\, N_{n_1,n_2,n_3}$$

tensor-product decomposition multiplicities of irreps of $su(2)$ of dimensions m_k - **ground ring – the representation ring of** $su(2) \times su(2)$

• $n \geq 4-$ **point functs** - free field OPE further deformed by the tachyons $T^{\epsilon}_{\alpha} = \int V^{\epsilon}_{\alpha}$ in the correlator serving as "screening charges"

$$a_- \; W^+_{\alpha_2} \; T^+_{\alpha_3} \, (T^-_0 T^-_{1/b})^k \sim W^+_{\alpha_2+\alpha_3-\frac{b}{2}+\frac{n}{b}} \,, \qquad k = 0, 1, \ldots$$

OPE coeffs $C^{(\epsilon)}_{-b/2\,\alpha_2\,\alpha_3}{}^{\alpha}$ - 4-point functions computed by Coulomb gas

" **inhomogeneous associativity eqs"** - string analog of the locality eqs

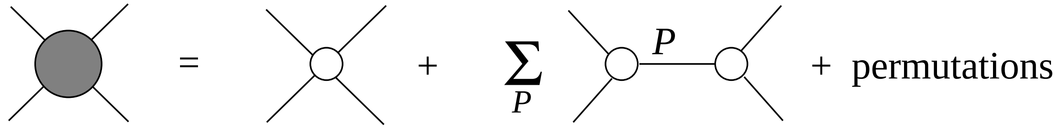
$$\sum_{\pm} G^{(\epsilon)}_4(\alpha_1 \pm b/2, \alpha_2, \alpha_3, \alpha_4) + \sum_{\alpha} C^{(\epsilon)}_{-b/2\,\alpha_1\alpha_3}{}^{\alpha} \; G^{(\epsilon)}_3(\alpha, \alpha_2, \alpha_4)$$
$$= \sum_{\pm} G^{(\epsilon)}_4(\alpha_1, \alpha_2 \pm b/2, \alpha_3, \alpha_4) + \sum_{\alpha} C^{(\epsilon)}_{-b/2\,\alpha_2\alpha_3}{}^{\alpha} \; G^{(\epsilon)}_3(\alpha_1, \alpha, \alpha_4)$$

finite number of **contact terms** if

- 1. one of the fields corresponds to a **degenerate Vir representation**, or

- 2. the four momenta are restricted by a (matter) **charge conservation condition** $\quad \sum_{i=1}^{4} P_i = 2e_0 - 2mb + \frac{2n}{b} \,, \quad m, n \in \mathbf{Z}_{\geq 0} \,,$

• General form of the **solutions**



• Ground ring relations derived for the **fixed chirality** $\{\epsilon_i\,, i = 1, 2, 3, 4\}$ **correlators** - unphysical set - partially symmetric

"locality" requirement - symmetrised correlators, consistent with the matter fusion rules, formally P replaced by $|P|$

$$\sum_P \; N_{P_1,P_2,P} \; |P| \; N_{P,P_3,P_4}$$

• the local correlators satisfy complicated difference eqs, depending on the momenta

• for the case of (one) degenerate field - different, more constructive method [A. Belavin, Al. Zamolodchikov]

II. Microscopic, discrete realisation of 2d gravity
- generalization of rational ADE string models [*Kostov '91*]

- target space - graph, $x = 1, 2, 3, \ldots$

A_n $\bullet\bullet\bullet\bullet\ldots\bullet\bullet$ $\rightarrow A_\infty$ **SOS (height) model on fluctuating lattice**

loop gas expansion ,

$\rightarrow$ dual formulation as a **matrix chain model** M_x, $N \rightarrow \infty$ (scaling) limit

$\rightarrow$ collective field formulation in terms of a $c = 1$ **chiral field** $\Psi_x(z)$ **on a Riemann surface** - infinite branch cover of the spectral plane

operator solution of Virasoro constraints - **finite diagram technique** for evaluation of n - loop amplitudes $\rightarrow$ local field correlators

• However different interpretation of the **matter interaction**
- only "order operators" $P = \pm m(1/b - b)$, closed under fusion

no underlying "pure matter" CFT

$\Rightarrow$ introduce unconventional "**diagonal**" perturbation of Liouville gravity

matter screening charges replaced by tachyons of matter charge $e_0 = 1/b - b$

$$e^{-2bi\chi},\ e^{\frac{2}{b}i\chi} \ \rightarrow \ e^{2e_0 i\chi + 2b\phi},\ e^{2e_0 i\chi + \frac{2}{b}\phi}$$

adapt the ground ring structure:

perturbed ground ring operator $A = a_- a_+$ projects to shifts of momenta by $\pm e_0$

• 4-point tachyon correlators coincide with the expressions obtained by the Feynman diagram technique of the matrix model