Gluon scattering amplitudes at strong coupling

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Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

- It can give non trivial information about more realistic theories but is more tractable. In the last years, many tools become available.
- The strong coupling regime can be study, by means of the gauge/string duality, through a weakly coupled string sigma model.
- Higher loop (MHV) amplitudes appear to have a remarkable iterative structure, leading to a proposal for all loops n-point amplitudes.

Aim of this talk

Prescription for computing gluon scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills at strong coupling by using the AdS/CFT correspondence.

Gauge theory results

Bern, Dixon, Smirnov, (Anastasiou, Carrasco, Johansson, Kosover, Roiban,...)

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}}...T^{a_{\rho(n)}})A_n^{(L)}(\rho(1),...,\rho(2))$$

- Leading N_c color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- The amplitudes are divergent so we need to introduce a regulator.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon)$
- Scale out the tree amplitude $M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$. Up to few loops, $M_n^{(L)}(\epsilon)$ can be written in terms of lower order amplitudes!

$$M_4^{(2)}(\epsilon) = \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_4^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)$$

Motivated by the infrared behavior (plus additional studies) of multi-loop amplitudes...

MHV amplitudes: all loops proposal!

$$\mathcal{M}_n \equiv 1 + \sum_{L=1} \alpha^L M_n^{(L)}(\epsilon) = exp\left[\sum_{\ell=1}^{\infty} \alpha^\ell \left(f^\ell(\epsilon) M_n^{(1)}(\ell\epsilon) + C^{(\ell)} + \mathcal{O}(\epsilon) \right)\right]$$

$$\alpha \sim \frac{\lambda \mu^{2\epsilon}}{8\pi^2}, \qquad f^\ell(\epsilon) = f_0^\ell + \epsilon f_1^\ell + \epsilon^2 f_2^\ell$$

We will perform explicit computations for $n = 4$.

4 point amplitude

$$\mathcal{A} = \mathcal{A}_{tree} \left(\mathcal{A}_{div,s}\right)^2 \left(\mathcal{A}_{div,t}\right)^2 \exp\left\{ \frac{f(\lambda)}{8} (\ln s/t)^2 + const \right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{ -\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) - \frac{1}{4\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) \right\}$$

- Divergent piece plus dynamical part of finite term are characterized by two functions.
- $\left(\lambda \frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda)$: Cusp anomalous dimension, controls leading divergence.
- $\lambda \frac{d}{d\lambda} g^{(-1)}(\lambda) = g(\lambda)$: Subleading divergence.

AdS/CFT duality

Consider a stack of $D3$-branes in type IIB string theory. Two equivalent descriptions of the same system.

- Low energy theory (quantum field theory) for the degrees of freedom of the branes.
- String theory on a curved background, induced by the matter density of the branes.

AdS/CFT duality (Maldacena)

Four dimensional maximally SUSY Yang-Mills $\Leftrightarrow$ Type IIB string theory on $AdS_5 \times S^5$.

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'}$$

$$\Downarrow$$

We can study a strongly coupled gauge theory by means of a weakly coupled sigma model

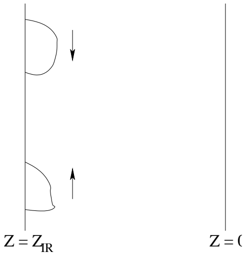
We will study scattering amplitudes at strong coupling by using the AdS/CFT duality.

- Set up the computation: Use a $D - brane$ as IR cut-off.
- Actual computations: Dimensional regularization.

String theory set up

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$

- Place a D-brane extended along x_{3+1} and located at some large z_{IR}.



- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings.
- The proper momentum of these strings, $k_{pr} = k\frac{z_{IR}}{R}$ is very large, so we are interested in the regime of fixed angle and very high momentum.

This regime was considered in flat space (Gross and Mende)

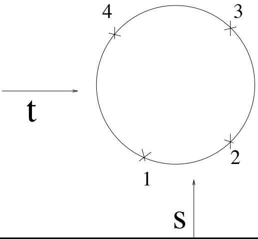
Key feature

The amplitude is dominated by a saddle point of the classical action.

$\Downarrow$

We need to consider a classical string on AdS

World-sheet with the topology of a disk with vertex operator insertions (corresponding to external states)



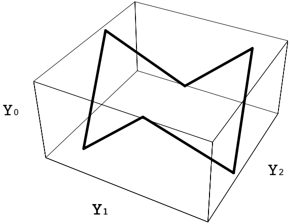
- Near each vertex operator, the momentum of the external state fixes the form of the solution.
- In the boundary of the world-sheet $z = z_{IR}$

- "T-duality": $ds^2 = w^2(z)dx_\mu dx^\mu \rightarrow \partial_\alpha y^\mu = i w^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu$
- Boundary conditions: x^μ carries momentum $k^\mu \rightarrow y^\mu$ has winding $\Delta y^\mu = 2\pi k^\mu$.
- After a change of coordinates $r = R^2/z$ we end up again with AdS_5

$$ds^2 = R^2 \frac{dy_{3+1}^2 + dr^2}{r^2}$$

World-sheet whose boundary is located at $r = R^2/z_{IR}$ and is a particular line constructed as follows...

- For each particle with momentum k^μ draw a segment joining two points separated by $\Delta y^\mu = 2\pi k^\mu$



- Concatenate the segments according to the ordering of the insertions on the disk (particular color ordering)
- Momentum conservation: Closed diagram.

- As $z_{IR} \rightarrow \infty$ the boundary of the world-sheet moves to $r = 0$.
- Exactly same computation as when computing the expectation value of a Wilson-Loop given by a sequence of light-like segments!

Prescription

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

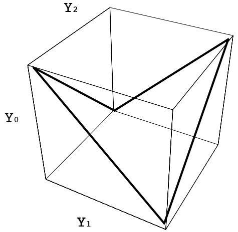
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- Prefactors are subleading in $1/\sqrt{\lambda}$, and we don't compute them.
- In particular our computation is blind to helicity, etc.

Four point amplitude at strong coupling

Consider $k_1 + k_3 \to k_2 + k_4$

- The simplest case $s = t$.



Need to find the minimal surface ending on such sequence of light-like segments

$$
\begin{aligned}
r(y_1, y_2) &= \sqrt{(1-y_1^2)(1-y_2^2)} \\
y_0 &= y_1 y_2
\end{aligned}
$$

- (T-dual) conformal symmetry takes this solution to the most general one.

Let's compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Theory in $D = 4 - 2\epsilon$ dimensions but with 16 supercharges.
- For integer D this is exactly the low energy theory living on $Dp-$branes $(p = D - 1)$

Gravity dual

$$ds^2 = h^{-1/2} dx_D^2 + h^{1/2}\left(dr^2 + r^2 d\Omega_{9-D}^2\right), \qquad h = \frac{c_D \lambda_D}{r^{8-D}}$$

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^{\epsilon}} \qquad c_D = 2^{4\epsilon}\pi^{3\epsilon}\Gamma(2+\epsilon)$$

T-dual coordinates

$$ds^2 = \sqrt{\lambda_D c_D}\left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}}\right) \to S_\epsilon = \frac{\sqrt{\lambda_D c_D}}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- Presence of ϵ will make the integrals convergent.
- The eoms will depend on ϵ but if we plug the original solution into the new action, the answer is accurate enough.
- plugging everything into the action...

$$iS = -\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon}\left(\frac{\pi\Gamma\left[-\frac{\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]}\,{}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2},\frac{1-\epsilon}{2};b^2\right)+1\right)+\mathcal{O}(\epsilon)$$

- Just expand in powers of ϵ...

Final answer

$$\mathcal{A} = e^{iS} = \exp\left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C}\right]$$

$$S_{div} = 2S_{div,s} + 2S_{div,t}$$

$$S_{div,s} = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1-\log 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}}$$

- Should be compared to the field theory answer

$$\mathcal{A} \sim \left(\mathcal{A}_{div,s}\right)^2\left(\mathcal{A}_{div,t}\right)^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + const\right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2}f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon}g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\}$$

Wilson loops vs. Scattering amplitudes

- This computation shows a relation between Wilson loops and scattering amplitudes.
- This relation holds also at weak coupling!

Write BDS on a slightly different way

$$\log \mathcal{M}_n = Div_n + \frac{f(\lambda)}{4} a_1(k_1, k_2, ..., k_n) + h(\lambda) + nk(\lambda)$$

Scattering amplitudes from WL (Drummond, Korchemsky, Sokatchev, Brandhuber,...)

$$< W_{k_i} >= 1 + \lambda \left(Div + w_1(k_1, ..., k_n) + c + n\tilde{c} \right)$$

$$\Downarrow$$

$$w_1(k_1, ..., k_n) = a_1(k_1, ..., k_n)$$

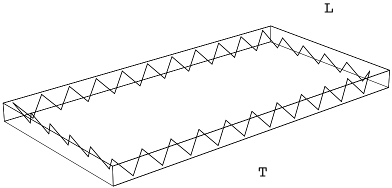
- BDS $\Rightarrow$ $a_{strong} = f^{strong} a_1(k_1, ..., k_n)$
- WL vs. Amplitudes at strong coupling $\Rightarrow a_{strong} = w_{strong}$
- WL vs. Amplitudes at strong coupling $\Rightarrow a_1 = w_1$

$$\Downarrow$$

$$w_{strong} = f^{strong} w_1(k_1, ..., k_n)$$

- For $n = 4$ and $n = 5$ that is the case! but fixed by symmetries.
- We need to take $n > 5$, what about $n = \infty$?

We choose a zig-zag configuration that approximates the rectangular Wilson loop.



- $\log < W_{rect}^{weak} >= \frac{\lambda}{8\pi}\frac{T}{L}$
- $\log < W_{rect}^{strong} >= \sqrt{\lambda}\frac{4\pi^2\sqrt{2}}{\Gamma(\frac{1}{4})^4}\frac{T}{L}$

- The strong coupling result is not what we would expect from the BDS ansatz, hence something needs to be revised...

What have we done?

- A prescription for computing planar scattering amplitudes on $\mathcal{N}=4$ SYM at strong coupling by using the AdS/CFT duality.
- We have done detailed computations for $n=4$ but the prescription is valid for any number of gluons.
- Our results agree in all detail with the conjecture of Bern, Dixon and Smirnov for $n=4$, but the conjecture may need to be revised for large number of gluons.
- A small step towards understanding the iterative structures for gluon amplitudes from the string theory point of view.

What things need to be done?

- Try to make explicit computations for $n > 4$.
- We haven't assume/use at all the machinery of integrability.
- Subleading corrections in $1/\sqrt{\lambda}$? Information about helicity of the particles, etc. (see Abel, Forste, Khose; Kruczenski, Roiban, Tirziu, Tseytlin)
- Gross and Mende computed higher genus amplitudes (in flat space) using similar ideas, can we do the same?
- Can we repeat the computation in other backgrounds?
- Deeper relation among Wilson loops and scattering amplitudes?
- Some powerful alternative to BDS? maybe using the non-abelian exponentiation theorem...