The holomorphic anomaly for open string moduli

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Based on G.B. and A.Tanzini, arXiv:0708.2627

Topological Open Strings are interesting for:

- String theorists who like all genera exact results

- Math.-Phys. people: those like to calculate string amplitudes at all genera, count cycles, topological invariants and other stuffs

- String Pheno. people: these like to get gauginos masses, Yukawa couplings and other stuffs upon superstring CY compactification

The last two items are connected via the topological twist of the superstring

$$[\mathrm{F-terms}] \quad \Longleftrightarrow \quad [\mathrm{topological} \quad \mathrm{obs.}]$$

Holomorphic anomaly equations (HAE)

- BCOV (hep-th/9309140) for closed strings.

$$\partial_{\bar{t}_{\bar{i}}} F_g = \frac{1}{2} C_{\bar{i}}^{jk} \left[\sum_{g_1+g_2=g} D_j F_{g_1} D_k F_{g_2} + D_j D_k F_{g-1} \right]$$

- for open strings and frozen open moduli (see Walcher arXiv:0705.4098)

$$\partial_{\bar{t}_{\bar{i}}} F_{g,h} = \frac{1}{2} C_{\bar{i}}^{jk} \left[\sum_{\substack{g_1+g_2=g \\ h_1+h_2=h}} D_j F_{g_1,h_1} D_k F_{g_2,h_2} + D_j D_k F_{g-1,h} \right] - \Delta_{\bar{i}}^{j} D_j F_{g,h-1}$$

where $\Delta_{\bar{i}}^{j} = g^{j\bar{j}} \int_0^1 dr \langle \phi_{\bar{i}}(0) \phi_{\bar{j}}^{[1]}(r) \rangle_{\Sigma_{0,1}}$

New issue : switch on open moduli in HAE's

Abstract of the talk

- Main result:Complete the holomorphic anomaly equations for topological strings with their dependence on open moduli.

- How: by standard path integral arguments generalizing the analysis of BCOV to strings with boundaries and open moduli.

- In particular: study anti-holomorphic dependence of string partition functions on open moduli and on closed moduli in presence of Wilson lines.

- Math. spin-off: compactification à la Deligne-Mumford of the moduli space of Riemann surfaces with boundaries. Actually: the open holomorphic anomaly equations are structured on the (real codimension one) boundary components of this space.

Anti-holomorphic dependence on open moduli was already noticed:

- I. Antoniadis, K. S. Narain and T. R. Taylor, "Open string topological amplitudes and gaugino masses,"

- D. Cremades, L. E. Ibanez and F. Marchesano, "Computing Yukawa couplings from magnetized extra dimensions,"

- M. Marino, "Open string amplitudes and large order behavior in topological string theory,"

- R. Russo and S. Sciuto, "The twisted open string partition function and Yukawa couplings,"

(later)

- M. Billo et al., "Instantons in N=2 magnetized D-brane worlds," "Instanton effects in N=1 brane models and the Kahler metric of twisted matter"

Boundary marginal deformations

$$F_{g,h} = \int_{\bar{\mathcal{M}}_{g,h}} \langle \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}}$$

$$\delta S_B = \bar{Q}_+ \bar{Q}_- \int_{\Sigma_{g,h}} \delta t^{\bar{i}} \phi_{\bar{i}} + \int_{\Sigma_{g,h}} Q_+ Q_- \delta t^i \phi_i +$$

$$+ Q \oint_{\partial\Sigma_{g,h}} \left(\delta t^{\bar{\alpha}} \Theta_{\bar{\alpha}} + \delta t^{\bar{i}} \Psi_{\bar{i}} \right) + \oint_{\partial\Sigma_{g,h}} \bar{Q} \left(\delta t^{\alpha} \Theta_{\alpha} + \delta t^i \Psi_i \right)$$

Open moduli span fibers over closed moduli

For the B-model for example:

$$\phi_{\bar{i}} = (w_{\bar{i}})_{IJ} (X) \rho_z^I \rho_{\bar{z}}^J, \qquad \phi_i = (\bar{w}_i)_{\bar{I}\bar{J}} (X) \eta^{\bar{I}} \theta^{\bar{J}},$$

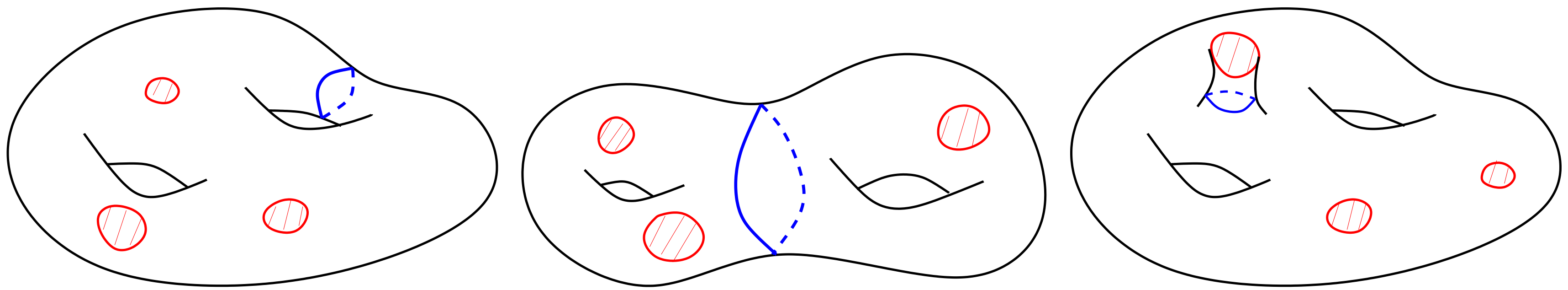
$$\Theta_{\bar{\alpha}} = \left(\delta A_{\bar{\alpha}}^{(1,0)} \right)_I (X) \left(\rho_z^I + \rho_{\bar{z}}^I \right), \qquad \Theta_{\alpha} = \left(\delta A_{\alpha}^{(0,1)} \right)_{\bar{I}} (X) \eta^{\bar{I}},$$

$$\Psi_{\bar{i}} = \left[(w_{\bar{i}})_I^{\bar{J}} A_{\bar{J}}^{(0,1)} \right] (X) \left(\rho_z^I + \rho_{\bar{z}}^I \right), \qquad \Psi_i = \left[(w_i)_{\bar{I}}^J A_J^{(1,0)} \right] (X) \eta^{\bar{I}}.$$

Main points to proceed:

- F-terms are calculated by Mandelstam diagrams where all intermediate states are at zero energy.
- Therefore one can reduce the integrals over the Moduli space of Riemann surfaces to the boundary.
- Hence, for open strings also degenerating open channels are relevant.
- Need to study Riemann surfaces with colliding boundaries (that is long strips)

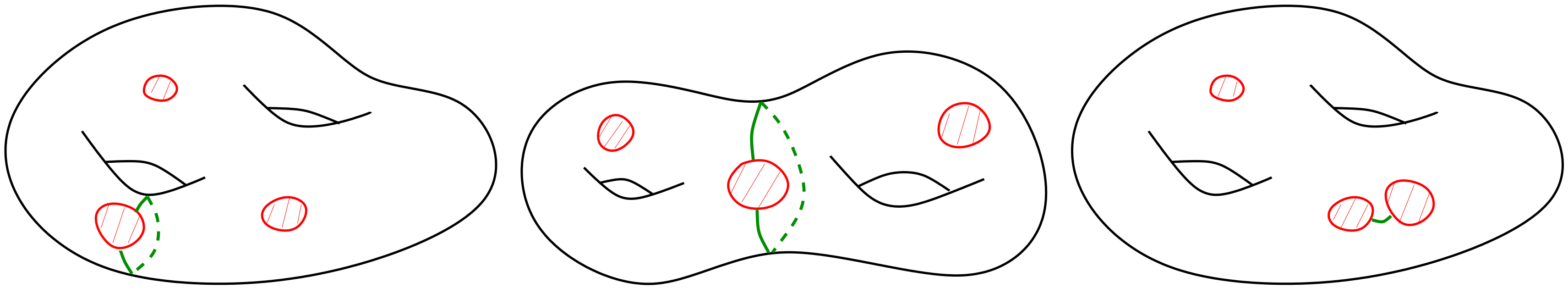
For closed strings channels: two topologically distinct ways:



[pinching] [dividing] [in particular, shrinking]

$$\partial_c \mathcal{M}_{g,h,n,\mathtt{m}} = \mathcal{M}_{g-1,h,n+2,\mathtt{m}} \cup \coprod_{\substack{g_1+g_2=g \\ h_1+h_2=h \\ n_1+n_2=n+2 \\ \mathtt{m}_1 \oplus \mathtt{m}_2 = \mathtt{m}}} \mathcal{M}_{g_1,h_1,n_1,\mathtt{m}_1} \times \mathcal{M}_{g_2,h_2,n_2,\mathtt{m}_2} \tag{1}$$

For open strings channels: three topologically distinct ways:



[pinching] [dividing] [colliding]

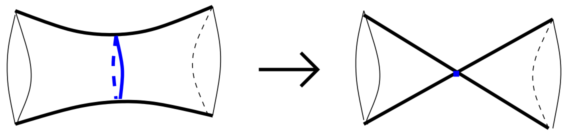
$$\partial_o \mathcal{M}_{g,h,n,\mathtt{m}} =$$

$$\mathcal{M}_{g-1,h+1,n,\hat{\mathtt{m}}\oplus(m_l+1,m_r+1)} \cup \coprod \mathcal{M}_{g_1,h_1,n_1,\mathtt{m}_1} \times \mathcal{M}_{g_2,h_2,n_2,\mathtt{m}_2} \cup \mathcal{M}_{g,h-1,n,\hat{\hat{\mathtt{m}}}\oplus(\mathtt{m}+\mathtt{m}'+2)}$$

$$g_1 + g_2 = g$$
$$n_1 + n_2 = n$$
$$h_1 + h_2 = h + 1$$
$$\mathtt{m}_1 \oplus \mathtt{m}_2 = \hat{\mathtt{m}} \oplus (m_l + 1, m_r + 1)$$

Zooming on the degenerating areas:

[closed channel]

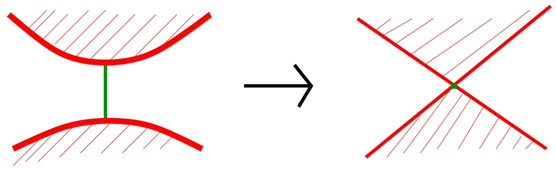


$$[zw = \epsilon, \ \epsilon \in \mathbf{C}]$$

[real codimension two (but the shrinking!)]

[conformal to a long tube]

[open channel]



$$[Re(z)Im(z) > \varepsilon, \ \varepsilon \in \mathbf{R}^+]$$

[real codimension one]

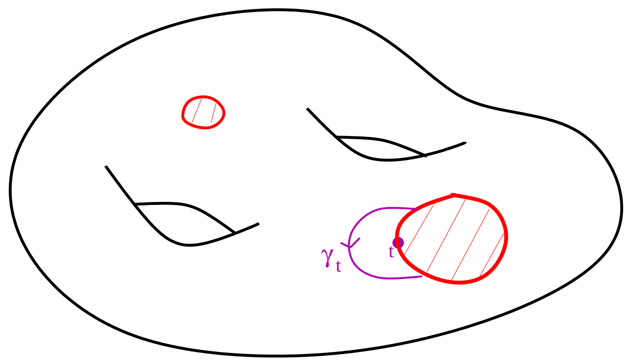
[conformal to a long strip]

Open moduli holomorphic anomaly

$$\partial_{\bar{t}_{\bar{\alpha}}} F_{g,h} = \int_{\mathcal{M}_{g,h}} \langle Q \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}}$$

where

$$Q \oint_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} = \oint_{\partial\Sigma_{g,h}} dt \int_{\gamma_t} dt' \left(G^+ + \bar{G}^+ \right) (t') \Theta_{\bar{\alpha}}(t) \quad ,$$



$$\partial_{\bar{t}_{\bar{\alpha}}} F_{g,h} = \int_{\mathcal{M}_{g,h}} \langle Q \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}}$$

- To calculate: pull **the** supercharge against the measure.
- use superconformal algebra $QG^- = T$
- use $\partial_n \langle X \rangle = \langle X \int T \cdot \nu_n \rangle$

$$\partial_{\bar{t}_{\bar{\alpha}}} F_{g,h} = \int_{\mathcal{M}_{g,h}} \left\{ \sum_{j=1}^{3g-3+h} \frac{\partial}{\partial m_j} \langle \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} (\bar{\mu}_j, \bar{G}^-) \prod_{k \neq j} |(\mu_k, G^-)|^2 \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}} \right.$$

$$\left. + \text{ cplx.conj. } + \sum_{b=1}^{h} \frac{\partial}{\partial l_b} \langle \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \prod_{a \neq b} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}} \right\} \quad .$$

$$\partial_{\bar{t}_{\bar{\alpha}}} F_{g,h} = \int_{\mathcal{M}_{g,h}} \Bigg\{ \sum_{j=1}^{3g-3+h} \frac{\partial}{\partial m_j} \langle \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}}(\bar{\mu}_j, \bar{G}^-) \prod_{k\neq j} |(\mu_k, G^-)|^2 \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}}$$

$$+ \text{ cplx.conj. } + \sum_{b=1}^{h} \frac{\partial}{\partial l_b} \langle \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \prod_{a\neq b} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}} \Bigg\} \quad .$$

Now use Stoke's theorem reducing to the real codimension one component of the boundary of the moduli space.

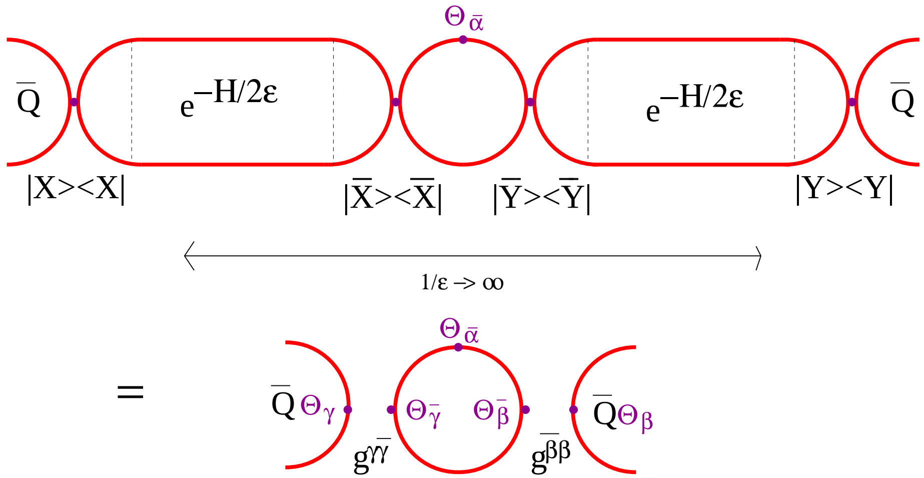
We stay with two contributions: [open strings degenerations] + [shrinking holes].

To calculate [open strings degenerations]:

isolate the Beltrami differentials corresponding to the boundary punctures

$$\int_{\partial_o \mathcal{M}_{g,h}} \langle \int_{\partial\Sigma_{g,h}} \Theta_{\bar{\alpha}} \int_{\gamma_{t_1}} (G^- + \bar{G}^-) \int_{\gamma_{t_2}} (G^- + \bar{G}^-) \prod (m', G^-) \rangle_{\Sigma_{g,h}}$$

In the vicinity of $\partial_o \mathcal{M}_{g,h}$ the Riemann surface develops a long strip. We calculate the path integral via the following CFT prescription

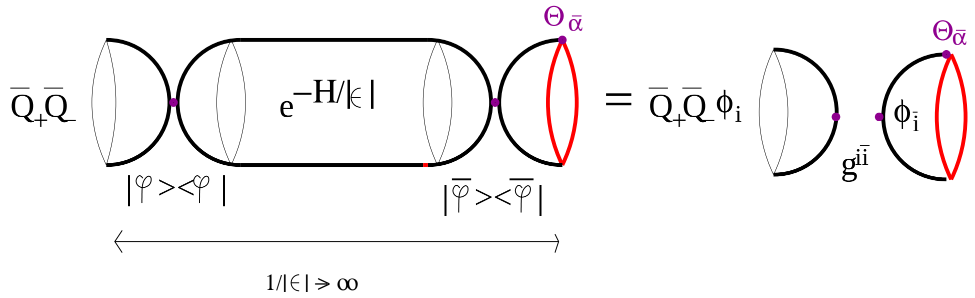


and we stay with

$$\int_{\partial_o \mathcal{M}_{g,h}} \langle \Theta_\beta \oint_{\partial \Sigma_{0,1}} \Theta_{\bar{\alpha}} \Theta_\gamma \rangle_{\Sigma_{0,1}} \langle \bar{Q} \Theta_\beta \bar{Q} \Theta_\gamma \prod (m', G^-) \rangle_{\Sigma_{singular}}$$

(there was a second term [shrinking holes])

- isolate the Beltrami differential concentrated around the node
- prescribe the CFT along this long tube on the Riemann surface as



$$\int_{\mathcal{M}_{g,h-1}} g^{\bar{i}i} \langle \oint_{\partial\Sigma_{0,1}} \Theta_{\bar{\alpha}} \phi_{\bar{i}} \rangle_{\Sigma_{0,1}} \langle \int_{\Sigma_{g,h-1}} \bar{Q}^{+} \bar{Q}^{-} \phi_i \prod (m', G^{-}) \rangle_{\Sigma_{g,h-1}}$$

(summing [open strings degenerations] + [shrinking holes].)

- rewrite the $Q\Theta_\alpha$ and $\bar{Q}^+\bar{Q}^-\phi_i$ as **holomorphic covariant derivatives**

The connection contributes the contact terms as in BCOV

- finally sum up

$$\partial_{\bar{t}_{\bar{\alpha}}} F_{g,h} = \frac{1}{2} g^{\bar{\beta}\beta} g^{\bar{\gamma}\gamma} \Delta_{\bar{\beta}\bar{\alpha}\bar{\gamma}} \left[D_\beta D_\gamma F_{g-1,h+1} + \sum_{g_1+g_2=g, h_1+h_2=h+1} D_\beta F_{g_1,h_1} D_\gamma F_{g_2,h_2} + D_\beta D_\gamma F_{g,h-1} \right] +$$
$$+ g^{i\bar{i}} \Pi_{\bar{\alpha}\bar{i}} D_i F_{g,h-1}$$

where

- $g_{\alpha\bar{\alpha}}$ is the open string moduli metric
- $\Pi_{\bar{\alpha}\bar{i}} = \langle \Theta_{\bar{\alpha}} \phi_{\bar{i}} \rangle_{\Sigma_{0,1}}$ is the overlap function.

Closed moduli with open string background

We have to calculate

$$\partial_{t_{\bar{i}}} F_{g,h} = \int_{\mathcal{M}_{g,h}} \langle \left(\bar{Q}_+ \bar{Q}_- \int_{\Sigma_{g,h}} \phi_{\bar{i}} + Q \oint_{\partial\Sigma_{g,h}} \Psi_{\bar{i}} \right) \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}}$$

where we have a [bulk contribution] + [boundary contribution].

- [boundary contribution]: it is equal in form to what we just calculated, but with $\Psi_{\bar{i}}$ boundary insertions
- [bulk contribution]: BCOV procedure has to be generalized because of the boundaries (J. Walcher still zero open moduli) and because of the non trivial open moduli background.

To proceed: just rewrite $\bar{Q}_+ \bar{Q}_- = QQ'$ where Q is the preserved supercharge and Q' the broken one. Pull **both** the supercharges against the measure. Q is standard. Pulling Q' against the measure $\to$ the breaking term $Q' S_B = \int_{\partial\Sigma} J'$ to add.

[bulk contribution]=[BCOV]+[Walcher's]+[$\int_{\partial\Sigma} J'$-insertion] (calculating [bulk contribution to closed moduli HAE])

The new term [$\int_{\partial\Sigma} J'$-insertion] reads

$$\int_{\mathcal{M}_{g,h}} \langle \int_{\Sigma_{g,h}} \phi_{\bar{i}} \left(\frac{1}{2} \int_{\partial\Sigma_{g,h}} J' \right) \left(Q \prod_{k=1}^{3g-3+h} |(\mu_k, G^-)|^2 \right) \prod_{a=1}^{h} (\lambda_a, G^-) \rangle_{\Sigma_{g,h}}$$

and again localizes on the **real codimension one** component of $\partial\mathcal{M}_{g,h}$.

Therefore, summing up with the [bouldary contribution] we had we get the complete extended HAE's which reads

$$\partial_{\bar{t}_{\bar{i}}} F_{g,h} = \frac{1}{2} C_{\bar{i}}^{jk} \left[\sum_{\substack{g_1+g_2=g \\ h_1+h_2=h}} D_j F_{g_1,h_1} D_k F_{g_2,h_2} + D_j D_k F_{g-1,h} \right] - (\Delta + \Delta')_{\bar{i}}^{j} D_j F_{g,h-1} +$$
$$+ \frac{1}{2} (\Delta' + B)_{\bar{i}}^{\beta\gamma} \left[D_\beta D_\gamma F_{g-1,h+1} + D_\beta D_\gamma F_{g,h-1} + \sum_{\substack{g_1+g_2=g \\ h_1+h_2=h+1}} D_\beta F_{g_1,h_1} D_\gamma F_{g_2,h_2} \right] \quad (2)$$

where C is the sphere 3–pt function and Δ, Δ', B are appropiate disk functions.

Summary: via exact CFT arguments

- Open moduli HAE's
- Closed moduli HAE's in presence of a non trivial open background
- Structure of the equations

$$\bar{\partial}_{open} F = (D_{open} F)^2 + O(t_{open}) D_{closed} F$$

$$\bar{\partial}_{closed} F = (D_{closed} F)^2 + O(t_{open}) (D_{open} F)^2$$

- at frozen open moduli $t_{open} = 0$ and vanishing open moduli derivatives reduce to BCOV $\bar{\partial}_{closed} F = (D_{closed} F)^2$.
- all coefficients structure is on the one of the boundary decomposition of $\mathcal{M}_{g,h}$

Open Issues

- complete with low topologies
- open moduli tt^*–geometry
- holomorphic ambiguity
- test in particular cases (tori, quintic)
- open-closed dualiy (geometric transition)
- matrix model dual picture (a lá Eynard-Marino-Orantin)
- pheno appl. F-terms in open string compactifications
- .....