The E_7 Invariant and Measures of Entanglement

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Introduction

Introduction

We have seen a correspondence between the entanglement of 3-qubits and the black hole entropy of the STU model

$$a_{ABD}|ABD\rangle \Longleftrightarrow \text{STU model}$$

$$\text{Det } a \Longleftrightarrow W(p,q)$$

Generated numerous intriguing connections between the different disciplines of Quantum Information and Supergravity

1. Attractor mechanisms and optimal entanglement distillation processes
2. Special geometry and error correction protocols [P. Levay, arXiv:hep-th/0603136, arXiv:hep-th/0707.2799]
3. Black hole classes and quantum equivalence classes
4. Quantum corrections and concurrences [R. Kallosh and A. Linde, arXiv:hep-th/0602061]

The correspondence has been extended to the $\mathcal{N}=8$ case relating Cartan's unique E_7 quartic invariant to the entanglement of a uniquely entangled 7-qubit state

Tripartite entanglement of 7-qubits
[M. Duff, S. Ferrara, arXiv:hep-th/0609227]

$$\begin{aligned} |\Psi\rangle &= a_{ABD}|ABD\rangle + b_{BCE}|BCE\rangle + c_{CDF}|CDF\rangle + d_{DEG}|DEG\rangle \\ &+ e_{EFA}|EFA\rangle + f_{FGB}|FGB\rangle + g_{GAC}|GAC\rangle \end{aligned} \tag{1}$$

The insights gained in the STU case where made possible by both having a complete correspondence and on the relatively complete understanding of 3-qubit entanglement

However, the particularly special state related to the $\mathcal{N}=8$ case is not well understood from the quantum information theoretic perspective.
In order to develop a similar understanding of the black hole - qubit analogy to that of the STU example we need to better understand this state, it's explicit relationship to the BH entropy and it's entanglement properties

Plan

1. Black Holes, Jordan Algebras and the Freudenthal Triple System (FTS)
   - The Jordan algebra and FTS representations of charges, in $d = 5$ and $d = 4$, and the role of the division algebras
   - $\mathcal{N} = 8$, Cartan's invariant I_4 and the FTS
2. The Qubit Interpretation of I_4
   - Relationship to the STU model and therefore Cayley's Hyperdet
   - Understanding the $\mathcal{N} = 4$ subsectors in terms of qubits
   - Bhargava's Cube Construction
3. Constructing the explicit dictionary
   - From black hole charges to state vector coefficients
   - The entanglement measure
4. Questions the FTS picture may still clarify

Black Holes, Jordan Algebras and the Freudenthal Triple System

Jordan Algebras

A *Jordan algebra* $\mathcal{J}$ over a field $\mathbb{F}$ (here $\mathbb{F} = \mathbb{R}$ always) char$\neq 2$ is a vector space over $\mathbb{F}$ with Jordan product $\circ$ s.t.

$$x \circ y = y \circ x; \qquad x^2 \circ (x \circ y) = x \circ (x^2 \circ y) \qquad \forall x, y \in \mathcal{J} \tag{2}$$

A Cubic Jordan algebra has an admissible cubic form $N\colon \mathcal{J} \to F$ such that

$$N(\alpha x) = \alpha^3 N(x) \quad \text{for} \quad \alpha \in \mathbb{F}, x \in \mathcal{J} \tag{3}$$

a trace bilinear form $(\cdot,\cdot)\colon \mathcal{J} \times \mathcal{J} \to F$ and a quadratic adjoint map, $\sharp\colon \mathcal{J} \to \mathcal{J}$ which satisfies

$$(x^\sharp)^\sharp = N(x)x, \qquad \forall x \in \mathcal{J}$$

The Jordan Algebras of 3×3 Hermitian Matrices

Here $\mathcal{J}$ is the space $J_3(\mathbb{A})$ of 3×3 Hermitian matrices over one of the nicely normed division algebras, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ or their split cousins, $\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s$

$$A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \qquad \text{where} \quad a, b, c \in \mathbb{R} \quad \text{and} \quad x, y, z \in \mathbb{A}$$

Here the Jordan product is given by $A \circ B = \frac{1}{2}(AB + BA)$ and the cubic norm, bilinear trace form and quadratic adjoint are given respectively by

$$N(A) = I_3(A) = abc - a\ \mathbf{n}(x) - b\ \mathbf{n}(y) - c\ \mathbf{n}(z) + (xy)z + \bar{z}(\bar{y}\bar{x}) \tag{4}$$

$$(A, B) = \mathrm{Tr}(A \circ B) \tag{5}$$

$$A^\sharp = A^2 - \mathrm{Tr}(A)A + \frac{1}{2}(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))I$$

$d = 5$ Black Hole Entropy

Under the norm preserving group Str_0, leaving $I_3(A)$ invariant, $A \in J_3(\mathbb{A})$ transform as the $(3\ \mathrm{dim}\mathbb{A} + 3)$ dimensional representation of $SL(3, \mathbb{A})$ where $\mathrm{dim}\mathbb{A} = 1, 2, 4, 8$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively.

In other words as the $6, 9, 15, 27$ of $SL(3, \mathbb{R}), SL(3, \mathbb{C}), SU^*(6), E_{6(-26)}$, respectively.

These are the symmetries of the magic $\mathcal{N} = 2, D = 5$ supergravities. There is one-to-one correspondence between the vector fields (and there charges) and the elements of $J_3(\mathbb{A})$. For the electric black holes, we have the conjugate Jordan matrix

$$J_3(Q) = \begin{pmatrix} q_1 & Q_v & \bar{Q}_c \\ \bar{Q}_v & q_2 & Q_s \\ Q_c & \bar{Q}_s & q_3 \end{pmatrix}$$

and the entropy is

$$S = \pi |I_3(Q)|$$

[S. Ferrara and M. Gunaydin, arXiv:hep-th/9708025]

$d = 4$ Black Holes and the FTS

For $d = 4$ this correspondence is extended, now between the charges and the *Freudenthal triple system* $\mathcal{F}(\mathcal{J})$ where $\mathcal{J}$ is a Jordan algebra

$$\begin{pmatrix} -q_0 & J_3(P) \\ J_3(Q) & p^0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}$$

here α and β are real and $A, B \in J_3(\mathbb{A})$

The p^0 is the charge coming from the $d = 4$ graviphoton field strength which derives from the vector of the $d = 5$ graviton

The charges in $J_3(P)$ derive from the field strengths already present in $d = 5$ [S. Ferrara and M. Gunaydin, arXiv:hep-th/9708025, S. Bellucci, S. Ferrara, M. Gunaydin, A. Marrani, arXiv:hep-th/0606209]

Feudenthal Triple System

Consider the vector space $\mathcal{F}(\mathcal{J})$ constructed as follows

$$\mathcal{F}(\mathcal{J}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathcal{J} \oplus \mathcal{J} \qquad \text{where } \mathcal{J} \text{ is a cubic Jordan algebra over } \mathbb{R}$$

So dim$\mathcal{F}(\mathcal{J}) = 2\ \det\mathcal{J} + 2$. We may write an arbitrary element $x \in \mathcal{F}(\mathcal{J})$ as a "2×2 matrix"

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \qquad \text{where } \alpha, \beta \in \mathbb{R} \quad A, B \in \mathcal{J}$$

Bilinear antisymmetric quadratic form $\{x, y\}$ and quartic norm form $q(x)$

$$\{x, y\} = \alpha\delta - \beta\gamma + (A, D) - (B, C)$$

$$q(x) = -[\alpha\beta - (A, B)]^2 - 4[\alpha N(A) + \beta N(B) - (A^\sharp, B^\sharp)] \tag{6}$$

where $N, \sharp$ and $(\cdot, \cdot)$ are inherited from $\mathcal{J}$ [S. Krutelevich, arXiv:math.NT/0411104]

Invariance Group of $\mathcal{F}(\mathcal{J})$

The invariance group $\mathrm{Inv}(\mathcal{F}(\mathcal{J}))$ is defined to be the set of all $\mathbb{R}$-linear transformations leaving $\{x, y\}$ and $q(x)$ invariant

In the case, $\mathcal{J} = J_3(\mathbb{A}$ or $\mathbb{A}^s)$, $\mathrm{Inv}(\mathcal{F}(\mathcal{J}))$ is generated by the transformations

1. $\phi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + (B, C) + (A, C^\sharp) + \beta N(C) & A + \beta C \\ B + a \times C + \beta C^\sharp & \beta \end{pmatrix}$
2. $\psi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & A + B \times C + \alpha D^\sharp \\ B + \alpha D & \beta + (A, D) + (B, D^\sharp) + \alpha N(D) \end{pmatrix}$
3. $T(s) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1}\alpha & s(A) \\ s^{*-1}(B) & \lambda\beta \end{pmatrix}$

Here, $s \in Str(\mathcal{J})$, $N(s(A)) = \lambda N(A)$, and $A \times B = (A + B)^\sharp - A^\sharp - B^\sharp$
[S. Bellucci, S. Ferrara, M. Gunaydin, A. Marrani, arXiv:hep-th/0606209]
[S. Krutelevich, arXiv:math.NT/0411104]

FTS and $d = 4$ Black Hole Entropy

For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, $\text{Inv}(\mathcal{F}(J_3(\mathbb{A})) = Sp(, \mathbb{R}), SU(3,3), SO^*(12), E_{7(-25)}$.

These are the symmtries of the magic $\mathcal{N} = 2, d = 4$ supergravities

$$\begin{pmatrix} -q_0 & J_3(P) \\ J_3(Q) & p^0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}$$

$$J_3(Q) = \begin{pmatrix} q_1 & Q_v & \bar{Q}_c \\ \bar{Q}_v & q_2 & Q_s \\ Q_c & \bar{Q}_s & q_3 \end{pmatrix}, \qquad J_3(P) = \begin{pmatrix} p_1 & P_v & \bar{P}_c \\ \bar{Q}_v & p_2 & P_s \\ P_c & \bar{P}_s & p_3 \end{pmatrix}$$

The charge representations have dimensions $(6\dim \mathbb{A} + 8)$ and correspond to the threefold antisymmetric traceless tensor (14') of $Sp(6, R)$, the threefold antisymmetric self-dual tensor (20) of $SU(3,3)$, the chiral spinor (32) of $SO^*(12)$ and the fundamental (56) of $E_{7(-25)}$
[S. Ferrara and M. Gunaydin, arXiv:hep-th/9708025]

$\mathcal{N} = 8$ and the FTS

The $\mathcal{N} = 8$ case with U-duality group $E_{7(7)}$ then also follows by using the split octonions, $\mathbb{A} = \mathbb{O}^s$. We have

$$E_{7(7)} \supset E_{6(6)}$$

under which

$$56 \rightarrow 1 + 1 + 27 + 27^{'}$$

In all cases the black hole entropy is

$$S = \pi\sqrt{|\mathrm{I}_4|} \tag{7}$$

where $I_4(x) = q(x)$ is the quartic norm

$$\begin{aligned} I_4(p^0, P; q_0, Q) = \quad & - \quad [p^0 q_0 + \mathrm{tr}(J_3(P) \circ J_3(Q))]^2 \\ & + \quad 4[-p^0 I_3(Q) + q_0 I_3(P) + \mathrm{tr}({J_3}^{\#}(P) \circ {J_3}^{\#}(Q))] \end{aligned} \tag{8}$$

[S. Ferrara and M. Gunaydin, arXiv:hep-th/9708025]

The Qubit Interpretation of I_4

The STU Model in the FTS 1

In the simple case, where we put P, Q all to zero, then

$$\mathcal{F}(J_3(\mathbb{O}^s)) = \begin{pmatrix} p^0 & p^i \\ q_i & -q_0 \end{pmatrix} \tag{9}$$

$$I_3(P) = p^1p^2p^3, \qquad I_3(Q) = q_1q_2q_3 \tag{10}$$

and

$$J_3{}^{\#}(P) = \begin{pmatrix} p^2p^3 & 0 & 0 \\ 0 & p^1p^3 & 0 \\ 0 & 0 & p^1p^2 \end{pmatrix} \quad J_3{}^{\#}(Q) = \begin{pmatrix} q^2q^3 & 0 & 0 \\ 0 & q^1q^3 & 0 \\ 0 & 0 & q^1q^2 \end{pmatrix} \tag{11}$$

and I_4 becomes

$$\begin{aligned} I_4 \quad &= \quad -(p\cdot q)^2 + 4\Big((p^1q_1)(p^2q_2) + (p^1q_1)(p^3q_3) + (p^3q_3)(p^2q_2)\Big) \\ &\quad -4p^0q_1q_2q_3 + 4q_0p^1p^2p^3 \end{aligned} \tag{12}$$

[S. Ferrara and R. Kallosh, arXiv:hep-th/0603247]

The STU Model in the FTS 2

If we make the identifications

$$\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_4 \\ a_7 \\ a_6 \\ a_5 \\ a_3 \end{bmatrix} \tag{13}$$

we recover Cayley's hyperdeterminant

$$I_4 = a_0^2 a_7^2 + a_1^2 a_6^2 + a_2^2 a_5^2 + a_3^2 a_4^2$$

$$-2(a_0 a_1 a_6 a_7 + a_0 a_2 a_5 a_7 + a_0 a_4 a_3 a_7 + a_1 a_2 a_5 a_6 + a_1 a_3 a_4 a_6 + a_2 a_3 a_4 a_5)$$

$$+4(a_0 a_3 a_5 a_6 + a_1 a_2 a_4 a_7)$$

The STU Model in the FTS 3

Using the transformation between P, Q and p, q :

$$\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P^0 - P^2 \\ Q_0 + Q_2 \\ -P^1 + P^3 \\ -P^1 - P^3 \\ Q_0 - Q_2 \\ -P^0 - P^2 \\ -Q_1 + Q_3 \\ -Q_1 - Q_3 \end{bmatrix} \tag{14}$$

This transformation gives us the relations:

$$P^2 = 2(p^2 p^3 - p^0 q_1), \quad P \cdot Q = p \cdot q - 2p^1 q_1, \quad Q^2 = 2(p^1 q_0 + q_2 q_3) \tag{15}$$

hence we find

$$I_4 = P^2 Q^2 - (P \cdot Q)^2 \tag{16}$$

$\mathcal{N}=4$ Subsector 1

$\mathcal{N}=4$ supergravity coupled to m vector multiplets where the symmetry is $SL(2,\mathbb{Z})\times SO(6,6+m,\mathbb{Z})$ where the black holes carry charges belonging to the $(2,12+m)$ representation [M. Duff, S. Ferrara, arXiv:hep-th/0609227]

Here $m=0$. We keep only 24 of the 56 charges but still have the $SL(2)^7$ subgroup. This corresponds to keeping only (p^i, q_i) and (P^s, Q_s) in $\mathcal{F}(J_3)$, then I_4 becomes

$$I_4 = -(p^0q_0 + p^iq_i + 2(P^s\cdot Q_s))^2 + 4[q_0(p^1p^2p^3 - p^1P^{s2}) - p^0(q_1q_2q_3 - q_1{Q_s}^2) + (p^2p^3 - P^{s2})(q_2q_3 - {Q_s}^2) + p^1p^3q_1q_3 + p^1p^2q_1q_2 + 2p^1q_1(P^s\cdot Q_s)] \quad (17)$$

where

$$P^{s2} = P^s\bar{P}^s, \quad P^s.Q_s = \frac{1}{2}(P^s\bar{Q}_s + \bar{Q}_sP^s), \quad {Q_s}^2 = Q_s\bar{Q}_s \quad (18)$$

$\mathcal{N}=4$ Subsector 2

Hence we find

$$I_4 = P^2Q^2 - 2P^2{Q_s}^2 - 2Q^2{P^s}^2 + 4{P^s}^2{Q_s}^2$$

$$-(P.Q)^2 - 4P.QP^s.Q_s - 4(P^s.Q_s)^2 \tag{19}$$

So if we identify

$$\begin{aligned} P^s &= \tfrac{1}{\sqrt{2}}(P^4, P^5, P^6, P^7, P^8, P^9, P^{10}, P^{11}) \\ Q_s &= \tfrac{1}{\sqrt{2}}(Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_{11}) \end{aligned} \tag{20}$$

then

$$I_4 = P^2Q^2 - (P.Q)^2 \tag{21}$$

where indices now run over $0, \ldots 11$, which is manifestly invariant under $SL(2) \times SO(2,10)$ or $SL(2) \times SO(6,6)$ according as we use the octonions or split octonions

Towards a Qubit Interpretation

The above suggests away to associate the 56 black hole charges to the 56 $a_{ABD}, b_{BCE}, c_{CDF}, d_{DEG}, e_{EFA}, f_{FGB}, g_{GAC}$. Using

$$E_{6(6)} \supset SO(4,4), \qquad 27 \to 1+1+1+8_s+8_c+8_v$$

we have

$$56 \to 1+1+1+1+1+1+1+1+8_s+8_c+8_v+8_s+8_c+8_v$$

under $E_{7(7)} \supset SO(4,4)$. If we recall

$$E_{7(7)} \supset SL(2,R)_A \times SL(2,R)_B \times SL(2,R)_D \times SO(4,4) \tag{22}$$

under which

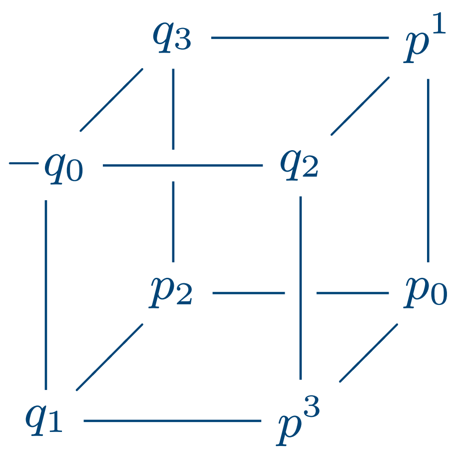
$$56 \to (2,2,2,1)+(2,1,1,8_v)+(1,2,1,8_s)+(1,1,2,8_c) \tag{23}$$

we are led to identify the a_{ABD} with the 8 singlets and the pairs (e_{EFA}, g_{GAC}), (b_{BCE}, f_{FGB}), (b_{BCE}, f_{FGB}) with the pairs of $8_v, 8_s, 8_c$

Bhargava's Cube Construction

Conisder the quantum STU model. Charges live in the space $V = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$

Specifically, we arrange the 8 (p^i, q_i) of the FTS in the following cube



If we denote the standard basis in $\mathbb{Z}^2$ by $\{|0\rangle, |1\rangle\}$ we have the element of V above described by

$$-q_0|000\rangle + q_2|001\rangle + q_1|010\rangle + p^3|011\rangle$$

$$+q_3|100\rangle + p_1|101\rangle + p_2|110\rangle + p_0|111\rangle$$

[M. Bhargava, Ann. of Math. 159 (2004), 217-250]

Fundamental Slicings 1

By slicing the cube across its three planes we find the following pairs of matrices:

$$M_1 = \begin{pmatrix} -q_0 & q_2 \\ q_1 & p^3 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} q_3 & p^1 \\ p^2 & p^0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} -q_0 & q_1 \\ q_3 & p^2 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} q_2 & p^3 \\ p^1 & p^0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} -q_0 & q_3 \\ q_2 & p^1 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} q_2 & p^2 \\ p^3 & p^0 \end{pmatrix}$$

The action of $\Gamma = SL_1(2,\mathbb{Z}) \times SL_2(2,\mathbb{Z}) \times SL_3(2,\mathbb{Z})$ on the cube is given by

$$(M_i, N_i) \mapsto (rM_i + sN_i, tM_i + uN_i)$$

for an element, $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$, of the i^{th} factor of Γ

Fundamental Slicings 2

Then construct the three binary quadratic forms given by

$$f_i(x,y) = \det(M_i x - N_i y), \qquad 1 \leq i \leq 3 \tag{24}$$

Explicitly

$$f_1 = -(q_2 q_1 + p^3 q_0)x^2 + (p.q - 2p^3 q_3)xy - (p^2 p^1 - p^0 q_3)y^2, \tag{25}$$

$$f_2 = -(q_1 q_3 + p^2 q_0)x^2 + (p.q - 2p^2 q_2)xy - (p^1 p^3 - p^0 q_2)y^2, \tag{26}$$

$$f_3 = -(q_3 q_2 + p^1 q_0)x^2 + (p.q - 2p^1 q_1)xy - (p^3 p^2 - p^0 q_1)y^2 \tag{27}$$

Note that the form f_1 is invariant under the subgroup

$$\{id_1\} \times SL_2(2,\mathbb{Z}) \times SL_3(2,\mathbb{Z}) \subset \Gamma$$

Cayley's Hyperdet

The remaining factor still acts in the standard way and the unique invariant under all 3 $SL(2)$'s is Cayley's Hyperdet

Taking the det of the Hessian of each of the quadratic forms gives precisely Cayley's Hyperdet (the BH entropy of the STU model)

$$\det H(f_i) = \det \begin{pmatrix} (f_i)_{xx} & (f_i)_{xy} \\ (f_i)_{yx} & (f_i)_{yy} \end{pmatrix} \tag{28}$$

$$= -(p \cdot q)^2 + 4\Big((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)\Big)$$
$$-4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3$$

Qubits and the Hessian

In fact, the Hessian of each of the quadratic forms is precisely related to one γ_a^i's in the following manner

$$H(f_i) = \begin{pmatrix} (f_i)_{xx} & (f_i)_{xy} \\ (f_i)_{yx} & (f_i)_{yy} \end{pmatrix} = \gamma_a^i \tag{29}$$

where

$$(\gamma_a^1)_{A_1 A_2} = \epsilon^{B_1 B_2} \epsilon^{D_1 D_2} a_{A_1 B_1 D_1} a_{A_2 B_2 D_2} \tag{30}$$

$$(\gamma_a^2)_{B_1 B_2} = \epsilon^{D_1 D_2} \epsilon^{A_1 A_2} a_{A_1 B_1 D_1} a_{A_2 B_2 D_2} \tag{31}$$

$$(\gamma_a^3)_{D_1 D_2} = \epsilon^{A_1 A_2} \epsilon^{B_1 B_2} a_{A_1 B_1 D_1} a_{A_2 B_2 D_2} \tag{32}$$

$$\det \gamma_a^i = -\mathrm{Det}\ a, \qquad \forall\ i$$

Back to the FTS and I_4 1

Consider keeping (p^i, q_i) and only one of (P^s, Q_s), (P^v, Q_v) or (P^c, Q_c). This yields three analogous equations:

1. $(\mathbf{P^s}, \mathbf{Q_s})$

$$I_4 = 4(p^2p^3 - p^0q_1)(q_2q_3 + p^1q_0) - 4(p.q - 2p^1q_1)^2 + 4{P^s}^2{Q_s}^2 - (P^s.Q_s)^2$$
$$-4(p^2p^3 - p^0q_1){Q_s}^2 - 4(q_2q_3 + p^1q_0){P^s}^2 - 4(p.q - 2p^1q_1)P^s.Q_s$$

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2. $(\mathbf{P^v}, \mathbf{Q_v})$

$$I_4 = 4(p^2p^1 - p^0q_3)(q_2q_1 + p^3q_0) - 4(p.q - 2p^3q_3)^2 + 4{P^v}^2{Q_v}^2 - (P^v.Q_v)^2$$
$$-4(p^2p^1 - p^0q_3){Q_v}^2 - 4(q_2q_1 + p^3q_0){P^v}^2 - 4(p.q - 2p^3q_3)P^v.Q_v$$

3. $(\mathbf{P^c}, \mathbf{Q_c})$

$$I_4 = 4(p^1p^3 - p^0q_2)(q_1q_3 + p^2q_0) - 4(p.q - 2p^2q_2)^2 + 4{P^c}^2{Q_c}^2 - (P^c.Q_c)^2$$
$$-4(p^1p^3 - p^0q_2){Q_c}^2 - 4(q_1q_3 + p^2q_0){P^c}^2 - 4(p.q - 2p^2q_2)P^c.Q_c$$

Back to the FTS and I_4 2

By considering each case separately one gains some insight as to how to associate each of the P^s etc with the remaining 44 $b, c, \ldots$

Considering case (1) terms corresponding to a_{ABD} are related to γ^3

$$2(p^2p^3 - p^0q_1) = -(\gamma_a^3)_{00} \tag{33}$$

$$2(q_2q_3 + p^1q_0) = -(\gamma_a^3)_{11} \tag{34}$$

$$p.q - 2p^1q_1 = (\gamma_a^3)_{01} = (\gamma_a^3)_{10} \tag{35}$$

That is, the index corresponding to qubit D is selected as special

From (P^s, Q_s) to (c_{CDF}, d_{DEG})

By considering which of the $\mathcal{N} = 4$ subsectors is given by the common qubit D one is led to $(a_{ABD}, c_{CDF}, d_{DEG})$

Thus, remembering

$$-4(p^2p^3 - p^0q_1){Q_s}^2 - 4(q_2q_3 + p^1q_0){P^s}^2 - 4(p.q - 2p^1q_1)P^s.Q_s$$

(P^s, Q_s) ought to be related to c_{CDF} and d_{DEG} in a manner ensuring

$$P^{s\,2} = (\gamma_c^2)_{00} + (\gamma_d^1)_{00} \tag{36}$$

$$Q_s^{\,2} = (\gamma_c^2)_{11} + (\gamma_d^1)_{11} \tag{37}$$

$$P^s.Q_s = (\gamma_c^2)_{01} + (\gamma_d^1)_{01} \tag{38}$$

Results

Explicit Dictionary

$$(P^s = \tfrac{1}{\sqrt{2}} P^i e_i, \quad Q^s = \tfrac{1}{\sqrt{2}} Q^i e_i) \qquad 0 \leq i \leq 7$$

$$\begin{array}{rclrcl}
P^0 & = & \frac{1}{\sqrt{2}}(-c_0 - c_5) & Q^0 & = & \frac{1}{\sqrt{2}}(-c_2 - c_7) \\
P^1 & = & \frac{1}{\sqrt{2}}(c_4 - c_1) & Q^1 & = & \frac{1}{\sqrt{2}}(c_6 - c_3) \\
P^2 & = & \frac{1}{\sqrt{2}}(c_0 - c_5) & Q^2 & = & \frac{1}{\sqrt{2}}(c_7 - c_2) \\
P^3 & = & \frac{1}{\sqrt{2}}(-c_1 - c_4) & Q^3 & = & \frac{1}{\sqrt{2}}(c_3 + c_6) \\
P^4 & = & \frac{1}{\sqrt{2}}(-d_0 - d_3) & Q^4 & = & \frac{1}{\sqrt{2}}(-d_4 - d_7) \\
P^5 & = & \frac{1}{\sqrt{2}}(d_1 - d_2) & Q^5 & = & \frac{1}{\sqrt{2}}(d_5 - d_6) \\
P^6 & = & \frac{1}{\sqrt{2}}(d_0 - d_3) & Q^6 & = & \frac{1}{\sqrt{2}}(d_7 - d_4) \\
P^7 & = & \frac{1}{\sqrt{2}}(-d_1 - d_2) & Q^7 & = & \frac{1}{\sqrt{2}}(d_5 + d_6)
\end{array}$$

where P^i and Q^i have signature $(+,+,-,-,+,+,-,-)$, $e_0 = 1$ and $e_i^2 = 1$ for $i = \{2, 3, 6, 7\}$

The Entanglment

This implies that, for any of the above $\mathcal{N} = 4$ subsector, I_4 takes the neat form

$$\begin{aligned} I_4(a_{ABD}, e_{EFA}, g_{GAC}) &= \det(\gamma_a^1 + \gamma_g^2 + \gamma_e^3) \\ &= -[a^4 + e^4 + g^4 + 2(a^2e^2 + a^2g^2 + e^2g^2)] \end{aligned} \tag{39}$$

where the superscript on the γ's specifies the position of the common qubit and corresponding γ matrix

$$a^2e^2 = -\frac{1}{2}\epsilon^{B_1B_2}\epsilon^{D_1D_2}\epsilon^{E_3E_4}\epsilon^{F_3F_4}\epsilon^{A_1A_4}\epsilon^{A_2A_3}a_{A_1B_1D_1}a_{A_2B_2D_2}e_{E_3F_3A_3}e_{E_4F_4A_4}$$

This holds through out, we may choose any one of the 7 $\mathcal{N} = 4$ subsectors, determined by a common qubit and write down the entropy using the above ideas

Conculsions

It does seem that studying the special qubit entanglement from the Freudenthal perspective is a fruitful approach and we can expect further insights

1. For the STU model Lévay has developed an understanding of the relationship between the attractor mechanism determining the black hole entropy and quantum distillation processes and certain error correction protocols. The FTS picture should aid the $\mathcal{N} = 8$ understanding of these features

2. FTS should give us a better picture of the relationship between 3-qubits and the special tripartite entanglement of 7-qubits

3. Bhargava's Higher composition laws: have seen a relationship between the Cube and $SL(2)^3$, the higher composition laws relate the Cube to the symmetries of the FTS