Quantum evolution across singularities

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DISCLAIMER:

Space-like and light-like singularities in gravitational and/or geometrical systems

⟷ ?

Time-dependent quantum Hamiltonians developing a singularity as a function of time

Case-by-case evidence based on a series of examples

A toy example: free scalar field on the compactified Milne space

$$ds^2 = -dt^2 + t^2 dx^2$$ (x compact: two cones joining at the tip)

$$S = \int dt\, dx\, t \left(\frac{\dot{\phi}^2}{2} - \frac{\phi'^2}{2t^2} - \frac{m^2\phi^2}{2} \right)$$

$$t(\partial_t \phi)^2 \longrightarrow \pi_\phi^2 / t$$ in the Hamiltonian

1/t singularity in the time-dependence of the Hamiltonian originating from the space-time singularity in the background metric.

A more ambitious example:
Matrix Big Bang

$$ds^2 = e^{2\alpha x^+}(-2dx^+dx^- + (dx^i)^2) + e^{2\beta x^+}(dx^a)^2$$

a light-like singularity at x^+=-infinity (finite proper distance)
a light-like isometry along x^-

matrix model description follows from the M-theory – type IIA string theory duality conjecture

may be a consistent theory of quantum gravity for space-times asymptotic to the original background

the matrix model Hamiltonian displays a 1/t singularity in its time dependence at the location of the original space-time singularity

transition rules???

Quantum Hamiltonians with a singular time dependence

General:

$$H = \hat{H}(t,\varepsilon), \qquad \exists \lim_{\varepsilon\to 0} \hat{H}(t \neq t^*, \varepsilon), \qquad \nexists \lim_{\varepsilon\to 0} \hat{H}(t = t^*, \varepsilon)$$

too complicated…

Less general:

$$H = \sum_i f_i(t,\varepsilon)\hat{H}_i$$

still complicated…
more later

$$H = f(t,\varepsilon)\hat{h}$$

very simple

Single operator structure

$$H = f(t,\varepsilon)\hat{h} \qquad \exists \lim_{\varepsilon\to 0} f(t \neq t^*, \varepsilon), \qquad \nexists \lim_{\varepsilon\to 0} f(t = t^*, \varepsilon)$$

General solution: $$U(t,t') = \exp[-i\int_t^{t'} dt\, f(t)\hat{h}]$$

Integrability properties of singular functions are conveniently described by the theory of distributions.

<u>Reminder:</u> distributions are (possibly singular) generalizations of functions defined by their integration properties in convolution with smooth "test functions" (examples: δ(x), P(1/x), 1/(x+i0)^2, etc)

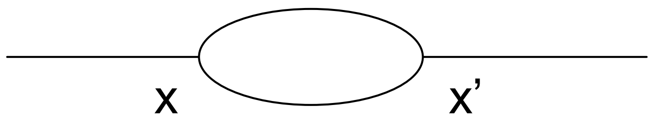
$$\int \delta(x)\mathcal{F}(x)dx = \mathcal{F}(0) \qquad \int P(1/x)\mathcal{F}(x)dx = P\int \frac{\mathcal{F}(x)}{x}dx \qquad \int (1/(x+i0)^2)\mathcal{F}(x)dx = \int_{C^-} \frac{\mathcal{F}(x)}{x^2}dx$$

U(t,t') **will exist if *f(t)* is a distribution (perhaps up to terms that only affect the value of *U(t,t')* at t'=t*, such as *δ'(t-t*)* with an infinite coefficient)**

Subtractions/renormalization

For any function *f(t)* with an isolated singularity at t* (not stronger than a power), it is possible to construct a distribution equal *f(t)* everywhere away from t*.

The procedure is familiar in the context of renormalization of conventional field theories.



The analogy is evident if renormalization is performed in coordinate (not momentum) representation

$D(x,x')^2$ is not a distribution, and one makes it a distribution by subtracting counterterms proportional to $\delta(x-x')$, etc with divergent coefficients.

The Hamiltonian *f(t)h* with *f(t)* singular at *t=t is renormalized by subtracting $\delta(t-t^*)$, etc to make *f(t)* a distribution**

Free scalar field on the Milne space

$$ds^2 = -dt^2 + t^2 dx^2 \qquad S = \int dt\, dx\, |t| \left(\frac{\dot{\phi}^2}{2} - \frac{\phi'^2}{2t^2} - \frac{m^2\phi^2}{2} \right)$$

$$H = \frac{1}{2|t|} \int dx \, \left(\pi_\phi^2 + \phi'^2 \right) + \frac{m^2|t|}{2} \int dx\, \phi^2$$

1/|t| needs to be renormalized, which can be accomplished by subtracting δ(t) with a divergent coefficient.

$$\left[\frac{1}{|t|}\right] = \lim_{\varepsilon \to 0} \left(\frac{1}{\sqrt{t^2+\varepsilon^2}} - \delta(t) \int_{-\mu}^{\mu} \frac{1}{\sqrt{t^2+\varepsilon^2}} dt \right) \sim \lim_{\varepsilon \to 0} \left(\frac{1}{\sqrt{t^2+\varepsilon^2}} - 2 \ln \frac{\mu}{\varepsilon}\, \delta(t) \right)$$

- **free dimensionful parameter μ (typical of renormalization)**
- **compatible with the** ***oribifold matching prescriptions*** **(though it requires different values of μ for different oscillators of the field)**

Multiple operator structures

$$H = \sum_i f_i(t,\varepsilon)\hat{H}_i \qquad U(t,t') = T\left\{\exp\left[-i\int_t^{t'} dt\, H(t)\right]\right\}$$

Magnus expansion:

$$\exp\left(-i\int_t^{t'} dt\, H(t) + \int dt\, dt'\, [H(t), H(t')] + i\int dt\, dt'\, dt''\, [H(t), [H(t'), H(t'')]] + \cdots\right)$$

(schematic……)

If all f_i's have a limit as distributions, all terms in the Magnus expansion are finite.

Even if f_i's do not have a limit as distributions, the limit of U may exist due to cancellations in the Magnus expansion.

Operator-valued generalizations of distributions…

Null-brane and the parabolic orbifold

$$ds^2 = -dX^+dX^- + \frac{X^2R^2}{(R^2+X^{+2})^2}dX^{+2} - \frac{4XR}{\sqrt{R^2+X^{+2}}}d\Theta\, dX^+ + (R^2+X^{+2})d\Theta^2 + dX^2$$

$R \to 0 \qquad ds^2 = -dX^+dX^- + X^{+2}d\Theta^2 + dX^2$ $\qquad \Theta$ **compact**

In these coordinates, the Hamiltonian of a scalar field is non-singular for non-zero R and develops an isolated singularity at $X^+=0$ for R=0:

$$H \sim \int dX\, dX^-\, d\Theta \left(\frac{X^2R^2}{(R^2+X^{+2})^2}\partial_-\pi_\phi\partial_-\phi + \frac{XR}{(R^2+X^{+2})^{3/2}}\partial_-\pi_\phi\partial_\Theta\phi + \frac{1}{R^2+X^{+2}}\partial_\Theta\pi_\phi\partial_\Theta\phi + \partial_X\pi_\phi\partial_X\phi + \cdots \right)$$

The time dependences in the Hamiltonian do not have a limit as distributions.

Nevertheless, the evolution operator U(t,t') can be computed and it does have a limit for R going to 0.

If U(R=0) is written as $\exp[-i\int H_{eff}(t)dt]$, **$H_{eff}$ turns out to be expressed through distributions.**

The divergences in the Magnus expansion must have cancelled…

Not yet

- More general understanding of Hamiltonian singularities involving multiple operator structures (operator-valued generalizations of conventional distributions, with a proper account of time-ordering)
- Singular time-dependence renormalization for systems with more complicated dynamics (interacting field theories, scattering singularities on the parabolic orbifold, focusing, etc)
- Transition through space-time singularities in the context of time-dependent matrix models (can possibly give a picture of genuine quantum-gravitational effects in the near-singular region)