# *Geometric Lagrangians*

# *for*

# *massive higher-spin fields**


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Valencia - $3^{rd}$ RTN Workshop

October, 4 - 2007


*D. F. - *in preparation*

# *Introduction I: geometry for higher-spin fields?*

Central object in Maxwell, Yang-Mills (spin 1) and Einstein (spin 2) theories

is

$$\text{the curvature : } \begin{cases} A_\mu \; \rightarrow F_{\mu\,\nu}\,, \\ h_{\mu\nu} \rightarrow \mathcal{R}_{\mu\,\nu,\,\rho\,\sigma}\,. \end{cases}$$

It gives us both *dynamical informations* and *geometrical meaning*.

Is it possible to find any similar description for higher-spins?

# *Introduction II: massive fields - the Fierz system*

Free propagation of *massive irreps* of the Poincaré group can be described by

spin-$s$, massive boson $\longrightarrow$ symmetric rank-$s$ tensor $\varphi_{\mu_1 \ldots \mu_s}$

spin-$(s+1/2)$, massive fermion $\longrightarrow$ symmetric rank-$s$ spinor-tensor $\psi^{a}_{\mu_1 \ldots \mu_s}$

satisfying the conditions [*Fierz,* 1939]:

$$(\Box - m^2)\,\varphi_{\mu_1 \ldots \mu_s} = 0\,, \qquad (i\gamma^{\alpha}\partial_{\alpha} - m)\,\psi^{a}_{\mu_1 \ldots \mu_s} = 0\,,$$

$$\partial^{\alpha}\varphi_{\alpha\,\mu_2 \ldots \mu_s} = 0\,, \qquad \partial^{\alpha}\psi^{a}_{\alpha\,\mu_2 \ldots \mu_s} = 0\,,$$

$$\varphi^{\alpha}{}_{\alpha\,\mu_3 \ldots \mu_s} = 0\,, \qquad \gamma^{\alpha}\,\psi^{a}_{\alpha\,\mu_2 \ldots \mu_s} = 0\,.$$

At the Lagrangian level the problem is

➙ how to implement the conditions $\begin{cases} \begin{cases} \partial^{\alpha}\varphi_{\alpha\,\mu_2 \ldots \mu_s} = 0 \\ \varphi^{\alpha}{}_{\alpha\,\mu_3 \ldots \mu_s} = 0 \end{cases} \\ \begin{cases} \partial^{\alpha}\psi^{a}_{\alpha\,\mu_2 \ldots \mu_s} = 0 \\ \gamma^{\alpha}\,\psi^{a}_{\alpha\,\mu_2 \ldots \mu_s} = 0 \end{cases} \end{cases}$ ?

Typically, known solutions

➙ involve *auxiliary fields*,

➙ are *not simple deformations* of the massless theory.

# Introduction III: the examples of spin 1 and spin 2

Spin 1 and spin 2: massive theory as

*quadratic deformation of the geometric theory:*

- Spin 1 [*Proca*]

$$\mathcal{L}(m=0) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}(m) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\,,$$

$$\partial_\nu\, F^{\mu\nu} - m^2 A^\mu = 0,$$

$$\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0 \rightarrow \partial_\mu A^\mu = 0$$

- Spin 2 [*Fierz-Pauli*]

$$\mathcal{L}(m=0) = \frac{1}{2}h_{\mu\nu}\,(\mathcal{R}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,\mathcal{R})$$

$$\mathcal{L}(m) = \frac{1}{2}h_{\mu\nu}\,\{(\mathcal{R}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,\mathcal{R}) - m^2 \underbrace{(h^{\mu\nu} - \eta^{\mu\nu}\,h^\alpha{}_\alpha)}_{Fierz-Pauli\ mass\ term}\}$$

$$\partial_\mu\,(\mathcal{R}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\,\mathcal{R}) \equiv 0 \rightarrow \partial_\mu\,(h^{\mu\nu} - \partial^\nu\,h^\alpha{}_\alpha) = 0$$

$$\boxed{\partial_\mu\,h^{\mu\nu} - \partial^\nu\,h^\alpha{}_\alpha = 0}$$ *Fierz-Pauli constraint*

# Introduction IV: difficulties of the Fronsdal's theory

"Canonical" description of higher-spin gauge fields encoded in the *Fronsdal's equation* (1978):

$$\mathcal{F} \equiv \Box\varphi - \partial\,\partial\cdot\varphi + \partial^2\varphi' = 0$$

- gauge invariant under $\delta\,\varphi = \partial\,\Lambda$ *iff* $\Lambda'\,(\equiv\,\Lambda^{\alpha}{}_{\alpha}) \equiv 0;$
- Lagrangian description *iff* $\varphi''\,(\equiv\,\varphi^{\alpha\,\beta}{}_{\alpha\,\beta}) \equiv 0\,.$

Gauge invariance with a *traceless* parameter

$\Rightarrow$

the "Einstein tensor" does not need to be (and *is not*) divergenceless

$$\partial\cdot\{\mathcal{F} - \frac{1}{2}\eta\,\mathcal{F}'\} = \frac{1}{2}\eta\,\mathcal{F}',$$

*not possible to extend the results of spin 1 and spin 2*

[*Aragone-Deser-Yang 1987*]

## *Introduction V: what do we need to generalise the spin 1 and spin 2 cases?*

To try and reproduce the "geometric construction" we need the following:

- Candidate tensors to play the role of *higher-spin curvatures*.
- Candidate *Ricci tensors* and *Dirac tensors*, to define the free equations of motion. These have to be consistent with the known result for the non-geometric, constrained theory of Fronsdal.
- *Bianchi identities* to be satisfied by the generalised Ricci tensors.
- Suitable *mass deformations*, such that the on-shell consequence of the Bianchi identity imply that the system reduces to the Fierz form.

# *Plan*

I. Higher-spin curvatures

II. Geometric massive theory I: bosons $\begin{cases} \rightarrow \text{“}Ricci\text{''} \ \ tensors, \\ \rightarrow Bianchi \ \ identities, \\ \rightarrow mass \ \ deformations. \end{cases}$

III. Geometric massive theory II: fermions $\begin{cases} \rightarrow the \ \ example \ \ of \ \ spin \ 3/2, \\ \rightarrow mass \ \ deformations. \end{cases}$

IV. Propagators and uniqueness

# *I. Higher-spin curvatures*

# *Higher-spin curvatures*

Following de Wit and Freedman (1980), introduce higher-spin curvatures:

- *Spin 1 [Maxwell]*: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ s. t. $\delta F_{\mu\nu} = 0$ under $\delta\, A_\mu \,=\, \partial_\mu \Lambda$ ;

$$\partial \cdot F^{\mu} = 0$$

  *(but also $s = 3/2$)*

- *Spin 2 [Einstein]*: $\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = \partial_\nu \Gamma^{\alpha}_{\beta\mu} - \partial_\mu \Gamma^{\alpha}_{\beta\nu}$ s. t. $\delta \mathcal{R}^{\alpha}{}_{\beta\mu\nu} = 0$ under $\delta\, h_{\mu\nu} \,=\, \partial_\mu\, \Lambda_\nu \,+\, \partial_\nu\, \Lambda_\mu$;

$$\mathcal{R}^{\alpha}{}_{\beta\alpha\nu} \equiv \mathcal{R}_{\mu\nu} = 0$$

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- *Spin 3 [de Wit - Freedman]*: $\delta\varphi_{\alpha\beta\gamma} = \partial_\alpha \Lambda_{\beta\gamma} + \partial_\beta \Lambda_{\alpha\gamma} + \partial_\gamma \Lambda_{\beta\alpha}$

$$\begin{aligned}
\Gamma^{(1)}_{\rho,\alpha\beta\gamma} &= \partial_\rho \varphi_{\alpha\beta\gamma} - (\partial_\alpha \varphi_{\rho\beta\gamma} + \partial_\beta \varphi_{\rho\alpha\gamma} + \partial_\gamma \varphi_{\alpha\beta\rho}) \ , \\
\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma} &= \partial_\rho \Gamma^{(1)}_{\sigma,\alpha\beta\gamma} - \frac{1}{2}(\partial_\alpha \Gamma^{(1)}_{\sigma,\rho\beta\gamma} + \ldots) \ , \\
\Gamma^{(3)}_{\rho\sigma\tau,\alpha\beta\gamma} &= \partial_\rho \Gamma^{(2)}_{\sigma\tau,\alpha\beta\gamma} - \frac{1}{3}(\partial_\alpha \Gamma^{(2)}_{\sigma\tau,\rho\beta\gamma} + \ldots) \ , \ \text{s. t. } \delta \Gamma^{(3)}_{\rho\sigma\tau,\alpha\beta\gamma} = 0 \ .
\end{aligned}$$

  *(but also $s = 7/2$)*

$\sim$

*Curvature*

$\equiv$ gauge-invariant top of this hierarchy of connexions:

$$\Gamma^{(3)}_{\rho\sigma\tau,\alpha\beta\gamma} \equiv \mathcal{R}^{(3)}_{\rho\sigma\tau,\alpha\beta\gamma} \quad \Rightarrow \quad EoM?$$

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*(but also $s = 7/2$)* (curvatures $\Rightarrow$ no constraints $\rightarrow$ *Fronsdal's theory is not geometric*)

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# *II. Geometric massive theory I: bosons*

♋

# Candidate Ricci tensors ($s = 3$)

- Spin-3 curvature: saturate three indices to restore the symmetries of $\varphi_{\mu_1\mu_2\mu_3}$:

$$\mathcal{R}_{\rho_1\rho_2\rho_3,\,\mu_1\mu_2\mu_3} \quad \longrightarrow \quad \partial\cdot\mathcal{R}'\,,$$

- restore dimensions of a relativistic wave operator, introducing *non-localities*:

$$\partial\cdot\mathcal{R}' \quad \longrightarrow \quad \frac{1}{\Box}\,\partial\cdot\mathcal{R}'\,,$$

so defining the *simplest* candidate Ricci tensor. [*D.F. - A. Sagnotti, 2002*]

- This possibility is *highly non-unique* $\rightarrow$ infinitely many -more singular- ones:

$$\frac{1}{\Box}\,\partial\cdot\mathcal{R}' \quad \longrightarrow \quad \mathcal{A}_{\varphi}\,(a) \equiv \frac{1}{\Box}\,\partial\cdot\mathcal{R}' + a\,\frac{\partial^2}{\Box^2}\,\partial\cdot\mathcal{R}''\,,$$

$$\Rightarrow$$

$$\boxed{\mathcal{A}_{\varphi}\,(a) = 0}$$

- *Meaning?*

- *Lagrangian origin (i.e. Bianchi identity)?*

# *Geometric Lagrangians for massless bosons*

$\sim$

Spin $s$: the most general candidate "Ricci" tensor

$$\mathcal{A}_\varphi(a_1, \dots a_k, \dots)$$

is such that, *for almost all choices* of $a_1, \dots a_k, \dots$ :

➽ (*CONSISTENCY*) the equation $\mathcal{A}_\varphi = 0$ implies the *compensator equation*

$$\mathcal{F} - 3\,\partial^3 \alpha_\varphi = 0\,,$$

with $\delta\,\alpha_\varphi = \Lambda'$ ➙ *Fronsdal form, after partial gauge-fixing*.

➽ (*LAGRANGIAN*) it is possible to define an *identically divergenceless Einstein tensor* $\mathcal{E}_\varphi$ s.t.

$$\mathcal{L} = \frac{1}{2}\varphi\,\mathcal{E}_\varphi \quad \Rightarrow \quad \mathcal{E}_\varphi = 0 \quad \Rightarrow \quad \mathcal{A}_\varphi = 0\,,$$

# *Mass deformation for bosons I*

- *Spin 2:*

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\mathcal{R} - m^2(h_{\mu\nu} - \eta_{\mu\nu}h') = 0\,.$$

$$\Rightarrow$$

$$\underbrace{M_h = h_{\mu\nu} - \eta_{\mu\nu}h'}_{Fierz-Pauli\ \ mass\ \ term}, \qquad \leftrightarrow \qquad \underbrace{\partial \cdot h_{\mu} - \partial_{\mu}h' = 0}_{Fierz-Pauli\ \ constraint},$$

- *Spin $s$:* General idea: *higher traces* should appear in the mass term:

$$\mathcal{E}_{\varphi}(a_1, \ldots a_k, \ldots) - m^2 M_{\varphi} = 0$$

$$\Rightarrow$$

$$\underbrace{M_{\varphi} = \sum \lambda_k \eta^k \varphi^{[k]}}_{generalised\ \ FP\ \ mass\ \ term}, \qquad \leftrightarrow \qquad \underbrace{\partial \cdot \varphi - \partial \varphi' = 0}_{FP\ \ constraint},$$

*How ?*

# *Mass deformation for bosons II: spin 4*

$$\boxed{\mu_\varphi \equiv \partial \cdot \varphi \, - \, \partial \, \varphi'}$$

- We look for $\lambda_1, \, \lambda_2$ s.t.

$$\partial \cdot M_\varphi \, = \, \partial \cdot \{ \varphi \, + \, \lambda_1 \, \eta \, \varphi' \, + \, \lambda_2 \, \eta^2 \, \varphi'' \} \, = \, \mu_\varphi \, + \, k \, \eta \, \mu'_\varphi \, ;$$

in this way

$$\partial \cdot \{ \mathcal{E}_\varphi \, (\{a_1, \, a_2\}) \, - \, m^2 \, M_\varphi (\lambda_1, \, \lambda_2) \} \, = \, 0 \;\; \rightarrow$$

$$\mu_\varphi \, + \, k \, \eta \, \mu'_\varphi \, = \, 0 \;\; \rightarrow$$

$$\mu'_\varphi \, = \, 0 \;\; \rightarrow$$

$$\boxed{\mu_\varphi \, = \, 0}$$

- *The condition* $\mu_\varphi \, = \, 0$ *implies in turn* $\quad \mathcal{A}'_\varphi \, (\{a_1, \, a_2\}) \, = \, 0 \quad \forall \, \{a_1, \, a_2\}$

- From the resulting equation $(\lambda_1 = -1, \;\; \lambda_2 = -1)$

$$\mathcal{A}_\varphi \, (\{a_1, \, a_2\}) \, - \, m^2 \, (\varphi - \eta \, \varphi' \, - \, \varphi'') = 0 \;\; \rightarrow$$

$$\varphi'' = 0 \;\; \rightarrow \;\; \varphi' = 0 \;\; \rightarrow$$

$$\boxed{\mathcal{A}_\varphi \, (\{a_1, \, a_2\}) \, = \, \Box \, \varphi} \quad \forall \, \{a_1, \, a_2\}$$

the last condition needed to put the system in the Fierz form.

# Mass deformation for bosons III

- The general solution is

$$\mathcal{L}(m) = \frac{1}{2}\varphi\{\mathcal{E}_\varphi(\{a_1, \dots a_k \dots\}) - m^2 M_\varphi\},$$

where, for $s = 2n$ or $s = 2n+1$

$$\boxed{M_\varphi = \varphi - \eta\varphi' - \eta^2\varphi'' - \frac{1}{3}\eta^3\varphi''' - \dots - \frac{1}{(2n-3)!!}\eta^n\varphi^{[n]}}.$$

- The structure of the mass term is to be understood such that

$$\partial \cdot M_\varphi = \mu_\varphi + k_1\eta\mu'_\varphi + \dots + k_m\eta^{[m]}\mu_\varphi^{[m]} + \dots,$$

so that $\partial \cdot M_\varphi = 0$ implies the basic, *Fierz-Pauli constraint*

$$\mu_\varphi = \partial\cdot\varphi - \partial\varphi' = 0,$$

together with all its *consistency conditions*: $\mu_\varphi^{[m]} = 0 \quad \forall\, m$ .

- *The massive theory is not unique*:
  - The Fierz-Pauli constraint implies $\mathcal{A}'_\varphi(\{a_k\}) = 0, \ \forall(\{a_k\})$
  - Under this condition *all traces of $\varphi$ can be shown to vanish* on-shell
  - this implies in turn $\mathcal{A}_\varphi(\{a_k\}) = \Box\varphi, \ \forall(\{a_k\})$, and then *any* Lagrangian equation can be reduced to the Fierz system.

## *III.* *Geometric massive theory II: fermions*

♋

# *The example of spin 3/2*

Spin-3/2 massive theory as

*quadratic deformation of the geometric, Rarita-Schwinger theory:*

- Spin 3/2

$$\mathcal{L}(m=0) = \frac{1}{2}\bar{\psi}_\mu \left(\mathcal{R}^\mu - \frac{1}{2}\gamma^\mu \not{\mathcal{R}}\right) + h.c.$$

$$\mathcal{L}(m) = \frac{1}{2}\bar{\psi}_\mu \left\{\left(\mathcal{R}^\mu - \frac{1}{2}\gamma^\mu \not{\mathcal{R}}\right) - m\left(\psi^\mu - \gamma^\mu \not{\psi}\right)\right\} + h.c.$$

$$\partial_\mu \left(\mathcal{R}^\mu - \frac{1}{2}\gamma^\mu \not{\mathcal{R}}\right) \equiv 0 \rightarrow \partial_\mu \left(\psi^\mu - \gamma^\mu \not{\psi}\right) = 0$$

$$\boxed{\partial_\mu \psi^\mu - \not{\partial}\, \not{\psi} = 0}$$

*(fermionic) Fierz-Pauli constraint*

# Mass deformation for fermions

In the general case the *fermionic analogue of the Fierz-Pauli constraint* is

$$\boxed{\mu_\psi \equiv \partial \cdot \psi - \not{\partial}\, \not{\psi} - \partial\, \psi' = 0}$$

- *Spin 5/2:* We look for a mass term $M_\psi$ s. t.

$$\mathcal{E}_\psi - m \underbrace{(\psi - \lambda_1 \gamma\, \not{\psi} - \lambda_2 \eta\, \psi')}_{M_\psi} = 0\,, \quad \Rightarrow \quad \partial \cdot M_\psi = \mu_\psi + \rho_1 \gamma\, \not{\mu}_\psi = 0\,.$$

The *unique* solution is

$$\mathcal{L} = \frac{1}{2} \bar{\psi} \{\mathcal{E}_\psi - m\, (\psi - \gamma\, \not{\psi} - \eta\, \psi')\} + h.c.\,.$$

- *Spin $s + 1/2$:* The same procedure leads to the general solution

$$\boxed{\mathcal{L} = \frac{1}{2} \bar{\psi} \{\mathcal{E}_\psi - m\, (\psi - \sum_{j=0}^{s} \frac{1}{(2j-1)!!} \gamma\, \eta^j\, \not{\psi}^{[j]} - \sum_{i=1}^{s} \frac{1}{(2i-3)!!} \eta^i\, \psi^{[i]})\} + h.c.}$$

- The generalised Fierz-pauli constraint implies the Fierz system *for all* geometric Einstein tensors (including those with more singular terms) $\Rightarrow$ *issue of uniqueness* (already at the simplest level).

# *IV. Propagators & uniqueness*

♋

# *Massless propagators*

*Are all those solutions really equivalent?*

- *Propagator* from Lagrangian equation with an external current:

$$\mathcal{E}_\varphi(a_1, \ldots a_k \ldots) = \mathcal{J} \quad \Rightarrow \quad \varphi = \mathcal{G} \cdot \mathcal{J}$$

- *Current exchange* $\mathcal{J} \cdot \varphi = \mathcal{J} \cdot \mathcal{G} \cdot \mathcal{J} \rightarrow$ consistency conditions on the *polarisations flowing*:

*almost all geometric theories give the wrong result, but one*.

The correct theory has a simple structure:

- The kinetic tensor has the compensator form $\mathcal{A}_\varphi = \mathcal{F} - 3\,\partial^3\,\gamma(\varphi)$;
- It satisfies the identities : $\begin{cases} \partial \cdot \mathcal{A}_\varphi - \frac{1}{2}\partial\,\mathcal{A}'_\varphi \equiv 0 \\ \mathcal{A}''_\varphi \equiv 0 \end{cases}$, and the Lagrangian is*

$$\boxed{\mathcal{L} = \frac{1}{2}\varphi\,(\mathcal{A}_\varphi - \frac{1}{2}\eta\,\mathcal{A}'_\varphi + \eta^2\,\mathcal{B}_\varphi) - \varphi \cdot \mathcal{J}}$$

*[ *D.F. - J. Mourad - A. Sagnotti, 2007*]

# Massive propagators - spin 4

- Consider the Lagrangian

$$\mathcal{L}(m) = \frac{1}{2}\varphi\{\mathcal{E}_\varphi(a_1, a_2) - m^2 M_\varphi\} - \varphi \cdot \mathcal{J},$$

where $\mathcal{J}$ is a *conserved* current. The on-shell condition $\mathcal{A}'_\varphi(a_1, a_2) = 0$ reduces the equation of motion to

$$\mathcal{A}_\varphi(a_1, a_2) - m^2(\varphi - \eta\,\varphi' - \varphi'') = \mathcal{J}.$$

where

$$\mathcal{A}_\varphi(a_1, a_2) \overset{o.s.}{=} \Box\,\varphi - \frac{\partial^2}{\Box}\varphi' - 3\,\frac{\partial^4}{\Box^2}\varphi'' = 0.$$

$$\Rightarrow$$

| *The whole structure of the propagator is encoded in the coefficients of* $M_\varphi$ |
|---|

- Inverting the equation of motion

$$\mathcal{J}\cdot\varphi = \frac{1}{p^2 - m^2}\{\mathcal{J}\cdot\mathcal{J} - \frac{6}{D+3}\,J'\cdot J' + \frac{3}{(D+1)(D+3)}\,J''\cdot J''\}$$

while the corresponding computation for the massless case gives

$$\mathcal{J}\cdot\varphi = \frac{1}{p^2}\{\mathcal{J}\cdot\mathcal{J} - \frac{6}{D+2}\,J'\cdot J' + \frac{3}{D(D+2)}\,J''\cdot J''\}$$

thus showing the (generalised) *vDVZ discontinuity* for higher-spin fields.

# "KK reduction" and uniqueness

➻ How to understand the origin of the Fierz-Pauli mass-term, for $s = 2$?

KK reduction ($\Box \rightarrow \Box - m^2$):

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R} \sim \Box(h - \eta h') + \ldots,$$

➻ How to perform a KK reduction of a *non local* theory?

Maybe not clear ($\frac{1}{\Box} \rightarrow \frac{1}{\Box - m^2}$), but still it is unambiguously defined the "pure massive" contribution of the resulting reduction:

$$\mathcal{E}_{\varphi} = \Box(\varphi + k_1 \eta \varphi' + k_2 \eta^2 \varphi'' + \ldots) + \ldots,$$

➻ Is it possible to find a geometric theory whose "box" term encodes the coefficients of the generalised FP mass term?

➻ up to spin 4 (at least) *it is the one selected by the analysis of the current exchange.*

➻ Why the mass term works well with *all* geometric tensors?

Not too strange, also true for spin 2: the non-local theory defined by the eom

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R} + \lambda(\eta - \frac{\partial^2}{\Box})\mathcal{R} - m^2(h - \eta h'),$$

reduces to the Fierz system, and gives the correct current exchange!

# *Comments & Conclusions*

- Foregoing locality, *linear dynamics of higher-spin gauge fields in geometric fashion*; infinitely many formulations are indeed available, at the free level.

- relationship with a parallel, *local*, unconstrained formulation*, or analysis of the propagator, shows that there is *one preferred geometric theory*.

*[ *D.F. - A. Sagnotti, '05, '06*]

$\sim$

- Generalisation of the Fierz-Pauli mass term, for bosons and fermions, involving *all* $(\gamma-)$*traces* of the field.

- Description of massive theory, for spin greater than two, that *does not involve auxiliary fields* (and no dimension-dependent coefficients).

- *Issue of uniqueness*: $\begin{cases} \textit{Mass term} \quad \longrightarrow \quad \textit{unique} \\ \textit{Massive theory} \quad \longrightarrow \quad \textit{degenerate} \\ \textit{``memory'' of the correct theory, by KK analysis ?} \end{cases}$

- The van Dam - Veltman - Zakharov discontinuity is shown to be present for any spin.