Star-Triangle relation in AdS/CFT

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Talk outline

- Review of planar, integrable AdS/CFT
- Algebraic construction of S-matrix
- Integrability encoded as star-triangle relation
- Kramers-Wannier duality
- Conclusions

Review of planar, integrable AdS/CFT

AdS/CFT:

$\mathcal{N} = 4$ SYM in $d = 4$

String theory in $AdS_5 \times S^5$

$$\frac{4\pi\lambda}{N_c} = g_S \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}$$

$$\lambda \equiv g_{YM}^2 N_c$$

Review of planar, integrable AdS/CFT

AdS/CFT: Scaling dimensions of CFT = Energies of string theory

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle \sim \delta_{ab}|x-y|^{-2\Delta_a}$$

Want to check AdS/CFT by comparing scaling dimensions and string energies

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Planar Limit: $N_c \to \infty$, λ generic

$\Rightarrow g_S = 0$

Planar gauge theory

Dilatation operator D

$$[D, \mathcal{O}_a] = -\Delta_a \mathcal{O}_a$$

$\Rightarrow$ Want to diagonalize D

Consider the family of operators $\text{Tr}\,(\chi_1 \chi_2 \cdots \chi_n)$

In the limit $N_c \to \infty$

$$[D,\ \text{Tr}\,(\chi_1 \chi_2 \cdots \chi_n)] = \sum a_k \text{Tr}(\cdots)$$

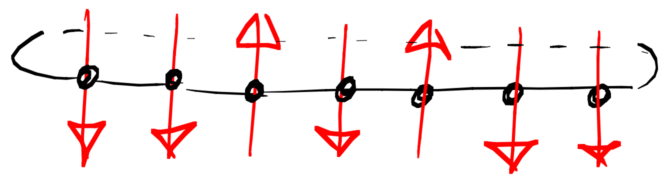
Represent $\text{Tr}(\cdots)$ as a spin chain with periodic boundary conditions

Review of planar, integrable AdS/CFT

Example: su(2), fields ϕ^1, ϕ^2

$\mathrm{Tr}\left(\phi^1\phi^1\phi^2\phi^1\phi^2\phi^1\phi^1\right) \Longleftarrow D$

$\Updownarrow \qquad\qquad \Updownarrow$

 $\Longleftarrow \mathcal{H}$

Review of planar, integrable AdS/CFT

- At 1-loop in the gauge theory $\Rightarrow$ Heisenberg spin chain

  $\Rightarrow$ Integrable, Bethe Ansatz $\Rightarrow$ Magnons $\Rightarrow$ S-matrix

- At 3-loops $\Rightarrow$ also integrable spin chain, Bethe Ansatz $\Rightarrow$ S-matrix

- Assuming integrable spin-chain for all loops

  $\Downarrow$

  Beisert: the algebra fixes the S-matrix, up to a global factor, for all values of λ

  (at least for asymptotically long spin-chains)

String theory

Planar limit $\Rightarrow g_S = 0.$

$$\sqrt{\lambda} = \frac{R^2}{\alpha'}$$

Can do perturbation theory when λ large

Compute world-sheet scattering amplitudes $\Rightarrow$ S-matrix

Beiserts S-matrix coincides with this one (at least up to 2-loop)

Algebraic construction of S-matrix

Relevant algebra $\quad \mathrm{su}(2|2) \oplus \mathrm{su}(2|2) \subset \mathrm{psu}(2,2|4)$

Study $\mathrm{su}(2|2)$ separately

Must enlarge: $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$

$$\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathfrak{P},$$

$$\{\mathfrak{S}^a{}_\alpha, \mathfrak{S}^b{}_\beta\} = \varepsilon^{ab}\varepsilon_{\alpha\beta}\mathfrak{K},$$

Fundamental representation : $\phi^1, \phi^2, \psi^1, \psi^2$

$$\mathfrak{Q}^\alpha{}_a|\phi^b\rangle = a\,\delta_a^b|\psi^\alpha\rangle,$$
$$\mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle = b\,\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b\rangle,$$
$$\mathfrak{S}^a{}_\alpha|\phi^b\rangle = c\,\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\beta\rangle,$$
$$\mathfrak{S}^a{}_\alpha|\psi^\beta\rangle = d\,\delta_\alpha^\beta|\phi^a\rangle.$$

$\Rightarrow$

$$\mathfrak{P}|\chi\rangle = ab|\chi\rangle$$
$$\mathfrak{K}|\chi\rangle = cd|\chi\rangle$$
$$\mathfrak{C}|\chi\rangle = \frac{1}{2}(ad+bc)|\chi\rangle$$

Energy

Closure: $ad - bc = 1$

Algebraic construction of S-matrix

Want to determine the 2-body $\to$ 2-body S-matrix by demanding invariance under the algebra.

Co-product: $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$[\Delta \mathcal{J}, S_{12}] = 0, \quad \forall \mathcal{J} \in \mathcal{A}$$

Assuming integrability :

$$\Delta \mathfrak{P} = \mathfrak{P} \otimes \mathcal{Z} + 1 \otimes \mathfrak{P}$$

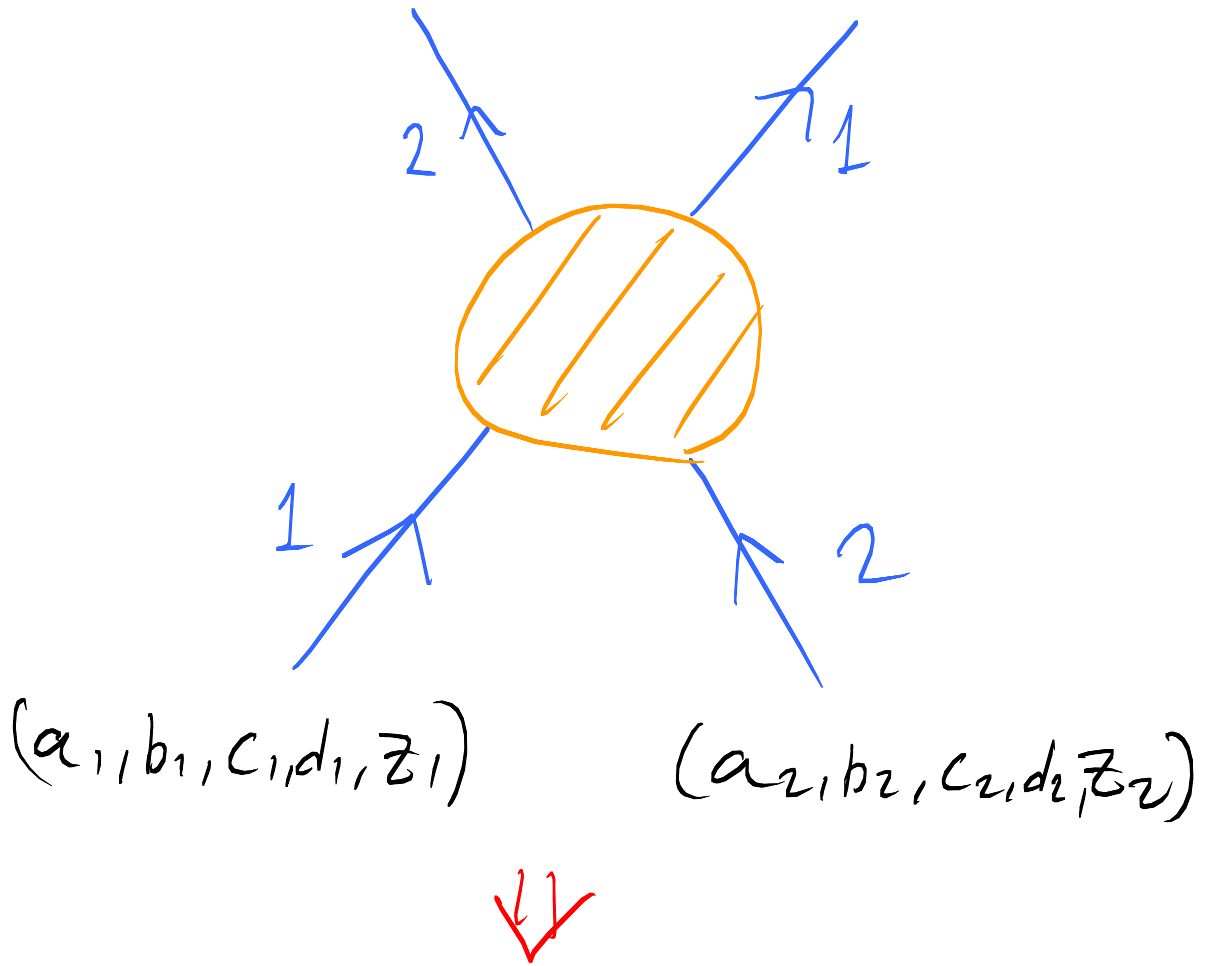
$$\Delta \mathfrak{K} = \mathfrak{K} \otimes \mathcal{Z}^{-1} + 1 \otimes \mathfrak{K}$$

$\mathcal{Z}$ is a new central charge (will see $\mathcal{Z} = e^{i\hat{P}}$) $\rightarrow$ returning z

Co-product of $\mathfrak{Q}$ and $\mathfrak{S}$ must be compatible with this

Take $\Delta \mathfrak{Q} = \mathfrak{Q} \otimes \mathcal{Z}^{1/2} + 1 \otimes \mathfrak{Q}$

Algebraic construction of S-matrix



S-matrix in terms of $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, z_1, z_2$.

(up to a global factor)

Algebraic construction of S-matrix

$\Delta_{12}\mathfrak{P} = \Delta_{21}\mathfrak{P}$ and $\Delta_{12}\mathfrak{K} = \Delta_{21}\mathfrak{K}$ $\Rightarrow$

$$a_k b_k = \alpha(1 - z_k)$$
$$c_k d_k = \beta(1 - z_k^{-1})$$

Eliminate b_k and d_k

But a_k and c_k are related via closure:

$$a_k d_k - b_k c_k = 1$$

Also, Energy $C_k = \frac{1}{2}\sqrt{1 - 16\alpha\beta(1 - z_k)(1 - z_k^{-1})}$

Compare with the SYM dispersion relation $E \sim \sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)}$

$\Rightarrow$

$$\alpha\beta = g^2$$
$$z_k = e^{ip_k}$$

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

Algebraic construction of S-matrix

Traditionally, a_k, b_k, c_k, d_k, are parameterized in terms of $x_k^\pm$, γ_k, α_B, g

$$z_k = e^{ip} = \frac{x_k^+}{x_k^-}$$

$$\alpha = g\alpha_B, \quad \beta = g/\alpha_B$$

Then $a_k d_k - b_k c_k = 1$ $\Rightarrow$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$$

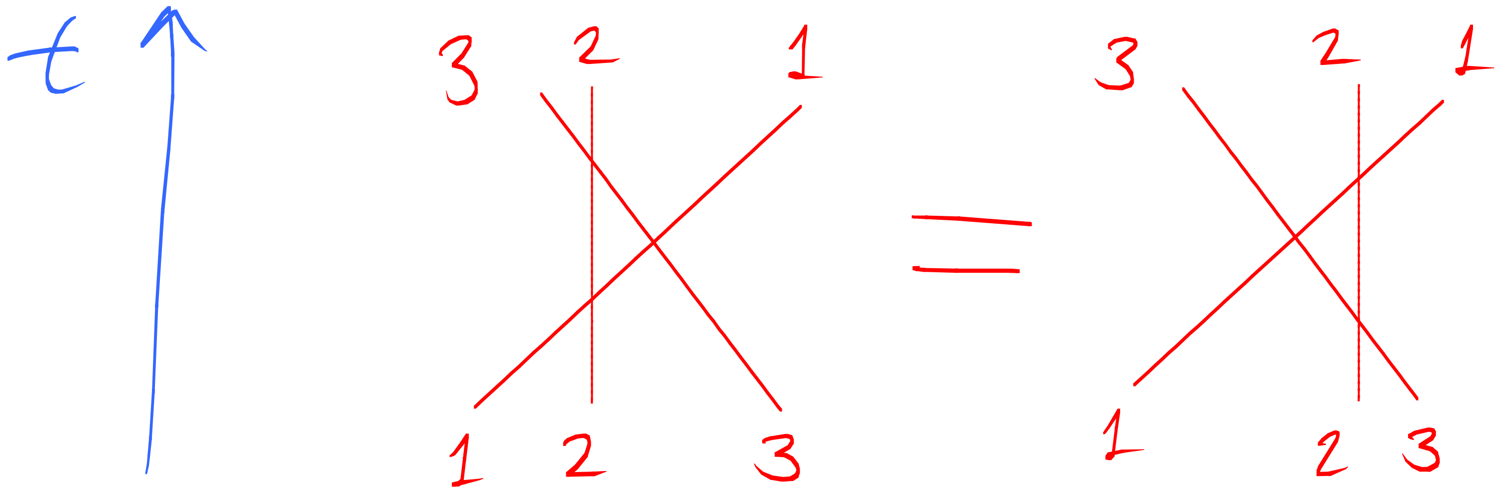
Unitarity fixes γ_k and α_B in terms of $x^\pm$ $\Rightarrow$

$$\Rightarrow S(x_1^\pm, x_2^\pm)$$

Integrability encoded as a star-triangle relation

The constructed S-matrix satisfies the Yang-Baxter equation

$$S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}$$



if $x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}$, $k = 1, 2, 3$,

but is this a necessary condition?

Integrability encoded as a star-triangle relation

Our analysis shows that

$$x_1^+ + \frac{1}{x_1^+} - x_1^- - \frac{1}{x_1^-} = x_2^+ + \frac{1}{x_2^+} - x_2^- - \frac{1}{x_2^-} = x_3^+ + \frac{1}{x_3^+} - x_3^- - \frac{1}{x_3^-}$$

But this common value is completely arbitrary!

Integrability encoded as a star-triangle relation

Integrability condition for the model:

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}, \qquad \text{for general } g \qquad (1)$$

This is a genus 1 elliptic curve.

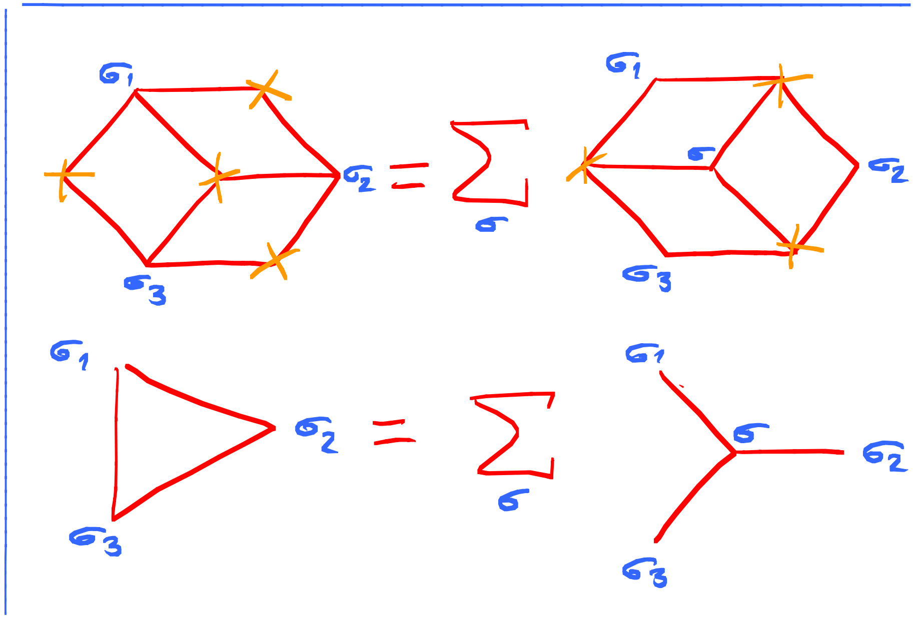
Not many known models have such an integrability condition

One example: 2D Ising model

introducing L, K by $\quad x^{\pm} = e^{-2L} e^{\pm 2K}$

$$(1) \to \qquad \sinh(2K)\sinh(2L) = k^{-1}, \quad k = 4ig$$

Ising model integrability condition = Star-triangle relation



Kramers-Wannier duality

Ising model has a well-known high-low temperature duality

$$\sinh(2K^*) = \frac{1}{\sinh(2L)}, \quad \sinh(2L^*) = \frac{1}{\sinh(2K)}, \quad k^* = k^{-1}$$

$$\sinh(2K)\sinh(2L) = k^{-1} \longrightarrow \sinh(2K^*)\sinh(2L^*) = k^{*-1}$$

In more familiar terms :

$$k = 4ig \;\Rightarrow\; g^* = -\frac{1}{16g}$$

$$e^{ip} = \frac{x^+}{x^-} = e^{4K} \;\Rightarrow\; 2K = i\frac{p}{2}$$

Define p^* such that $2K^* = i\frac{p^*}{2}$

$$x^{\pm} = e^{-2L}e^{\pm 2K}$$

$$\Rightarrow \sin\frac{p^*}{2} = 4ig\sin\frac{p}{2}$$

Kramers-Wannier duality

Curios fact: Spin chain rapidity $u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$

su(2)-sector of S-matrix is $\frac{u_1-u_2+i}{u_1-u_2-i}$

Satisfies $u^* = \frac{1}{4ig}u$

This also works in reverse

Imposing Kramers-Wannier: $u^* = h(g)u$

and 1-loop result: $u = \frac{1}{2}\cot\frac{p}{2}$

$$\Rightarrow u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}f(g), \qquad \text{where } f(g)\to 1 \text{ for } g\to 0$$

Taking the minimal result $f(g)=1$ we see that the S-matrix

is obtained completely in the su(2)-sector by imposing Kramers-Wannier.

Could be useful in proving the all-loop S-matrix on the gauge theory side.

Conclusions

- Planar ($N_c \to \infty$) AdS/CFT seems to be integrable
- The 2-body $\to$ 2-body S-matrices of the two sides of the theory are determined, up to a global phase, by the algebra and the co-algebra
- The integrability condition for the theory can be identified with a Star-Triangle relation
- A Kramers-Wannier duality can be defined
- Kramers-Wannier determines the all-loop spin-chain rapidity (up to a momentum independent factor) and thus the S-matrix (at least in the su(2) sector)