New vortices in non-commutative gauge theory

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Motivations

D-branes as classical solutions

- Very difficult to find multiple D-branes in OSFT
- N.C. field theory captures some essential feature of open string (field) theory
- How the shape of a brane emerges?
- Are there new degrees of freedom when two D-branes become coincident?

N.C. gauge theory will give a precise gauge invariant description of the above

The Setting

- Put a Dp-brane on the flat space time. (U(1) gauge theory)
- Put a B-field on two spacelike world-volume direction (n.c. U(1) gauge theory)
- Describe the Dp-2 branes obtained by tachyon condensation on the n.c. direction

Plan of the talk

- Non commutative Moyal Plane
- Non commutative scalars at infinite θ
  - Projectors as D-branes
  - Partial Isometry
- Working at finite θ
  - Gauge field and covariantization
  - Moduli and gauge invariance (moduli are positions)
- Adding commutative directions
  - Solutions by superposition of Moyal planes
- Coincident D-branes
  - New physical degrees of freedom: puffing
  - Puffed vortices probed by Wilson lines
  - Evolution of puffed vortices
- Open Problems

MOYAL PLANE

Take R^2 and substitute the commuting coordinates with the Heisemberg Algebra

$$[x^1, x^2] = 0 \quad \longrightarrow \quad [\hat{x}^1, \hat{x}^2] = i\theta$$

algebra of commuting functions $\longrightarrow$ noncommutative associative algebra of operators.

Let $\hat{A}$ be some operator, by choosing symmetric ordering I can associate a function on R^2 (Weyl Transform)

$$A(x^1, x^2) = W_{\hat{A}}(x^1, x^2) = \int dy \langle x^1 - y|\hat{A}|x^1 + y\rangle \, e^{\frac{2iyx_2}{\theta}}$$

Operator product is mapped to the Moyal product by the Weyl Transform

$$W_{\hat{A}\,\hat{B}}(x^1, x^2) = W_{\hat{A}} * W_{\hat{B}}(x^1, x^2)$$

where

$$A * B(x^1, x^2) = A(x^1, x^2) \, \mathrm{exp} \left(\overleftarrow{\partial}_i \, \theta^{ij} \, \overrightarrow{\partial}_j \right) B(x^1, x^2)$$

Relevant example

$$w = \frac{1}{\sqrt{\theta}}(x^1 + ix^2)$$
$$\longrightarrow \quad [w, \bar{w}] = 1$$
$$\bar{w} = \frac{1}{\sqrt{\theta}}(x^1 - ix^2)$$

A basis for operators is given by $|n\rangle\langle m|$, the whole noncommutative algebra can be written as

$$|n\rangle\langle m| \cdot |p\rangle\langle q| = \delta_{mp}\,|n\rangle\langle q|$$

using h.o. eigenfunctions

$$\langle x^1|n\rangle = \mathcal{N}\, H_n\left(\frac{x^1}{\sqrt{\theta}}\right)\, e^{-\frac{x_1^2}{2\theta}}$$

one finds ($r^2 = |w|^2$, $\phi = arg(w)$)

$$\Lambda_{n,m}(x_1, x_2) = W_{|n\rangle\langle m|}(x_1, x_2)$$
$$= \frac{1}{\pi\theta}\, e^{-r^2}\sqrt{\frac{n!}{m!}}\,(2r^2)^{\frac{m-n}{2}}\, e^{i(n-m)\phi} L_n^{m-n}(2r^2)$$

basis for functions vanishing at infinity which obeys

$$\Lambda_{n,m} * \Lambda_{p,q}(w, \bar{w}) = \delta_{mp}\Lambda_{n,q}(w, \bar{w})$$

In particular I get a (numerable) infinite set of moyal projectors by Weyl transforming the density operators of the h.o. eigenstates $|n\rangle\langle n|$

Non commutative solitons at large θ
(Gopakumar, Minwalla, Strominger)

Consider the toy model of a scalar with a suppressed kinetic term in 2 n.c. dimensions

$$S[\phi] = \int d^2x \left(\frac{1}{2}\partial_i\phi\partial^i\phi - V_*(\phi)\right)$$
$$= \int dw d\bar{w} \left(\partial\phi\bar{\partial}\phi - \theta V_*(\phi)\right)$$

for very large θ only the potential term survives, in operator formalism

$$S[\hat{\phi}] = -Tr\left[V(\hat{\phi})\right]$$

The action has a $U(\infty)$ gauge symmetry $\phi \to U\phi U^\dagger$, with $U^\dagger U = UU^\dagger = 1$.

Eoms: $V'(\hat{\phi}) = 0$

Let $V(z)$ be a polynomial with k extrema

$$V'(z) = N\prod_{i=1}^{k}(z - z_i),$$

then for each rank one projector $|n><n|$ I have a solution $z_i|n><n|$ whose energy is just $V(z_i)$ (the shape of the potential gives no contribution, just the extrema do).

For a rank N projector $P_N = \sum_{n=0}^{N} |n><n|$ I have the solution $z^i P_N$ whose energy is just $N V(z_i)$.

All these solutions are represented by localized lumps of width θ in the noncommutative plane.

Note that in the commutative limit this solitons is not stable as it can shrink to zero size

Partial Isometry

To map eoms to eoms it is sufficient to ask $S^\dagger S = 1$ with $\phi \to S\phi S^\dagger$, but if $SS^\dagger \neq 1$ this is not a symmetry of the action

consider for example the shift operator

$$S = \sum_n |n+1 >< n|$$

we have

$$S^\dagger S = 1 \quad but \quad SS^\dagger = 1 - |0 >< 0|$$

and more generically

$$S^n (S^\dagger)^n = 1 - P_n$$

Starting from the trivial vacuum solution

$$\phi = \phi_* I$$

with

$$V(\phi_*) = 0$$

the absolute minimum, I can simply generate a rank n solution that is just

$$\hat{\phi}_n = \phi_* S^n (S^\dagger)^n,$$

with energy

$$-S[\hat{\phi}_n] = n$$

WORKING AT FINITE θ

- Derivatives

Under Weyl transform we have

$$\partial_i f(x_1, x_2) \quad \rightarrow \quad -i\theta_{ij}[\hat{x}^j, \hat{f}]$$

So, in operator formalism, derivatives are inner operators.

Going to complex coordinates $w, \bar{w}$

$$\partial = -\theta^{-1/2}[\bar{w}, \cdot]$$
$$\bar{\partial} = \theta^{-1/2}[w, \cdot]$$

The action for scalars is

$$S[\phi] = 2\pi\theta\, Tr\left[-\frac{1}{\theta}[w,\phi][\bar{w},\phi] - V_*(\phi)\right]$$

Note that the $U(\infty)$ is broken by the kinetic terms. To restore the symmetry we add a gauge field and introduce covariant derivatives (from now on $\theta = 1$)

$$D\phi = -[C, \phi]$$
$$\bar{D}\phi = [\bar{C}, \phi]$$

where

$$C = \bar{w} + iA$$
$$\bar{C} = w - i\bar{A}$$

we have

$$D\phi \rightarrow UD\phi\bar{U}$$

provided

$$A \quad \rightarrow \quad UA\bar{U} - iU[\bar{w}, \bar{U}]$$
$$\phi \quad \rightarrow \quad U\phi\bar{U}$$

N.C. GAUGE THEORY

The connection A has a curvature

$$F_{w\bar{w}} = \partial\bar{A} - \bar{\partial}A - i[A,\bar{A}] = -i([C,\bar{C}]+1)$$

Yang Mills action

$$S = 2\pi \mathrm{Tr}(-\frac{1}{4}F_{ij}F^{ij} - [C,\phi][\overline{C},\phi] - V(\phi))$$

gauge invariance $UU^\dagger = U^\dagger U = 1$

$$\delta\phi = U\phi U^\dagger$$
$$\delta C = UCU^\dagger$$

eoms:

$$[\overline{C},[C,\phi]] - V'(\phi) = 0$$

$$[\overline{C},[\overline{C},C]] - [\phi,[\overline{C},\phi]] = 0$$

Again one can start from the vacuum solution (zero energy)

$$\phi_0 = \phi_*\, I$$
$$C_0 = \bar{w} \ \rightarrow \ F = 0$$

and use partial isometry to generate higher rank solutions (vortices+solitons)

$$\phi_n = \phi_* S^n I (S^\dagger)^n = \phi_*(1-P_n)$$
$$C_n = S^n \bar{w} (S^\dagger)^n \ \rightarrow \ F = P_n$$

The total energy is thus given by

$$E = n(1+V(\phi_*))$$

Rank n vortices have a 2n (real) dimensional moduli space.

$$C_n = \sum_{i=0}^{n-1} \alpha_i |i><i| + S^n \bar{w} (S^\dagger)^n \ \rightarrow \ F = P_n$$

N.C. WILSON LINES

To understand the meaning of the complex moduli α_i we need to introduce Wilson lines. In function language

$$W(x,\mathcal{C}) = P_* \mathrm{exp}\left(i\int_x^{x+l} d\xi^i A_i\right)$$

under a gauge tranformation ($*$–rotation)

$$W(x,\mathcal{C}) \;\rightarrow\; U(x) * W(x,\mathcal{C}) * U_*^{-1}(x+l)$$

To get some gauge invariant quantity we need to integrate over the whole NC plane (trace). It is useful to give momentum k

$$W(k,\mathcal{C}) = \int d^2x W(x,\mathcal{C}) * e^{ik\cdot x}$$

under a $*$-rotation

$$W(k,\mathcal{C}) \;\rightarrow\; \int d^2x U(x) * W(x,\mathcal{C}) * U^\dagger(x+l) * e^{ik\cdot x}$$

But $*$ multiplication with plane waves gives translation (which hence are part of the gauge group!)

$$e^{ik\cdot x} * f(x) = f(x+k\cdot\theta) * e^{ik\cdot x}$$

In particular

$$W(k,\mathcal{C}) \;\rightarrow\; \int d^2x U(x) * W(x,\mathcal{C}) * e^{ik\cdot x} * U^\dagger(x+l-k\theta)$$

by ciclicity $W(k,\mathcal{C})$ is gauge invariant if $l = k\cdot\theta$.

Open Wilson lines are gauge invariant if the lenght and the momentum are related!

In operator formalism the gauge invariant Wilson line is given by

$$W(l = k\theta) = Tr\, e^{l \cdot C} = Tr\, e^{\bar{l}C - l\bar{C}}$$

Gross-Nekrasov: This object can be used to *define* the position of the vortices in an unambiguous way (as translation are gauge symmetries the naive concept of position, for example the core of the gaussian given by the projector, is meaningless)

Evaluating W on the rank n vortex $C = \sum_{i=0}^{n-1} \alpha_i |i><i| + S^n \bar{w} (S^\dagger)^n$

$$W(l) = \sum_{i=0}^{n-1} e^{\bar{l}\alpha_i - l\bar{\alpha}_i}$$

by Fourier transforming the momentum l to position space x we find delta-function distributions centered at the moduli α

$$W(x) = \frac{1}{(2\pi)^2} \int d^2 l\, W(l)\, e^{i(l\bar{x} + \bar{l}x)} = \sum_{k=0}^{n-1} \delta(x - i\alpha_k)$$

... Other gauge invariant observables (for example fermion condensates) can be used to prove that the moduli α represent the position of the vortices.

ADDING COMMUTATIVE DIRECTIONS

Take $R^{1,p} \times R^2_\theta$ and consider the gauge system as before (U(1) with adjoint matter)

$$S = \int dz^m d\overline{w} dw (-\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M \phi D^M \phi - V(\phi))$$

that can be written using 2d operator formalism as

$$\begin{aligned} S \;=\; \int dz^m \mathrm{Tr}\Big(& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \overline{C} D^\mu C - \frac{1}{2}([C,\overline{C}]+1)^2 \\ & + \frac{1}{2} D_\mu \phi D^\mu \phi - [C,\phi][\overline{C},\phi] - V(\phi)\Big) \end{aligned}$$

This now looks like a $U(\infty)$ gauge theory with adjoint matter in $m+1$ dimensions and

$$D_\mu(\phi, C) = \partial_\mu(\phi, C) + i[A_\mu, (\phi, C)]$$

the eoms are given by

$$\begin{aligned} D_\mu D^\mu \phi + [C,[\phi,\overline{C}]] + [\overline{C},[\phi,C]] + V'(\phi) &= 0 \\ D_\mu D^\mu \overline{C} + [\overline{C},[C,\overline{C}]] + [\phi,[\overline{C},\phi]] &= 0 \\ D_\mu D^\mu C + [C,[\overline{C},C]] + [\phi,[C,\phi]] &= 0 \\ D_\mu F^{\mu\nu} - i([C, D^\nu \overline{C}] + [\overline{C}, D^\nu C] + [\phi, D^\nu \phi]) &= 0 \end{aligned}$$

again solutions can be found using the partial isometry trick,

$$\phi = \phi_*(1-P_N), \quad C(z) = \sum_{i=0}^{N-1} \alpha_i(z) |i\rangle\langle i| + S^N a^\dagger \bar{S}^N, \quad A_\mu(z) = 0$$
$$\partial_\mu \partial^\mu \alpha_i(z) = 0$$

Note that the $\alpha_i(z)$ parametrize the worldvolume of the n vortices in the transverse directions and are constrained to be of minimal volume by the D'Alembert equation (non trivial shapes given by non trivial boundary conditions at infinity which cannot be changed by finite variations of the fields)

PUFFED VORTICES

Consider a rank 2 solution with degenerated eigenvalues (only the top part is shown)

$$C_2(z) = \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha(z) \end{pmatrix}.$$

However C is not hermitian, so it does not need to be diagonalizable. The most general gauge invariant ansatz is indeed given by the (non normalized) Jordan block

$$C_2(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ 0 & \alpha(z) \end{pmatrix}.$$

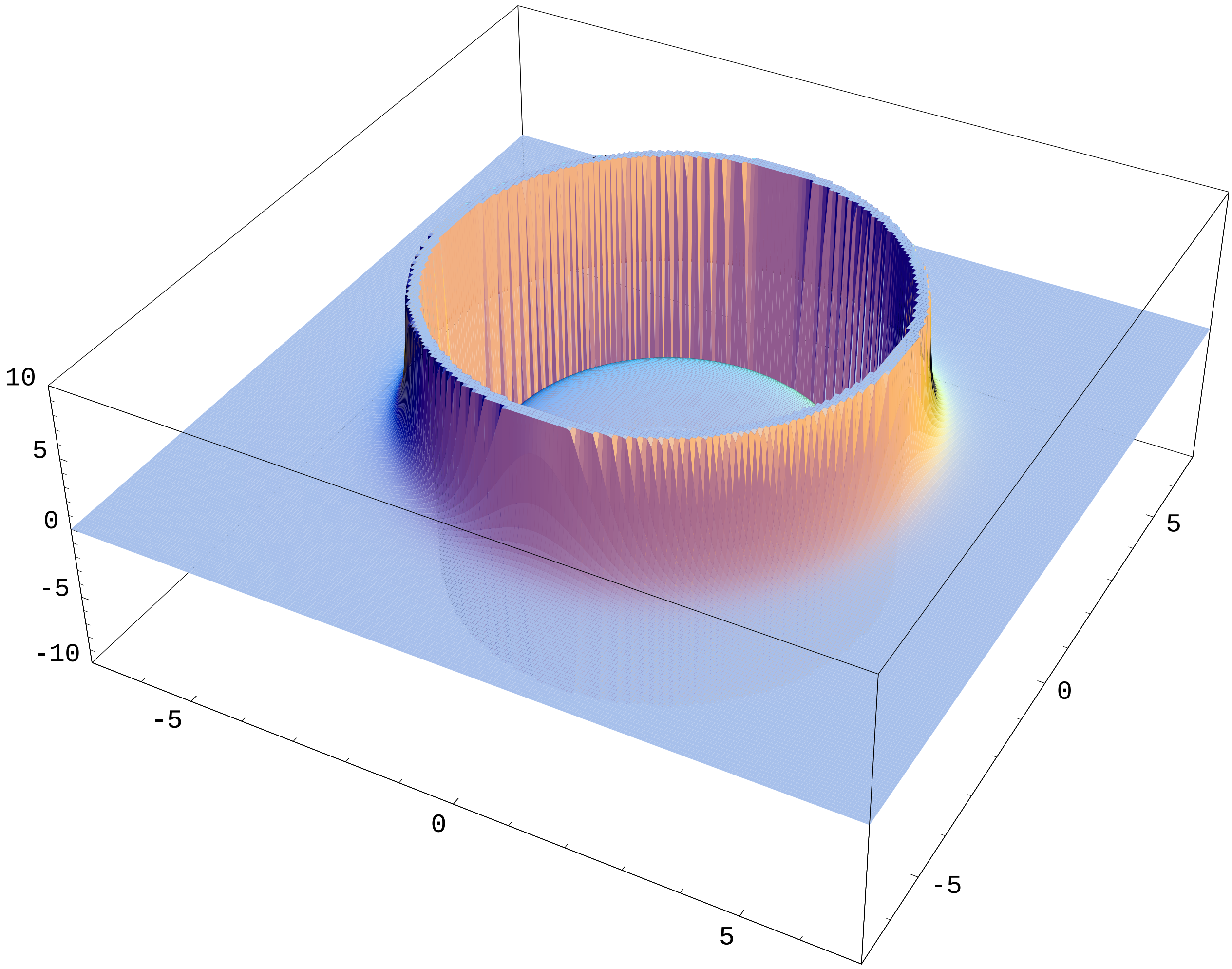
the phase of beta can be changed by gauge transformation, only the modulus is gauge invariant. The Wilson line on this background is given by

$$\begin{aligned} Tr\, e^{\bar{l}C - l\bar{C}} &= Tr\, e^{\bar{l}\alpha - l\bar{\alpha}} \cos(|l\,\beta|) \begin{pmatrix} 1 & \tan(|l\,\beta|)\frac{\bar{l}}{|l|} \\ -\tan(|l\,\beta|)\frac{l}{|l|} & 1 \end{pmatrix} \\ &= 2\, e^{\bar{l}\alpha - l\bar{\alpha}} \cos(|l\,\beta|). \end{aligned}$$

Fourier transforming to space we get

$$\begin{aligned} W(x) &= \frac{2}{(2\pi)^2} \int d^2q\, e^{i(\bar{q}x_\alpha + q\bar{x}_\alpha)} \cos(|q|\beta) \\ &= -\frac{1}{2\pi} \frac{\beta}{(\beta + r)^{\frac{3}{2}}} \left(\frac{1}{(\beta - r - i\epsilon)^{\frac{3}{2}}} + \frac{1}{(\beta - r + i\epsilon)^{\frac{3}{2}}} \right) \end{aligned}$$

the two pointlike vortices are puffed into a a shell with radius β!



Profile of a rank 2 puffed vortex as probed by Wilson lines

EVOLUTION OF THE SHELLS

This gauge invariant configuration is a solution to the equation of motion if

$$\Box_z \beta(z) = -2\beta^3(z)$$

We concentrate on two special cases

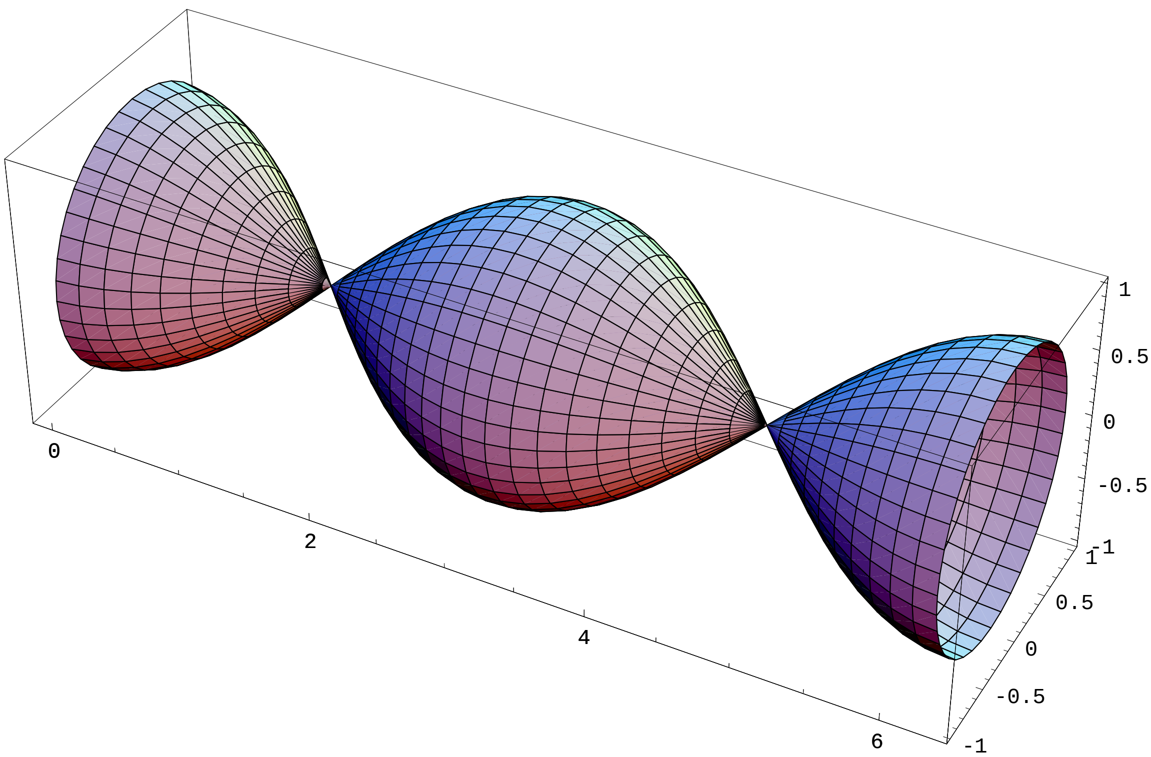
- Homogeneus time dependent solution: $\beta(z) = \beta(t)$

The equation is given by

$$\frac{d^2\beta}{dt^2} = -2\beta^3$$

The 2-parameters solution is given in term of a Jacobi Elliptic Function

$$\beta(t) = a\,\mathrm{dn}(a(t - t_0), 2)$$



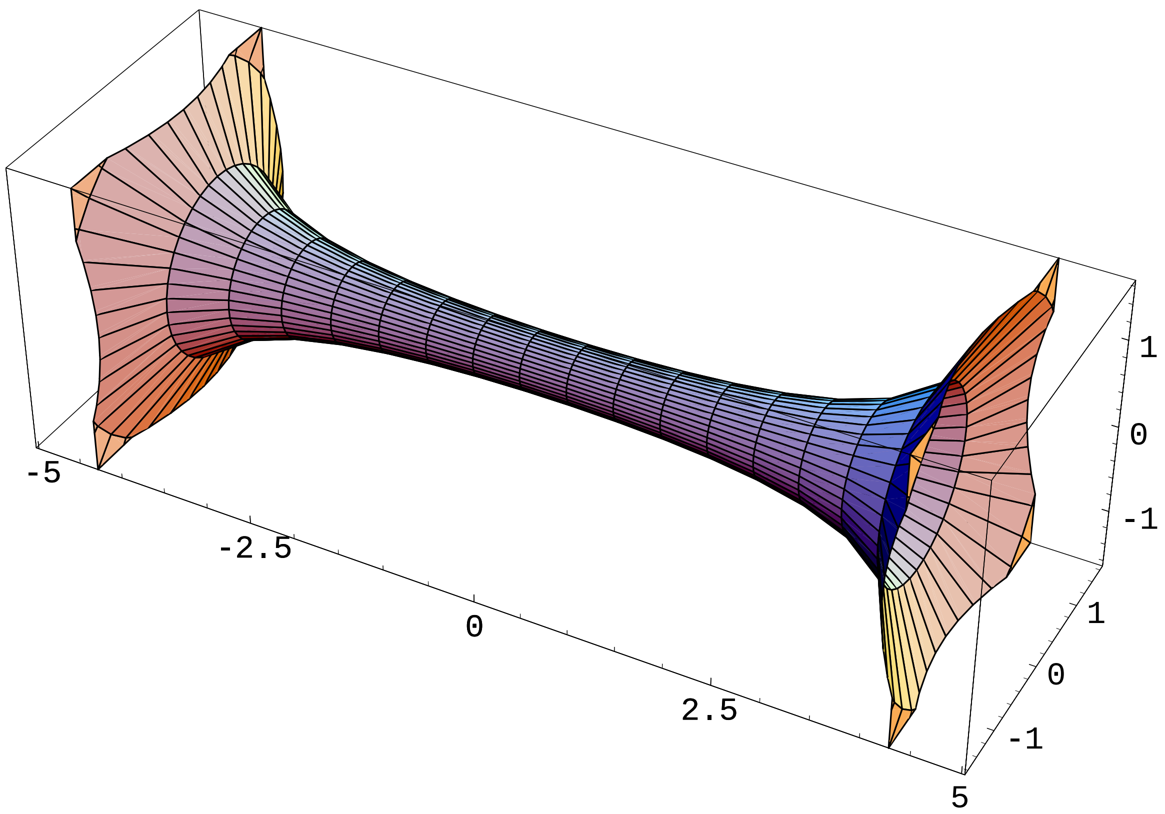
Bouncing solution

- Static solution which depends on just one space–like coordinate: $\beta(z) = \beta(x)$

$$\frac{d^2\beta}{dx^2} = 2\beta^3$$

The solution is (remarkably) given by the Wick rotation of the latter and it is still real

$$\beta(x) = b\,\mathrm{dn}(ib(x - x_0), 2)\,,$$



Vortex stretched between two domain walls

SUMMARY

- The position of n.c. vortices can be defined in a gauge invariant way by open Wilson lines
- This prescription gives to vortices minimal area world-volumes
- When two or more vortices coincide they can be ”puffed” to a shell which propagates according to the eoms
- In strict d=2 the radius of the shell is forced to vanish on shell, still it is a gauge invariant quantity
- Time dependent bouncing solutions
- Static solutions in which a vortex is stretched between two domain walls (codimension 1)

In progress

- Higher rank puffed vortices (do we need extra gauge invariant quantities to describe them?)
- The Type II case (Is this phenomenon present on the brane/anti–brane system?)