# *Universal Kounterterms in Lovelock AdS Gravity*

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3rd RTN Workshop, Valencia 2007

# Holographic Renormalization

(Henningson-Skenderis)

$$I = -\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{-G}(\hat{R}-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} d^d x\sqrt{-h}K$$

$$\Lambda = -\frac{d(d-1)}{2\ell^2}$$

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = N^2(\rho)d\rho^2 + h_{ij}(\rho,x)dx^i dx^j.$$

Radial Foliation

- For asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{g_{ij}(\rho,x)}{\rho}dx^i dx^j.$$

boundary $\rho = 0$

Fefferman-Graham expansion

$$g_{ij}(\rho,x) = g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \ldots$$

# (Dirichlet) Counterterms Method

$$I_{reg} = -\frac{1}{16\pi G}\int\limits_{M} d^{d+1}x\sqrt{-G}(\hat{R}-2\Lambda) - \frac{1}{8\pi G}\int\limits_{\partial M} d^{d}x\sqrt{-h}K + \int\limits_{\partial M} d^{d}x\mathcal{L}_{ct}\left(h, R(h), \nabla R(h)\right)$$

- Preserves general covariance.
- Dirichlet condition on the boundary metric.
- Finite Stress Tensor for AdS gravity.

$$T^{ij} = \frac{2}{\sqrt{-h}}\frac{\delta}{\delta h_{ij}}\left(I + I_{ct}\right)$$

- Conserved Quantities Definition for AAdS spacetimes.

$$
\begin{aligned}
\mathcal{L}_{ct} \quad = \quad & \frac{D-2}{\ell}\sqrt{-h} + \frac{\ell\sqrt{-h}}{2(D-3)}R + \frac{\ell^3\sqrt{-h}}{2(D-3)^2(D-5)}\left(R^{ij}R_{ij} - \frac{D-1}{4(D-2)}R^2\right) \\
& + \frac{\ell^5\sqrt{-h}}{(D-3)^3(D-5)(D-7)}\left(\frac{3D-1}{4(D-2)}RR^{ij}R_{ij} - \frac{(D-1)\,(D+1)}{16(D-2)^2}R^3\right. \\
& \left. -2R^{ij}R^{kl}R_{ijkl} - \frac{D-1}{4(D-2)}\nabla_i R\nabla^i R + \nabla^k R^{ij}\nabla_k R_{ij}\right) + \ldots
\end{aligned}
$$

- Full series for arbitrary dimension is unknown.
- Proliferation of possible terms for higher dimensions.
- They do not follow any particular pattern.
- Is there any other (more compact) counterterms series that regularizes the AdS action?

# Four-Dimensional Case

- $I_{EH} + Gauss - Bonnet$

$$I = \frac{1}{16\pi G}\int\limits_M d^4x\sqrt{-G}(\hat{R} - 2\Lambda) + \alpha\int\limits_M \mathcal{E}_4$$

- Aros, Contreras, R.O., Troncoso and Zanelli, PRL **84**, 1647 (2000). [gr-qc/9909105]

$$\begin{aligned}\mathcal{E}_4 &= \frac{1}{4}\sqrt{-G}\delta^{[\mu_1\ldots\mu_4]}_{[\nu_1\ldots\nu_4]}\hat{R}^{\nu_1\nu_2}_{\mu_1\mu_2}\hat{R}^{\nu_3\nu_4}_{\mu_3\mu_4} \\ &= \sqrt{-G}\left(\hat{R}_{\mu\nu\alpha\beta}\hat{R}^{\mu\nu\alpha\beta} - 4\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + \hat{R}^2\right)\end{aligned}$$

$$\delta I = \int\limits_{\partial M} d^{D-1}x\sqrt{-h}n_{\mu_1}\delta^{[\mu_1\ldots\mu_4]}_{[\nu_1\ldots\nu_4]}G^{\nu_2\beta}\delta\hat{\Gamma}^{\nu_1}_{\beta\mu_2}\left(\frac{1}{64\pi G}\delta^{[\nu_3\nu_4]}_{[\mu_3\mu_4]} + \alpha\hat{R}^{\nu_3\nu_4}_{\mu_3\mu_4}\right)$$

We fix $\alpha = \frac{\ell^2}{64\pi G}$ by the ALAdS (Asymptotically Locally AdS) condition

$$\boxed{\hat{R}^{\alpha\beta}_{\mu\nu} = -\frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]}}$$

- $D = 2n$ Dimensions

$$I = I_{E-H} + \alpha_{2n} \int\limits_{M} \mathcal{E}_{2n}$$

Aros, Contreras, R.O., Troncoso and Zanelli, PRD **62**, 044002 (2000).

[hep-th/9912045]

- Euler Term in $D = 2n$ Dimensions

$$\mathcal{E}_{2n} = \frac{\sqrt{-G}}{2^n} \delta^{[\nu_1 \ldots \nu_{2n}]}_{[\mu_1 \ldots \mu_{2n}]} \hat{R}^{\mu_1 \mu_2}_{\nu_1 \nu_2} \ldots \hat{R}^{\mu_{2n-1} \mu_{2n}}_{\nu_{2n-1} \nu_{2n}}$$

## Boundary Terms

$$I = I_{EH} + c_3 \int\limits_{\partial M} B_3 .$$

R.O., JHEP **0506**: 023 (2005). [hep-th/0504233]

## Euler Theorem

$$\int\limits_{M_4} \mathcal{E}_4 = 2\,(4\pi)^2\,\chi(M_4) + 2 \int\limits_{\partial M_4} B_3$$

$$B_3 = 2\varepsilon_{ABCD}\,\theta^{AB}\left(R^{CD} + \tfrac{1}{3}\,(\theta^2)^{CD}\right)$$

## Kounterterms

$$B_3 = 2\sqrt{-h}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K^{j_1}_{i_1}\,(R^{j_2 j_3}_{i_2 i_3}(h) - \tfrac{2}{3} K^{j_2}_{i_2} K^{j_3}_{i_3})$$

$$K_{ij} = -\tfrac{1}{2N}\partial_\rho h_{ij}, \quad ds^2 = N^2(\rho)d\rho^2 + h_{ij}(\rho, x)dx^i dx^j$$

$$c_3 = \frac{\ell^2}{64\pi G}$$

## D=2n Kounterterms

$$B_{2n-1} = 2n \int\limits_0^1 dt\ \sqrt{-h}\delta^{[j_1 \cdots j_{2n-1}]}_{[i_1 \cdots i_{2n-1}]} K^{i_1}_{j_1} \left(\tfrac{1}{2} R^{i_2 i_3}_{j_2 j_3}(h) - t^2 K^{i_2}_{j_2} K^{i_3}_{j_3}\right) \times \dots$$

$$\dots \times \left(\frac{1}{2} R^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}}(h) - t^2 K^{i_{2n-2}}_{j_{2n-2}} K^{i_{2n-1}}_{j_{2n-1}}\right)$$

$$c_{2n-1} = (-1)^n \frac{\ell^{2n-2}}{16 n \pi G (2n-2)!}$$

## Chern-Simons Forms

$$\boxed{< F^{n+1} >= dL_{CS}(A)}$$

$$\boxed{L_{CS}(A) = (n+1) \int_0^1 dt \quad < A F_t^n >} \qquad A_t = tA, \quad F_t = dA_t + A_t^2$$

## Transgression Forms

$$< F^{n+1} > - < \bar{F}^{n+1} > = dL_{TF}(A, \bar{A})$$

$$L_{TF}(A, \bar{A}) = (n+1) \int_0^1 dt \ \ < (A - \bar{A}) F_t^n >$$

$A_t = \bar{A} + t(A - \bar{A}), \quad F_t = dA_t + A_t^2$

# $D = 2n + 1?$

- No Topological Invariants of the Euler class in odd dimensions
- $D = 3$

$$I = -\frac{1}{16\pi G}\left[\int_M d^3x\,\sqrt{-G}\left(\hat{R} + \frac{2}{\ell^2}\right) + 2c_2\int_{\partial M} d^2x\,\sqrt{-h}K\right]$$

- Dirichlet Problem $c_2 = 1$

$$\delta I = \frac{1}{16\pi G}\int_{\partial M} d^2x\,\sqrt{-h}\left(K^{ij} - h^{ij}K\right)\delta h_{ij}\,.$$

- Balasubramanian-Kraus counterterm

$$I_{reg} = I_{Dirichlet} + \frac{1}{8\pi G}\int_{\partial M} d^2x\,\frac{\sqrt{-h}}{\ell}\,.$$

- A naive observation $c_2 = \frac{1}{2}$
  O. Miskovic and R.O., PLB **640**, 101 (2006), hep-th/0603092
  $$\delta I = \int\limits_{\partial M} \epsilon_{ab} \delta K_i^j e_j^a e_k^b dx^i \wedge dx^k + \epsilon_{ab} \left( K_i^j \delta e_j^a e_k^b - K_i^j e_j^a \delta e_k^b \right) dx^i \wedge dx^k$$

- F-G expansion
  $$h_{ij} = \frac{g_{(0)ij}}{\rho} + \ldots$$
  $$K_{ij} = \frac{1}{\ell} \frac{g_{(0)ij}}{\rho} + \ldots$$
  $$K_i^j = \frac{1}{\ell} \delta_i^j - \frac{\rho}{\ell} \left( g_{(1)} g_{(0)}^{-1} \right)_i^j + \ldots$$

- $K_i^j = \frac{1}{\ell} \delta_i^j \qquad\qquad \delta K_i^j = 0$

- factor $\frac{1}{2}$ from the C-S formulation of 3D gravity

$$I_{CS} = \left\langle AdA + \frac{2}{3}A^3 \right\rangle = I_{E-H} + \frac{1}{2} \int\limits_{\partial M} (G - H)$$

$I_{CS}$ is regularized

Banados & Mendez, PRD**58**:104014 (1998).

- $$I_{CS} = I_{Dirichlet} + \frac{1}{8\pi G\ell} \int_{\partial M} d^2x \left(\sqrt{-h} - 4\sqrt{-g_0}\, R_0\right)$$

- Five Dimensions (R.O., JHEP **0704** 073 (2007). hep-th/0610230 )

$$I=I_{EH}+c_{4}\int\limits_{\partial M}B_{4}$$

- Kounterterms

$$B_{4}=\sqrt{-h}\delta_{[j_{1}j_{2}j_{3}]}^{[i_{1}i_{2}i_{3}]}K_{i_{1}}^{j_{1}}\left(R_{i_{2}i_{3}}^{j_{2}j_{3}}-K_{i_{2}}^{j_{2}}K_{i_{3}}^{j_{3}}+\frac{1}{3\ell^{2}}\delta_{i_{2}}^{j_{2}}\delta_{i_{3}}^{j_{3}}\right).$$

$$c_{4}=\frac{\ell^{2}}{128\pi G}$$

- Conserved Charges $Q(\xi)=q(\xi)+q_{0}(\xi)$

$$\begin{aligned} q(\xi) &= \frac{\ell^{2}}{64\pi G}\int_{\Sigma}d^{3}x\sqrt{\sigma}\,u_{j}\,\xi^{i}\,\delta_{[i_{1}i_{2}\cdots i_{4}]}^{[jj_{2}\cdots j_{4}]}K_{i}^{i_{1}}\,\delta_{j_{2}}^{i_{2}}\left(\hat{R}_{j_{3}j_{4}}^{i_{3}i_{4}}+\frac{1}{\ell^{2}}\delta_{[j_{3}j_{4}]}^{[i_{3}i_{4}]}\right), \\ q_{0}(\xi) &= -\frac{\ell^{2}}{128\pi G}\int_{\Sigma}d^{3}x\,\sqrt{\sigma}\,u_{j}\,\xi^{i}\,\delta_{[i_{1}i_{2}\cdots i_{4}]}^{[jj_{2}\cdots j_{4}]}\left(\delta_{j_{2}}^{i_{2}}K_{i}^{i_{1}}+\delta_{i}^{i_{2}}K_{j_{2}}^{i_{1}}\right)\times \\ &\times\left(R_{j_{3}j_{4}}^{i_{3}i_{4}}-K_{j_{3}}^{i_{3}}K_{j_{4}}^{i_{4}}+\frac{1}{\ell^{2}}\delta_{i_{3}}^{j_{3}}\delta_{i_{4}}^{j_{4}}\right). \end{aligned}$$

- $D = 2n+1$ Dimensions

$$I = I_{EH} + c_{2n} \int\limits_{\partial M} B_{2n}$$

- Kounterterms

$$B_{2n} = 2n\sqrt{-h}\int_0^1 dt \int_0^t ds \delta^{[\dots]}_{[\dots]} K \left(\frac{1}{2}R - t^2 KK + \frac{s^2}{\ell^2}\delta\delta\right)^{n-1}$$

$$c_{2n} = -\frac{\ell^{2n-2}}{16\pi n(2n-1)!G}\left[\int_0^1 dt(t^2-1)^{n-1}\right]^{-1}$$

- Conserved Charges $Q(\xi) = q(\xi) + q_0(\xi)$

- $$L_{TF}(A,\bar{A}) = L_{CS}(A) - L_{CS}(\bar{A})) - d\Xi_{2n}\left(A,\bar{A}\right)$$

$$\Xi_{2n} = -n(n+1)\int_0^1 ds \int_0^1 dt \; s \; < A_t(A-\bar{A}) \; F_{st}^{n-1} >,$$

$$F_{st} = sF_t + s(s-1)A_t^2$$

## Lovelock gravity

$$I=\kappa\sum_{p=0}^{[(D-1)/2]}\alpha_p\, I^{(p)}$$

$$I^{(p)}=-\frac{(D-2p)!}{2^p}\int\limits_{M_D}d^Dx\,\sqrt{-G}\,\delta^{[\nu_1\cdots\nu_{2p}]}_{[\mu_1\cdots\mu_{2p}]}\,\hat{R}^{\mu_1\mu_2}_{\nu_1\nu_2}\cdots\hat{R}^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}}$$

## Einstein-Gauss-Bonnet AdS gravity

$$I_{EGB}=\frac{1}{16\pi G}\int_M d^Dx\sqrt{-G}\left[\hat{R}-2\Lambda+\alpha(\hat{R}_{\mu\nu\kappa\lambda}\hat{R}^{\mu\nu\kappa\lambda}-4\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}+\hat{R}^2)\right]$$

$\frac{1}{\ell_{eff}^2}=\frac{1\pm\sqrt{1+8\Lambda\alpha^*}}{2(D-1)(D-2)\alpha^*}$ and $\alpha^*=\alpha\frac{(D-3)(D-4)}{(D-1)(D-2)}$

## Standard regularization

$$I_{reg}=I_{EGB}+\kappa\int\limits_{\partial M}d^dx\,\beta_d+\int\limits_{\partial M}d^dx\,\mathcal{L}_{ct}(h,R(h),\nabla R(h))$$

## Kounterterms in EGB AdS gravity

$$I_{reg} = I_{EGB} + c_d \int_{\partial M} d^d x B_d$$

- G. Kofinas and R.O., PRD **74**, 084035 (2006). [hep-th/0606253]

  -$B_d$ is the same as in the EH case, but $\ell \longrightarrow \ell_{eff}$

  -coupling constant $c_d \longrightarrow c_d^{EGB}$

$$c_{2n-1}^{EGB} = (-1)^n \frac{\ell_{eff}^{2n-2}}{16n\pi G(2n-2)!} \left[1 - \frac{2\alpha}{\ell_{eff}^2}(D-2)(D-3)\right]$$

$$c_{2n}^{EGB} = -\frac{\ell_{eff}^{2n-2}}{16\pi n(2n-1)!G} \left[1 - \frac{2\alpha}{\ell_{eff}^2}(D-2)(D-3)\right] \left[\int_0^1 dt(t^2-1)^{n-1}\right]^{-1}$$

- # Kounterterms in Lovelock AdS gravity

$$I_{reg} = I_{LAdS} + c_d \int_{\partial M} d^d x B_d$$

- G. Kofinas and R.O. [arXiv:0708.0782]

  -the form of $B_d$ is universal (with the effective AdS radius $\ell_{eff}$)

  -coupling constant $c_d \to c_d^{LAdS}$

# Conclusions and Prospects

- An alternative, universal counterterms prescription
- Well-defined, finite action principle
- Correct Conserved Quantities (Vacuum Energy)
- Correct Black Hole Thermodynamics
- A general formula for a Finite Stress Tensor for AAdS Gravity?
  -Weyl anomaly
- Comparison to Standard Counterterm series?

  O. Miskovic and R.O. [arXiv:0708.0782], to appear in JHEP

  In two particular Lovelock theories (Chern-Simons-AdS/Born-Infeld-AdS), where $\mathcal{L}_{ct}$ can be obtained for any dimension. (Banados, R.O., Theisen, JHEP 0510:067,2005)