Topological Quantum Field Theories,

Hecke Algebras and Quantum Groups

by

Samson L. Shatashvili

Trinity College Dublin

and

IHES, Bures-Sur-Yvette

based on joint work with:

A. Gerasimov, hep-th/0609024

G. Moore and N. Nekrasov, hep-th/9712241

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$\underline{Topological\ \ Gauge\ \ Theories:}$

$\underline{Flat\ \ Connections}$

- Σ_h - Riemann surface of genus h
- **G** - compact group (**important**), $SU(N)$, in the rest
- $\mathcal{A}$ - space of all gauge fields on Σ
- $A \in \mathcal{A}$, F-curvature of A: $F = dA + A^2$
- $\varphi \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\mathbf{g})}$ - zero-form in $\mathrm{ad}_{\mathbf{g}}$ representation
- $\psi \in \mathcal{A}^1(\Sigma, \mathrm{ad}_{\mathbf{g}})$ - odd one-form taking values in $\mathrm{ad}_{\mathbf{g}}$

Infinitesimal gauge transformations:

$$\mathcal{L}_\alpha\, A = d\varphi + [A, \varphi]$$

$$\mathcal{L}_\alpha\, \psi = -[\varphi, \psi]$$

$$\mathcal{L}_\alpha\, \varphi = -[\alpha, \varphi]$$

In the space of fields A, φ, ψ **BRST** δ exists such that it squares to zero up to infinitesimal gauge transformations:

$$\delta^2 = -i\mathcal{L}_\varphi$$

$$\delta\, A = i\psi, \quad \delta\, \psi = -(d\varphi + [A, \varphi]), \quad \delta\, \varphi = 0,$$

Action functional:

$$\mathcal{S}_0 = \frac{1}{2\pi}\int_{\Sigma} Tr(i\varphi F + \frac{1}{2}\psi\wedge\psi)$$

Flat Connections - critical points of $\mathcal{S}_0$ with respect to φ.

Consider the (path)integral over all fields with this action functional:

$$Z_{YM}(\Sigma) = \frac{1}{(\mathcal{G}_\Sigma)}\int D\varphi\, DA\, D\psi\; e^{\frac{1}{2\pi}\int_\Sigma (i\mathrm{Tr}\,\varphi\,\mathrm{F(A)}+\frac{1}{2}\mathrm{Tr}\,\psi\wedge\psi)}$$

In this integral one also sums over all topological classes of the principal G-bundle over Σ_h.

This theory is called Topological 2d YM theory. The path integral is invariant under δ.

- **Observables - descend:** Given homogeneous invariant polynomial of $\varphi(P)$, $I(\varphi)$, local observable is given by:

$$o_I^{(0)}(P) = \mathrm{Tr}\, I(\varphi) \Rightarrow \mathcal{O}^{(0)} = \int_\Sigma (\mathrm{vol}_\Sigma) Tr I(\varphi)$$

since $o^{(0)}(P)$ is independent of position of P up to δ because:

$$do^{(k)} = -i\delta(o^{(k+1)}).$$

Non-local observables $\mathcal{O}^i, \quad \delta\mathcal{O}^i = 0$, are given by:

$$\mathcal{O}^{(0)} = \int_\Sigma (\mathrm{vol}_\Sigma) Tr I(\varphi)$$

$$\mathcal{O}^{(1)}_I = \int_\Gamma o^1 = \int_\Gamma \sum_{a=1}^{\mathrm{rank}(\mathbf{g})} \frac{\partial I(\varphi)}{\partial \varphi^a} \psi^a$$

$$\mathcal{O}^{(2)}_I = \int_\Sigma o^2 = \int_\Sigma \frac{1}{2} \int_\Sigma \sum_{a,b=1}^{\mathrm{rank}(\mathbf{g})} \frac{\partial^2 I(\varphi)}{\partial \varphi^a \, \partial \varphi^b} \psi^a \wedge \psi^b +$$

$$+ i \int_\Sigma \sum_{a=1}^{\mathrm{rank}(\mathbf{g})} \frac{\partial I(\varphi)}{\partial \varphi^a} F(A)^a$$

$\Gamma \in \Sigma$ - a closed curve on a two-dimensional surface Σ.

Physical obesrvables $\Leftrightarrow$ δ-equivariant cohomology classes.

Q: compute the correlators of these Observables

$$< \mathcal{O}^{i_1} ... \mathcal{O}^{i_n} > = \frac{1}{(\mathcal{G}_\Sigma)} \int DAD\psi D\varphi \mathcal{O}^{i_1} ... \mathcal{O}^{i_n} e^{-S_0}$$

- $\mathcal{M} : F = 0$ - space of flat connections, subspace in $\mathcal{A}$.
- $\Omega_k \in \mathrm{H}^*(\mathcal{M})$ - correspond to $\mathcal{O}^i$, one-to-one

Intersection theory on $\mathcal{M}$ - compute:

$$\int_{\mathcal{M}} \Omega_{n_1} \wedge \ldots \wedge \Omega_{n_k}$$

• Correlators of $\mathcal{O}$'s compute these intersections numbers:

$$Z(\Sigma_h; t_i, ...t_n) = \frac{1}{(\mathcal{G}_\Sigma)} \int DAD\psi D\varphi e^{-S(t_1,...,t_n)}$$

$$S(t_1, ..., t_n) = S_0 + \sum_{i=1}^{n} t_i \mathcal{O}^i$$

• Physical approach of computing the correlators - **Abelianization**

After all non-abelain components are integrated out $\Leftrightarrow$ integral over abelian fields from Cartan sub-algebra.

Abelian action - descends from invariant polynomial $I(\phi)$ and may have a Gauss-Bonet term:

$$S = \int_\Sigma \Big[\sum_k \left(\frac{\partial I(\varphi)}{\partial \varphi_k} \right) F_k + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 I(\varphi)}{\partial \varphi_i \partial \varphi_j} \right) \psi_i \wedge \psi_j \Big] +$$

$$+ \int_\Sigma T(\varphi) \frac{1}{8\pi} \mathcal{R}^{(2)}$$

$\psi_A^{ab} = \mathrm{diag}(\psi_1, \ldots, \psi_N)$, F_k is the curvature, corresponding to the k'th entry of A^{ab}, and $\mathcal{R}^{(2)}$ is the two dimensional scalar curvature.

One needs to find two functions of abelian fields $I(\varphi)$ and $T(\varphi)$. Plus - explicit form of observables in terms of abelian fields.

For 2d topological YM theory: $I(\varphi) = \sum_k \varphi_k^2, T(\varphi) = 0$

• The generating function for deformation with local observables only can be computed explicitly:

$$Z_{YM}(\Sigma_h; t_1, ..., t_n) =$$
$$= \left(\frac{\mathrm{Vol}(G)}{(2\pi)^{\dim(G)}} \right)^{2h-2} \sum_{\mu \in P_{++}} (\dim\, V_\mu)^{2-2h}\, e^{-\sum_{k=1}^{\infty} t_k\, p_k(\mu+\rho)}$$

• V_μ - unitary irreducible representation of $G = SU(N)$

• $\mu \in \mathcal{Z}_+$, $\mu_1 \geq \mu_2 \geq ... \geq \mu_N$-highest weight

• $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ is a half-sum of positive roots

• P_{++} is a subset of the dominant weights of G

Hamiltonian approach

• Consider Σ with boundary - a cylinder $R \times S^1$.

Action (after integrating out fermions):

$$S(\varphi, A) = \frac{1}{2\pi} \int dt d\sigma\ \mathrm{Tr}\left(\varphi \partial_{\mathrm{t}} \mathrm{A}_\sigma + \mathrm{A}_{\mathrm{t}}(\partial_\sigma \varphi + [\mathrm{A}_\sigma, \varphi])\right)$$

This has a form $\int dt(\sum_i p_i \dot{q}_i - \lambda \Phi(p,q))$ with:

• λ: lagrange multiplier $\to A_t$; $\sum_i \to \int_R$, $p_i \to \varphi(\sigma)$, $q_i \to A_\sigma(\sigma)$ and Φ - constraint:

$$\Phi(\varphi, A_\sigma) = \partial_\sigma \varphi + [A_\sigma, \varphi] = 0$$

So, infinite-dimensional phase space M with coordinates φ, A_σ is reduced with respect to constraint $\Phi = 0$

Action functional after this infinite-dimensional Hamiltonian reduction

$$\int_R dt \sum_i \tilde{p}_i \dot{\tilde{q}}_i = \int d^{-1}\omega$$

ω - symplectic form on reduced phase space $\tilde{M}$; $\tilde{p}_i, \tilde{q}_i$ - Darboux coordinates on $\tilde{M}$.

$\tilde{M}$ - finite-dimensional:

$$\tilde{M} = (T^* H)/W$$

H - Cartan sub-group, W - Weyl group.

Choose polarization associated with projection $\pi : T^* H \to H$.

$x = \sum_{j=1}^{\mathrm{rank}(\mathbf{g})} x_j e^j$ and $\{e^j\}$ is an orthonormal bases of Cartan sub-algebra.

The set of invariant operators descending onto $\tilde{M} \Leftrightarrow \mathrm{Ad}_G$-invariant polynomials of φ (commuting set)

$$\mathcal{O}^{(0)\,k}(\varphi) = \frac{1}{(2\pi)^k} \mathrm{Tr}\,\varphi^{\mathrm{k}} = \frac{1}{(2\pi \mathrm{i})^{\mathrm{k}}} \sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{\partial^{\mathrm{k}}}{\partial \mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}$$

trace is taken in the N-dimensional representation.

Bases of wave-functions:

$$\mathcal{O}^{(0)k}(\varphi)\Psi_\lambda(x_1,\cdots,x_N) = p_k(\lambda)\Psi_\lambda(x_1,\cdots,x_N),$$

$$\Psi_\lambda(x_{w(1)},\cdots,x_{w(N)}) = \Psi_\lambda(x_1,\cdots,x_N), \quad w \in W.$$

$\lambda = (\lambda_1,\cdots,\lambda_N)$ are elements of the weight lattice P of $\mathbf{g}$

Redefine

$$\Phi_i(x) = \Delta_G(e^{2\pi i x})\Psi_i(x)$$

Integration measure becomes a flat measure on H:

$$< \Psi_1, \Psi_2 >= \frac{1}{|W|}\int_H dx\,\overline{\Phi}_1(x)\,\Phi_2(x),$$

and wave-functions are expressed in terms of characters:

$$\Phi_{\mu+\rho}(g) = \Delta_G(e^{2\pi i x})\,{}_\mu(g), \qquad \mu \in P_{++},$$

$$\Phi_{\mu+\rho}(x) = \sum_{w\in W}(-1)^{l(w)}e^{2\pi i\,w(\mu+\rho)(x)},$$

$l(w)$ is a length of a reduced decomposition of $w \in W$.

- **Conclusion**:
  - local cohomologies $\mathcal{O}^i \Leftrightarrow$ Hamiltonians H_i
  - wave-functions $\Psi_{\{\lambda\}}(x) \Leftrightarrow$ specturm of H_i, characters
  - eigen-values - Casimirs $p_i(\lambda)$
  - Hilbert space - sum over all representations

Topological Gauge Theories:

Higgs Bundles

Higgs bundles were defined by Hitchin, more than 20 years ago, through:

$$F(A) - \Phi \wedge \Phi = 0$$
$$\nabla_A^{(1,0)}\, \Phi^{(0,1)} = 0$$
$$\nabla_A^{(0,1)}\, \Phi^{(1,0)} = 0$$

Here new "field" enters, Φ, one-form in adjoint representation of gauge group.

These are dimensional reduction of instanton equations $F^+ = 0$ in four dimensional space down to two dimensions.

We will need later its super-partner, Ψ, odd Grassman variable, also one-form in adjoint representation:

$$(\Phi, \psi_\Phi): \qquad \Phi \in \mathcal{A}^1(\Sigma, \mathrm{ad}_{\mathbf{g}}), \qquad \psi_\Phi \in \mathcal{A}^1(\Sigma, \mathrm{ad}_{\mathbf{g}})$$

the decompositions

$$\Phi = \Phi^{(1,0)} + \Phi^{(0,1)}$$
$$\psi_\Phi = \psi_\Phi^{(1,0)} + \psi_\Phi^{(0,1)}$$

will correspond to the decomposition of the space of one-forms $\mathcal{A}^1(\Sigma_h) = \mathcal{A}^{(1,0)}(\Sigma_h) \oplus \mathcal{A}^{(0,1)}(\Sigma_h)$ defined in terms of a fixed complex structure on Σ_h.

The space of the solutions has a natural hyperkähler structure and admits compatible $U(1)$ action.

• Write the QFT similar to topological Yang Mills theory where the flat connection condition is replaced by Hitchin equations for Higgs bundle.

• More general question - write the integral representation for integration over moduli space of linear hyperkähler quotients. Hitchin equation defines infinite-dimensional hyperkähler quotient.

Answer to both was given by MNS (Moore, Nekrasov and S. Sh.) in mid 90's.

Introduce extra fields φ_0 similar to φ in case of flat connections:

$$\varphi_0 \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\mathbf{g}})$$

In addition one needs to introduce more fields, φ_+, φ_-, 0-forms in adjoint representation, and their super-partners χ, odd Grassman variables also in adjoint representation:

$$(\varphi_\pm, \chi_\pm): \qquad \varphi_\pm \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\mathbf{g}}), \qquad \chi_\pm \in \mathcal{A}^0(\Sigma, \mathrm{ad}_{\mathbf{g}})$$

MNS defined Action functional and δ-operator acting on the space of these fields:

$$\boxed{\begin{gathered} S_0(\varphi_0,\varphi_\pm,A,\Phi,\psi_A,\psi_\Phi,\chi_\pm) = \\ = \frac{1}{2\pi}\int_{\Sigma_h} d^2z\,\mathrm{Tr}(\mathrm{i}\varphi_0\,(\mathrm{F(A)}-\Phi\wedge\Phi)-\mathrm{c}\Phi\wedge*\Phi+ \\ +\varphi_+\nabla_A^{(1,0)}\,\Phi^{(0,1)}+\varphi_-\nabla_A^{(0,1)}\,\Phi^{(1,0)})+fermions \end{gathered}}$$

δ-cohomology is defined through:

$$\delta A = i\psi_A, \quad \delta\psi_A = -\nabla_A\varphi_0, \quad \delta\varphi_0 = 0, \quad \delta\Phi = i\psi_\Phi$$

$$\delta\psi_\Phi^{(1,0)} = -[\varphi_0,\Phi^{(1,0)}]+c\Phi^{(1,0)}, \quad \delta\psi_\Phi^{(0,1)} = -[\varphi_0,\Phi^{(0,1)}]-c\Phi^{(0,1)},$$

$$\delta\chi_\pm = i\varphi_\pm, \quad \delta\varphi_\pm = -[\varphi_0,\chi_\pm]\pm c\chi_\pm.$$

and as before one has the action of vector field:

$$\mathcal{L}_v\varphi_\pm = \mp\varphi_\pm, \quad \mathcal{L}_v\chi_\pm = \pm\chi_\pm$$

$$\mathcal{L}_{\varphi_0}A = -\nabla_A\varphi_0, \quad \mathcal{L}_{\varphi_0}\psi_A = -[\varphi_0,\psi_A]$$

$$\mathcal{L}_{\varphi_0}\Phi = -[\varphi_0,\Phi], \quad \mathcal{L}_{\varphi_0}\psi_\Phi = -[\varphi_0,\psi_\Phi]$$

$$\mathcal{L}_{\varphi_0}\varphi_0 = 0, \quad \mathcal{L}_{\varphi_0}\varphi_\pm = -[\varphi_0,\varphi_\pm], \quad \mathcal{L}_{\varphi_0}\chi_\pm = -[\varphi_0,\chi_\pm],$$

But now, due to circle action of Hitchin there is one more vector field:

$$\mathcal{L}_v\Phi^{(1,0)} = +\Phi^{(1,0)}, \quad \mathcal{L}_v\Phi^{(0,1)} = -\Phi^{(1,0)},$$

$$\mathcal{L}_v\psi_\Phi^{(1,0)} = +\psi_\Phi^{(1,0)}, \quad \mathcal{L}_v\psi_\Phi^{(0,1)} = -\psi_\Phi^{(0,1)}.$$

δ squares to zero up to action of these two vector fields:

$$\delta^2 = i\mathcal{L}_{\varphi_0} + c\mathcal{L}_v$$

The action for YMH theory is sum of action of YM theory and δ-exact term:

$$S_{YMH}(\varphi_0, \varphi_\pm, \chi_\pm, A, \psi_A, \Phi, \psi_\Phi) = S_{YM}(\varphi_0, A, \psi_A) +$$

$$+\delta[\int_{\Sigma_h} d^2z \, \mathrm{Tr}\,(\frac{1}{2}\Phi \wedge \psi_\Phi + \varphi_+ \nabla_{\mathrm{A}}^{(1,0)} \Phi^{(0,1)} + \varphi_- \nabla_{\mathrm{A}}^{(0,1)} \Phi^{(1,0)})]$$

so it is obviously δ-invariant, as well as $\mathcal{L}_{\varphi_0}$ and $\mathcal{L}_v$ invariant.

• **Observables:** not surprisingly local δ-cohomology is generated by same operators as in YM theory as long as $c \neq 0$:

$$o_I^{(0)}(P) = \mathrm{Tr}\, I(\varphi_0) \Rightarrow \mathcal{O}^{(0)} = \int_\Sigma (\mathrm{vol}_\Sigma) Tr I(\varphi_0)$$

1-observable and 2-observable is defined exactly same way as in YM theory via descend procedure, but now with new operator δ.

Add to the action quadratic combination of scalars $\varphi_0, \varphi_\pm$:

$$S_{YMH} + \tau_1 \int_\Sigma Tr\varphi_0^2 + \tau_2 \int_\Sigma Tr\varphi_-\varphi_+$$

Bosonic part of the action after integration (elimination) of $\varphi_0, \varphi_\pm$ is:

$$\frac{1}{\tau_1^2}\int_\Sigma Tr|F(A)-\Phi\wedge\Phi|^2+\frac{1}{\tau_2^2}\int_\Sigma Tr\nabla_A^{(1,0)}\,\Phi^{(0,1)}\nabla_A^{(0,1)}\,\Phi^{(1,0)}-$$
$$-c\int_\Sigma Tr\Phi\wedge *\Phi$$

YMH theory corresponds to $\tau_1 = \tau_2 = 0$ and we see that contributions come only from solutions of:

$$F(A)-\Phi\wedge\Phi=0,\quad \nabla_A^{(0,1)}\,\Phi^{(1,0)}=0$$

Higgs bundle equations of Hitchin. We call the theory with $\tau_1 = \tau_2 = 0$ topological YMH theory.

Deform the action by adding the observables:

$$\tilde{S} = S + t_i\mathcal{O}^i$$

and evaluate the integral over all fields:

$$Z_\Sigma(t) =< e^{-\tilde{S}(t)} >$$

This integral in topological YMH theory is (equivariant) integral over moduli space of Higgs bundles and for $t_i = 0$ corresponds to the "regularized volume" of moduli space. Parameter c is equivariant $\Rightarrow$ regularization paramter.

Moduli space of Higgs bundles equivalently can be described and space of solutions of complexified flat connection:

$$F^c(A^c = A + i\Phi) = 0$$

This compexified equation is gauge invariant under complexified gauge transformations and the quitient is same as space of solutions to Hitchin equations modulo real gauge transformations.

Topological YMH theory depends on one important, equivariant parameter, c. There are two key limiting cases - $c \to \infty$ and $c \to 0$.

• $c - \infty$. c enters bosonic part of action through:

$$-c \int_{\Sigma} Tr|\Phi|^2$$

Thus for $c \to \infty$ only $\Phi = 0$ contributes and Φ together with its super-partner ψ_Φ drops out. We are left with topological YM theory for real group - case already studied.

• $c = 0$. Oposite limit, $c \to 0$, corresponds to flat connections for complexefied group thus if considerations of topological YM would apply to complexified (non-compact) groups answer should be described by representation theory of complex group. Unfortunately these are infinite-dimensional and answer should diverge - and it does, the theory is not regularized since $c = 0$.

• Topological YMH theory interpolates between representations of compact group and its complexification giving proper treatment of latter in $c \to 0$ limit.

Abelianization: abelianized action descends from functional:

$$I(\lambda) = \sum_{j=1}^{N}(\frac{1}{2}\lambda_i^2 - 2\pi n_j\lambda_j) + \sum_{k,j=1}^{N}\int_0^{\lambda_j-\lambda_k} \text{arctg}\ \lambda/c\, d\lambda$$

$$\lambda_i \equiv \varphi_0^i$$

which coincides with so called Yang functional introduced by C. N. Yang for Bethe Ansatz of NLS equation!

Last term can be written is $\lambda log\lambda$ and in that form sometimes is called Veneziano-Yankielovich superpotential.

Explicit form of the action also has non-zero Gauss-Bonet term. Bosonic part of action is:

$$S = \int_{\Sigma_h}\sum_{i=1}^{N}[(\varphi_0)_i + \sum_{j=1}^{N}\log\left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic}\right)]F(A)^i +$$

$$+\frac{1}{2}\int_{\Sigma_h}\sum_{i,j=1}^{N}\log\left(\frac{(\varphi_0)_i - (\varphi_0)_j + ic}{(\varphi_0)_i - (\varphi_0)_j - ic}\right)R^{(2)}\sqrt{g}$$

Zero observables are:

$$\mathcal{O}_k^0 = \sum_{i=1}^{N}{(\varphi_0)_i}^k$$

$$\tilde{S}(t) = S + \sum_k t_k\mathcal{O}_k^0$$

$$Z_{\Sigma_h}(t) = < e^{-\tilde{S}(t)} >$$

The generating function $Z_{\Sigma_h}(t)$ is still expressed in terms of infinite-dimensional integral over abelian fields but luckily this integral is exactly computable because of localization technique.

Define following functions, $\mu(\lambda)$, $\alpha_i(\lambda)$ and D_λ, of N real variables $\lambda_1, ..., \lambda_N$:

$$\mu(\lambda) = \det \| \frac{\partial^2 I(\lambda)}{\partial \lambda_i \, \partial \lambda_j} \|$$

$$e^{2\pi i \alpha_j(\lambda)} = \mathcal{F}_j(\lambda) \equiv e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j - ic}{\lambda_k - \lambda_j + ic}$$

Generating function is expressed in terms of these as:

$$Z_\Sigma(t) \sim \sum_{\lambda \in \mathcal{R}_N} D_\lambda^{2-2h} \, e^{-\sum_{k=1}^{\infty} t_k \; p_k(\lambda)}$$

$$D_\lambda = \mu(\lambda)^{1/2} \prod_{i<j} (\lambda_i - \lambda_j)(c^2 + (\lambda_i - \lambda_j)^2)^{1/2}$$

$$\mathcal{R}_N : \qquad e^{2\pi i \alpha_j(\lambda)} = e^{2\pi i \lambda_j} \prod_{k \neq j} \frac{\lambda_k - \lambda_j + ic}{\lambda_k - \lambda_j - ic} = 1$$

Projection to solutions of **NLS** Bethe Ansatz equation arises from following infinite sum in partitions function over topological classes of abelian connection:

$$\mu(\lambda) \sum_{(n_1, \cdots, n_N) \in \mathbf{Z}^N} e^{2\pi i \sum_j n_j \alpha_i(\lambda)} = \sum_{(\lambda_1^*, \cdots, \lambda_N^*) \in \mathcal{R}_N} \prod_j \delta(\lambda_j - \lambda_j^*)$$

C. N. Yang: this set can be enumerated by the multiplets of the integer numbers $(p_1, \cdots, p_N) \in \mathbf{Z}^N$ such that $p_1 \geq p_2 \geq \cdots \geq p_N$, $p_i \in \mathbf{Z}$. Thus, the sum in is the sum over the same set of partitions as in the case of flat connections.

• *Intersection forms on moduli space of Higgs bundels are one parameter, c, defomrations of those for flat connections.*

• *Bethe Ansatz for N-particle sector of NLS enters instead of highest weight representations of compact group.*

Hamiltonian picture

• Phase space of YMH theory for $c \neq 0$ is same as for YM theory. Bases of wave functions - deformed by c.

• For $c = 0$ local δ-cohomologies contain additional operators - phase space is T^*H^c/W instead of T^*H/W since $c = 0$ is complexified flat connection.

• Wave functions - S_N invariant functions on a torus H; equivalently the functions on $\mathbf{R^N}$ invariant under action of the semidirect product of the lattice $P_0 = \pi_1(H)$ and Weyl $W = S_N$ group $\Rightarrow$ affine Weyl group W^{aff}.

• The lattice P_0 can be interpreted as a lattice of the $\mathbf{R^N}$-valued constant connections on S^1 which are gauge equivalent to the zero connection.

• The corresponding gauge transformations act on the wave functions by the shifts $x_j \rightarrow x_j + n_j$, $n_j \in \mathbf{Z}$ of the argument of the wave functions in the chosen polarization.

• This infinte sum over π_1's is same as infinite sum in MNS partition function - sum over topological classes of A.

- **Wave-functions - periodic functions on Cartan: $\Phi^{(c)}_\lambda(x)$**
- **λ's - labels parametrizing the state in the spectrum**
- **Hamiltonians H_k- operators from δ-cohomology $\mathcal{O}^k$**

$$H_k \Phi^{(c)}_\lambda(x_1, \cdots, x_N) = p_k(\lambda) \Phi^{(c)}_\lambda(x_1, \cdots, x_N),$$

$$\Phi^{(c)}_\lambda(x_{w(1)}, \cdots, x_{w(N)}) = \Phi^{(c)}_\lambda(x_1, \cdots, x_N), \quad w \in W.$$

$$\Phi^{(c)}_\lambda(x_1, \cdots, x_j + 1, \cdots, x_N) = \Phi^{(c)}_\lambda(x_1, \cdots, x_j, \cdots, x_N)$$

$$(\lambda_1, \cdots, \lambda_N) \in \mathcal{R}_N$$

Exactly same as for **NLS** in N-particle sector! Identification ($\mathcal{D}_i$ - Dunkle operator, s_{ij} - permutation):

$$H_k = \sum_{i=1}^N \mathcal{D}_i^k; \qquad \mathcal{D}_i = -i \frac{\partial}{\partial x_i} + i \frac{c}{2} \sum_{j=i+1}^N (\epsilon(x_i - x_j) + 1) s_{ij}$$

$$H_2 = \frac{1}{2} \sum_{i=1}^N \mathcal{D}_i^2 = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \le i < j \le N} \delta(x_i - x_j)$$

$$\Phi^c_\lambda(x) = \mu(\lambda)^{-\frac{1}{2}} \sum_{w \in W} (-1)^{l(w)} \prod_{i<j} \left(\frac{\lambda_{w(i)} - \lambda_{w(j)} + ic\epsilon(x_i - x_j)}{\lambda_{w(i)} - \lambda_{w(j)} - ic\epsilon(x_i - x_j)} \right)^{\frac{1}{2}}$$

$$\times \exp(2\pi i \sum_k \lambda_{w(k)} x_k)$$

$\mathcal{O}^k = \sum_i (\phi_0)_i^k \leftrightarrow H_k \qquad \Leftrightarrow \qquad (\phi_0)_i \leftrightarrow \mathcal{D}_i$ (naively)

H_k - generators of the center of double affine Hecke algebra at degeneration (quantum deformation of affine Weyl group).

Double affine Hecke algebra - if further deform the YMH theory by one parameter, k. YM theory has one parameter deformation - Gauged G/G WZW theory $\Leftrightarrow$ replaces sum over all representations of group by sum over all representations of **KM** group: Verlinde formula.

This is also same as quantum mechanics on moduli space of flat connections - **CS** theory.

Topological YMH has gauged WZW deformation (Hall-Littlewood $\Phi^c_\lambda(x) \to$ Macdonald; **CS** for complex group G^c?):

$$Z_{\Sigma_h}(t_0 = e^c; t_1, ..., t_N) = \sum_{\lambda \in \mathcal{R}_q} D_q(\lambda)^{2-2h} e^{-\sum_{i=1}^N t_i p_i(\lambda)}$$

$$D_q(\lambda) = \mu_q^{-1/2}(\lambda) \prod_{i<j} (q^{1/2(\lambda_i - \lambda_j)} - q^{1/2(\lambda_j - \lambda_i)}) \times$$

$$\times \prod_{i<j} |t_0 q^{1/2(\lambda_i - \lambda_j)} - q^{1/2(\lambda_j - \lambda_i)}|$$

$$\mathcal{R}_q : \quad e^{2\pi i \beta_j(\lambda)} = e^{2\pi i \lambda_j (k+N)} \prod_{i \neq j} \frac{t_0 e^{2\pi i(\lambda_i - \lambda_j)} - 1}{t_0 e^{2\pi i(\lambda_j - \lambda_i)} - 1} = 1$$

$$\mu_q = det || \frac{\partial \beta_i(\lambda)}{\partial \lambda_j} ||, \qquad q = e^{\frac{2\pi i}{k+N}}$$

$\mathcal{R}_q$ - Bethe Ansatz equation for XXZ spin chain at $s \to \infty$.

• All this has further interpretation in terms of 4d topological gauge theory where Bethe Ansatz equations have interpretation in terms of Seiberg-Witten pre-potential.