Dilatation Operator

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Outline

Related Publications

- based on: JHEP09(2007)025
- BG: hep-th/0410010 (with S. Bellucci), hep-th/0506186, hep-th/0508056, hep-th/0608028, hep-th/0611274, hep-th/0701089

AdS/CFT correspondence

AdS/CFT correspondence (planar limit: $N \to \infty$):

$$(\mathcal{N} = 4 \text{ SYM})_{\mathcal{M}_{1,3}} \Leftrightarrow (\text{string theory})_{\mathrm{AdS}_5 \times S^5}$$

Recorded a considerable progress since '97. Can be mapped to a spin chain. Integrability... [Minahan-Zarembo, Staudacher-Beisert,...]
Can be extended to include $1/N$ corrections, which correspond to string interactions. Still can be mapped to a spin system: dynamical model, but no integrability!
Available tools: Statistical physics/Thermodynamics

Object of study: Dilatation operator/Mixing matrix
[Beisert-Kristjansen-Plefka-Semenoff-Staudacher]

Two approaches: $\left\{ \begin{array}{c} \text{Spin} \\ \text{Matrix} \end{array} \right\}$ parametrization.

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Spin vs. Matrix approach

Spin description:

- ✓ The planar limit is easy
- ✓ (Planar) Integrability is manifest
- ✗ Non-planar corrections are difficult
- ✗ Semi-classical limit is difficult

Matrix description:

- ✗ Planar limit requires some effort
- ✗ No integrability for any finite matrix size
- ✓ Non-planarity is natural
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Taking into account the non-planar contribution to AdS/CFT requires matrix model approach.

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  RG-flow $\Leftrightarrow$ Dynamics
- ...in terms of a differential operator can be directly interpreted as a Hamiltonian of QM system ➠ Matrix Model description.
- ...is well studied in (mostly compact sectors of) $\mathcal{N} = 4$ SYM.
- ...plays key role in theories with conformal invariance: $\mathcal{N} = 4$ SYM, β-deformed versions, Quiver theories as well as in ordinary gauge theories e.g. QCD.
- ...we need to know it for a generic gauge theory

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To propose a recipe to construct the dilatation operator as a differential operator acting on the space of normal symbols of local composite operators.

The procedure should...

- ...apply to renormalizable models with massless fields
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The Setup [Polyakov]

"Alphabet": $\{\phi_A\}$ localized fields and their derivatives in SYM:
$\{W_A\} = \{F_{\mu\nu}, \phi, \psi, \nabla F, \nabla\phi, \nabla\psi \dots\}$
"Language": (gauge invariant) combinations of letters
"Words": simplest gauge invariants, one-trace composite operators,

$$\mathcal{O}_{A_1 A_2 \dots A_L} = \operatorname{tr} W_{A_1} W_{A_2} \dots W_{A_L}$$

"Phrases":

$$\mathcal{O}_{A_1 A_2 \dots A_{L_1}} \mathcal{O}_{B_1 B_2 \dots B_{L_2}} \dots \mathcal{O}_{C_1 C_2 \dots C_{L_r}}$$

"Dual states":

$$\check{\mathcal{O}} = \mathcal{O}\Big|_{\phi \to \check{\phi}} \qquad \check{\phi}(x) \equiv \frac{\partial}{\partial\phi(x)}$$

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Renormalization & Operator mixing

Dimension is given by the renormalization scale.
Consider a set of composite local operators $\mathcal{O}_J$ closed under renormalization (mixing)

$$\mathcal{O}_J^{Ren} = Z(\Lambda)_J{}^I \mathcal{O}_I$$

Anomalous part of Dilatation Operator

$$\Delta = -Z^{-1} \cdot \frac{\partial Z(\Lambda)}{\partial \log \Lambda}$$

Anomalous dimensions

$$\Delta \mathcal{O}_\lambda = \lambda \mathcal{O}_\lambda$$

General Construction

We have to analyze the RG-transformation of a composite operator $\mathcal{O}$ in perturbation theory. Mixing matrix $Z(\Lambda)$ can be found considering divergent terms in correlators of $\mathcal{O}$ with another probe operator $\mathcal{O}'$,

$$\langle : \mathcal{O}'_y(\phi) :: \mathcal{O}_0 : \rangle = \langle : \mathcal{O}'_y : \mathrm{e}^{-\int :V(\phi):} : \mathcal{O}_0 : \rangle_0$$

The source of relevant divergences is the Wick expansion of products

$$e^{-\int :V(\phi):} : \mathcal{O}_0 : = \left(1 - \int : V(\phi) : + \frac{1}{2!} \iint : V(\phi) :: V(\phi) : + \dots \right) : \mathcal{O}_0 :$$

So, we should modify $\mathcal{O}_0$ in such a way to cancel divergences and find the scale dependence after the cancelation.

Tools

Wick expansion in functional form can be cast into [see Kleinert]

$$:\mathcal{O}'_y :: \mathcal{O}_x := \mathrm{e}^{\check{\phi}_{Ay} D_{AB}(y-x) \check{\phi}_{Bx}} \mathcal{O}'_y \mathcal{O}_x \equiv \mathcal{O}' * \mathcal{O}(x,y)$$

$*$ — star product resembles one in noncommutative theories!

$$\check{\phi}_{Ax} = \frac{\partial}{\partial \phi_A(x)}$$ Not a functional derivative!

e.g. Euclidean massless propagator for a scalar field,

$$D_{ab}(x-y) = \frac{1}{4\pi^2} \frac{\delta_{ab}}{(x-y)^2}$$

Functional Wick expansion can be generalized to the product of 3, 4,. . . factors

Tools

Differential regularization/renormalization scheme in real space allows to regularize singular expressions like [Freedman-Johnson-Latorre],

$$\frac{1}{x^{2k}} = -\frac{1}{4^{k-1}(k-1)!(k-2)!}\Box^{k-1}\frac{\ln \mu^2 x^2}{x^2}, \qquad k \geq 2$$

introduces a scale dependence:

$$\mu\frac{\partial}{\partial\mu}\left[\frac{1}{x^{2k}}\right]_{\mathrm{reg}} \equiv \left[\frac{1}{x^{2k}}\right] = \frac{8\pi^2}{4^{k-1}(k-1)!(k-2)!}\Box^{k-2}\delta(x)$$

where we used the property

$$\Box\frac{1}{x^2} = -4\pi^2\delta(x)$$

One vertex level

Regularizing the terms in the ▸ Wick Expansion we get for the first order in interaction potential

$$-\int \mathrm{d}y\,[V_{\mathrm{int}}(y)*] = -\int \mathrm{d}y\,\left[\mathrm{e}^{\check{\phi}_y\cdot D_y\cdot\check{\phi}}\right] V_y$$
$$= -\int \mathrm{d}y\left(\check{\phi}_y\cdot[D_y]\cdot\check{\phi} + \frac{1}{2}(\check{\phi}_y\otimes\check{\phi}_y)\cdot[D_y\otimes D_y]\cdot(\check{\phi}\otimes\check{\phi})\right.$$
$$\left. + \frac{1}{3!}(\check{\phi}^{\otimes 3})\cdot[D_y^{\otimes 3}]\cdot(\check{\phi}^{\otimes 3}) + \frac{1}{4!}(\check{\phi}^{\otimes 4})\cdot[D_y^{\otimes 4}]\cdot(\check{\phi}^{\otimes 4}) + \dots\right) V_y,$$

One vertex level

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\begin{aligned}
-\int \mathrm{d}y\,[V_{\mathrm{int}}(y)*] &= -\int \mathrm{d}y\,\left[\mathrm{e}^{\check{\phi}_y\cdot D_y\cdot\check{\phi}}\right] V_y \\
&= -\int \mathrm{d}y\,\Bigg(\frac{1}{2}(\check{\phi}_y\otimes\check{\phi}_y)\cdot[D_y\otimes D_y]\cdot(\check{\phi}\otimes\check{\phi}) \\
&\quad + \frac{1}{3!}(\check{\phi}^{\otimes 3})\cdot[D_y^{\otimes 3}]\cdot(\check{\phi}^{\otimes 3}) + \frac{1}{4!}(\check{\phi}^{\otimes 4})\cdot[D_y^{\otimes 4}]\cdot(\check{\phi}^{\otimes 4}) \Bigg) V_y,
\end{aligned}
$$

Two vertex level

Second level yields

$$\frac{1}{2!}\int \mathrm{d}y_1 \int \mathrm{d}y_2 [V_{\mathrm{int}}(y_1) * V_{\mathrm{int}}(y_2) *]$$

$$= \frac{1}{2}\int \mathrm{d}y_1 \int \mathrm{d}y_2 \times$$

$$\Big\{ (\check{\phi}_{y_1} \otimes \check{\phi}_{y_1} \otimes \check{\phi}_{y_2}) \cdot [D_{y_1} \otimes D_{y_1 - y_2} \otimes D_{y_2}] \cdot (\check{\phi} \otimes \check{\phi}_{y_2} \otimes \check{\phi}) +$$

$$(\check{\phi}_{y_1}^{\otimes 3} \otimes \check{\phi}_{y_2}) \cdot [D_{y_1}^{\otimes 2} \otimes D_{y_1 - y_2} \otimes D_{y_2}] \cdot (\check{\phi}^{\otimes 2} \otimes \check{\phi}_{y_2} \otimes \check{\phi}) + \dots \Big\} V_{y_1} V_{y_2} .$$

Example: Compact sector of $\mathcal{N}=4$ SYM at one loop

The compact sector is generated by scalar fields only and no derivatives,

$$\mathcal{O}_{a_1\dots a_{L_1};b_1\dots b_{L_2};\dots} = \mathrm{tr}\,\phi_{a_1}\dots\phi_{a_{L_1}}\,\mathrm{tr}\,\phi_{b_1}\dots\phi_{b_{L_2}}\dots$$

The one-loop part comes from the first term in of the 1st level Wick expansion:

$$\int \mathrm{d}y\,[V_{\mathrm{int}}(y)*] = \frac{1}{8\pi^2}\check{\delta}^2 V_{\mathrm{int}}(\phi)$$

where

$$\check{\delta}f(\phi) = \check{\phi}_a(f(\phi))\check{\phi}_a, \quad V(\phi) = \frac{g^2}{4}\,\mathrm{tr}[\phi_a,\phi_b]^2$$

Conclusions

- Non-planar gauge string duality can be described using matrix models if we know the mixing matrix as a differential operator
- The problem can be solved by (almost) no Feynman diagram computation of at each order
- Simple examples give perfect agreement with known results
- more work to be done: two loops, application to more explicit examples etc

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