Towards 5D curved projective superspace

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and work in progress

Outline

Introduction and motivations

- In studying supersymmetric theories a useful tool, beside a natural framework, is the use of superspace techniques.

In the case of SUSY with 8 real supercherges two off-shell formalisms in superspace used:

- Harmonic superspace (HS) [A. S. Galperin, E. A. Ivanov, S. N. Kalitsyn, V. Ogievetsky, E. Sokatchev (1984)], very powerful for quantum computations ($\mathcal{N}=2,4$ SYM)
- Projective superspace (PS) [A. Karlhede, U. Lindström, M. Roček (1984)], useful in studying sigma models and explicit construction of hyper-Kähler and quaternionic-Kähler metrics

Superspace SUGRA with 8 supercharges?

- SUGRA in harmonic superspace
  [A. S. Glaperin, N. A. Ky, E. Sokatchev (1987)];
  [A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, E. Sokatchev (1987)];
  has many unclear ingredients, only prepotential theory
- SUGRA in projective superspace is actually unknown

How to increase our understanding?
For AdS^5/CFT$_4$, brane-world and extra-dimensions physics:
5D superspace SUGRA. $\Longrightarrow$ Strategy:

- Study simplest curved case: 5D $\mathcal{N}=1$ AdS superspace (AdS$^{5|8}$)
- Then 5D $\mathcal{N}=1$ SUGRA
- Generalize to SUGRA with eight supercherges D$\leq$6

Here we will focus on projective superspace

5D $\mathcal{N}=1$ anti-de Sitter supergeometry

- $AdS^{5|8}$=SU(2,2|1)/SO(4,1)×U(1)
- Holonomy group: SO(4,1)×U(1)

Covariant derivatives:

Parametrization: $z^{\hat{M}}=(x^{\hat{m}},\theta^{\hat{\mu}}_i)$, $\hat{m}=0,..,4$, $\hat{\mu}=1,..,4$, $i=\underline{1},\underline{2}$

$$\mathcal{D}_{\hat{A}}=E_{\hat{A}}\;+\;\Phi_{\hat{A}}\,J\;+\;\frac{1}{2}\,\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}\,M_{\hat{b}\hat{c}}$$

$E_{\hat{A}}=E_{\hat{A}}{}^{\hat{M}}(z)\partial_{\hat{M}}$ — supervielbein

$M_{\hat{b}\hat{c}}$, $\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}$ — Lorentz $\mathrm{SO}(4,1)$ generators & connections

J, $\Phi_{\hat{A}}$ — U(1) antiHermitian generator & connection

$$[J,\mathcal{D}^i_{\hat{\alpha}}]=\mathrm{J}^i{}_j\mathcal{D}^j_{\hat{\alpha}}\;,\quad[M_{\hat{\alpha}\hat{\beta}},\mathcal{D}^i_{\hat{\gamma}}]=\tfrac{1}{2}\Big(\varepsilon_{\hat{\gamma}\hat{\alpha}}\mathcal{D}^i_{\hat{\beta}}+\varepsilon_{\hat{\gamma}\hat{\beta}}\mathcal{D}^i_{\hat{\alpha}}\Big)\;,$$

$$M_{\hat{\alpha}\hat{\beta}}=\tfrac{1}{2}(\Sigma^{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}}M_{\hat{a}\hat{b}}\;,\qquad\mathrm{J}^{ij}=\mathrm{J}^{ji}\;,\qquad\overline{\mathrm{J}^{ij}}=\mathrm{J}_{ij}$$

Algebra of covariant derivatives

$$
\begin{aligned}
\{\mathcal{D}^i_{\hat\alpha},\mathcal{D}^j_{\hat\beta}\} &= -2\mathrm{i}\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta}+3\mathrm{i}\omega\varepsilon^{ij}\varepsilon_{\hat\alpha\hat\beta}J+4\mathrm{i}\omega\mathrm{J}^{ij}M_{\hat\alpha\hat\beta}\ ,\\
[\mathcal{D}_{\hat a},\mathcal{D}^i_{\hat\beta}] &= \frac{1}{2}\omega\mathrm{J}^i{}_j(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma}\ , \qquad \omega=\mathrm{const}\ ,\\
[\mathcal{D}_{\hat a},\mathcal{D}_{\hat b}] &= -\omega^2\mathrm{J}^2M_{\hat a\hat b}\ , \qquad \mathrm{J}^2=-\frac{1}{2}\mathrm{J}^i{}_j\mathrm{J}^j{}_i>0\ ,
\end{aligned}
$$

- Find it from coset construction
- Or, ansatz so that: (i) Torsion covariantly constant; (ii) $\mathrm{SO}(4,1)\times\mathrm{U}(1)$ belongs to the automorphism group

$\Rightarrow$ Bianchi identities determine the algebra

Note: $-\omega^2\mathrm{J}^2$ is the constant negative curvature of AdS^5

Killing supervectors

- To costruct off-shell multiplets and action principles it is necessary to study the isometry group of $AdS^{5|8}$.
- The isometry group $\mathrm{SU}(2,2|1)$ is generated by those supervector fields $\xi^{\hat{A}}(z)E_{\hat{A}}$ which enjoy the property

$$\xi \equiv \xi^{\hat{A}}\mathcal{D}_{\hat{A}} = \xi^{\hat{a}}\mathcal{D}_{\hat{a}} + \xi^{\hat{\alpha}}_i\mathcal{D}^i_{\hat{\alpha}} = -\frac{1}{4}\xi^{\hat{\alpha}\hat{\beta}}\mathcal{D}_{\hat{\alpha}\hat{\beta}} + \xi^{\hat{\alpha}}_i\mathcal{D}^i_{\hat{\alpha}} ,$$

$$\delta_\xi\mathcal{D}_{\hat{A}} = -[(\xi + \rho J + \Lambda^{\hat{\beta}\hat{\gamma}}M_{\hat{\beta}\hat{\gamma}}), \mathcal{D}_{\hat{A}}] = 0 ,$$

fix $\xi^{\hat{a}}$, $\xi^{\hat{\alpha}}_i$, ρ, $\Lambda^{\hat{\beta}\hat{\gamma}}$ via the Killing supervector quations

- Isometry transformations of matter superfields on $\mathrm{AdS}^{5|8}$

$$\delta_\xi\chi \;\; = \;\; -(\xi + \rho J + \Lambda^{\hat{\alpha}\hat{\beta}}M_{\hat{\alpha}\hat{\beta}})\chi ,$$

all using covariant derivatives and U(1), SO(4,1) generators

Analytic subspace

- To study multiplets the most interesting part of the algebra is:

$$\{\mathcal{D}^i_{\hat\alpha},\mathcal{D}^j_{\hat\beta}\} = -2\mathrm{i}\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta}+3\mathrm{i}\omega\varepsilon^{ij}\varepsilon_{\hat\alpha\hat\beta}J+4\mathrm{i}\omega\mathrm{J}^{ij}M_{\hat\alpha\hat\beta}$$

- Introduce more structure: isospinors $u^{\pm}_i$ *inert* under J,

$$\{\mathcal{D}^+_{\hat\alpha},\mathcal{D}^+_{\hat\beta}\}=4\mathrm{i}\omega\mathrm{J}^{++}M_{\hat\alpha\hat\beta}\ ,$$

$$(u^+u^-)\equiv u^{+i}u^-_i\neq 0\ ,\quad \mathcal{D}^{\pm}_{\hat\alpha}\equiv u^{\pm}_i\mathcal{D}^i_{\hat\alpha}\ ,\quad \mathrm{J}^{\pm\pm}\equiv u^{\pm}_iu^{\pm}_j\mathrm{J}^{ij}$$

when Q is a Lorentz scalar superfield, consistently impose

$$\mathcal{D}^+_{\hat\alpha}Q(z,u^{\pm}) = 0\ , \qquad\qquad \text{analyticity condition}$$

Depending on the choice of $u^{\pm}_i$, and the properties of Q, such as his $u^{\pm}$ dependance, we will have HS or PS.

AdS$^{5|8}$ Projective Superspace

- Now consider the isospinors $u_i^{\pm}$ with $(u_i{}^-\,,\,u_i{}^+)\in \mathrm{GL}(2,\mathbb{C})$
- Projective superfields: *analytic* and *depends only on* u_i^+

$$\mathcal{D}^+_{\hat\alpha}Q^{(n)}(z,u^+)=0\ ,\quad D^{++}Q^{(n)}=0\ ,\quad z\in \mathrm{AdS}^{5|8}$$
$$\delta_\xi Q^{(n)}=-\big(\xi+\rho J\big)Q^{(n)}$$

- Need u_i^-: $(u^+u^-)\neq 0$ to consistently realize the action of J:

$$J\,Q^{(n)}=-\frac{1}{(u^+u^-)}\Big(\mathrm{J}^{++}D^{--}-n\,\mathrm{J}^{+-}\Big)Q^{(n)}\ ,$$
$$\frac{\partial}{\partial u^{-i}}J\,Q^{(n)}=0\ ,\ \Leftrightarrow\ Q^{(n)}(z,c\,u^+)=c^nQ^{(n)}(z,u^+)\ ,\quad c\in\mathbb{C}^*$$
$$D^{++}=u^{+i}\frac{\partial}{\partial u^{-i}}\ ,\quad D^{--}=u^{-i}\frac{\partial}{\partial u^{+i}}$$

- $Q^{(n)}(z,u^+)$ is a homogeneous function of u^+ of degree n
  $\Longrightarrow$ is a tensor field over $\mathbb{C}P^1$ $(u^+\sim cu^+)$
- we define $Q^{(n)}(z,u^+)$ as a projective multiplet of weight n

Important: no smoothness on the u^+ dependance of $Q^{(n)}(z,u^+)$

More on multiplets in projective superspace

- Restrict to the north chart of $\mathbb{CP}^1$: $u^{+\underline{1}} \neq 0$

$$Q^{(n)}(u^{+i}) = (u^{+\underline{1}})^n Q^{[n]}(\zeta) \ , \quad Q^{[n]}(\zeta) \equiv Q^{(n)}(1,\zeta)$$
$$Q^{[n]}(z,\zeta) = \sum_{k=-\infty}^{+\infty} Q_k^{[n]}(z)\zeta^k \ ,$$

where $Q_k^{[n]}(z)$ are in general unconstrained AdS$^{5|8}$ superfields.

- Now analyticity condition $\mathcal{D}^+_{\hat{\alpha}} Q^{(n)} = 0$:

$$\mathcal{D}^{\underline{2}}_{\hat{\alpha}} Q^{[n]}(\zeta) = \zeta \mathcal{D}^{\underline{1}}_{\hat{\alpha}} Q^{[n]}(\zeta) \ , \quad \mathcal{D}^{\underline{2}}_{\hat{\alpha}} Q_k^{[n]} = \mathcal{D}^{\underline{1}}_{\hat{\alpha}} Q_{k-1}^{[n]} \ .$$

If the expansion of $Q^{[n]}(\zeta)$ terminates from below or above then some $Q^{[n]}(\zeta)$ became constrained.

Classification of multiplets in terms of the beahviour of the series.

Projective action principle (flat case)

- In 5D flat PS: [S.M. Kuzenko (2006)] generalizing 4D PS [A. Karlhede, U.Lindström, M. Roček (1984)], [W. Siegel (1985)]

$$(\hat{D}^-)^4 = -\frac{1}{96}\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}D^-_{\hat\alpha}D^-_{\hat\beta}D^-_{\hat\gamma}D^-_{\hat\delta} \ ,$$
$$S = -\frac{1}{2\pi}\oint\frac{u^+_i \mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x(\hat{D}^-)^4 L^{++}(z,u^+)\Big| \ ,$$
$$D^+_{\hat\alpha}L^{++} = 0 \ , \quad L^{++}(z,cu^+) = c^2 L^{++}(z,u^+) \ ,$$

Invariant under projective transformations (S independ of u^-)

$$({u_i}^-\,,\,{u_i}^+) \ \rightarrow \ ({u_i}^-\,,\,{u_i}^+)\,R\ , \quad R \ = \ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$$

Projective action principle in AdS$^{5|8}$ (I)

- In AdS$^{5|8}$: $\mathcal{D}^+_{\hat\alpha}\mathcal{L}^{++}=0$, $\mathcal{L}^{++}(z,cu^+)=c^2\mathcal{L}^{++}(z,u^+)$, Ansatz ($(\hat{\mathcal{D}}^-)^2\equiv\mathcal{D}^{\hat\alpha -}\mathcal{D}^-_{\hat\alpha}$, $e=\det(e_{\hat m}{}^{\hat a})$):

$$
\begin{aligned}
S &= -\frac{1}{2\pi}\oint\frac{u^+_i\mathrm{d}u^{+i}}{(u^+u^-)^4}\int\mathrm{d}^5x\,e\Big[(\hat{\mathcal{D}}^-)^4 \\
&\qquad +\beta_1\,\omega\mathrm{J}^{--}(\hat{\mathcal{D}}^-)^2+\beta_2\,(\omega\mathrm{J}^{--})^2\Big]\mathcal{L}^{++}(z,u^+)\Big| \,,
\end{aligned}
$$

  with β_1 and β_2 some coefficients (to be determined).
- Work in Wess-Zumino gauge

$$
\mathcal{D}_{\hat a}| = \nabla_{\hat a} = e_{\hat a}{}^{\hat m}(x)\,\partial_{\hat m}+\frac{1}{2}\omega_{\hat a}{}^{\hat b\hat c}(x)\,M_{\hat b\hat c}
$$

  $\nabla_{\hat a}$ is the covariant derivatives of 5D AdS space

$$
[\nabla_{\hat a},\nabla_{\hat b}]=-\omega^2\mathrm{J}^2M_{\hat a\hat b}
$$

Projective action principle in $AdS^{5|8}$ (II)

- Impose projective invariance $\Longrightarrow \beta_1 = -\mathrm{i}\,25/24,\ \beta_2 = -18$
- Supersymmetry $\delta_\xi S = 0$ leads to the same restrictions (much more involved computation)

Then the projective and SU(2,2|1) invariant action is

$$S = -\frac{1}{2\pi}\oint \frac{u_i^+ \mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\, e\Big[(\hat{\mathcal{D}}^-)^4 - \frac{25}{24}\,\mathrm{i}\omega \mathrm{J}^{--}(\hat{\mathcal{D}}^-)^2 + 18\,(\omega\mathrm{J}^{--})^2\Big]\mathcal{L}^{++}(z,u^+)\Big|$$

Some models

Within the previous formalism, it is possible to formulate in $AdS^{5|8}$ projective superspace [S. M. Kuzenko, G. T.-M. (2007)]:

- Superconformal tensor multiplets models
  [A. Karlhede, U. Lindström & M. Roček (1984)], [N. Berkovits & W. Siegel (1996)], [B. de Wit, M. Roček & S. Vandoren (2001)]
  in D=5 flat projective superspace [S. M. Kuzenko (2006)]
- hyperkähler sigma-models on (tangent)cotangent bundles of Kähler manifolds.
  [S. M. Kuzenko (1998)], [S. J. Gates & S. M. Kuzenko (1999, 2000)], [M. Arai & M. Nitta (2006)], [M. Arai, S. M. Kuzenko & U. Lindström (2007)]
- Superconformal (charged) hypermultiplets sigma-model
  [S. M. Kuzenko (2006)]
- Vector multiplet, Chern-Simon couplings and vector-tensor dynamical systems

5D off-shell SUGRA in projective superspace?

How to generalize to a 5D $\mathcal{N}=1$ supergravity theory the previous construction?

- To our knowledge the off-shell geometry of 5D "minimal" SUGRA in superspace has been studied only in [Howe (1982)] generalizing 4D $\mathcal{N}=2$ minimal Poincaré SUGRA of [Breitenlohner, Sohnius (1980)]

Then let us start with the 5D SUGRA geometry of [Howe (1982)]

Geometry of 5D $\mathcal{N}=1$ off-shell Poincaré SUGRA (I)

- The tangent-space group is chosen to be $\mathrm{SO}(4,1)\times\mathrm{SU}(2)$
  The superspace covariant derivative $\mathcal{D}_{\hat{A}}=(\mathcal{D}_{\hat{a}},\mathcal{D}^i_{\hat{\alpha}})$ are

$$\mathcal{D}_{\hat{A}} \;=\; E_{\hat{A}}{}^{\hat{M}}\,\partial_{\hat{M}}+\frac{1}{2}\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}\,M_{\hat{b}\hat{c}}+\Phi_{\hat{A}}{}^{kl}\,J_{kl}+V_{\hat{A}}Z$$

  - $E_{\hat{A}}{}^{\hat{M}}(z)$ is the supervielbein,
  - $\Omega_{\hat{A}}{}^{\hat{\beta}\hat{\gamma}}(z)$ is the Lorentz connection,
  - $[J_{kl},\mathcal{D}^i_{\hat{\alpha}}]=-\delta^i_{(k}\mathcal{D}_{l)\hat{\alpha}}$, and $\Phi_{\hat{A}}{}^{kl}(z)$ is the SU(2)-connection,
  - Z a real gauged central charge with connection $V_{\hat{A}}(z)$
- The covariant derivatives obey the anti-commutation relations

$$[\mathcal{D}_{\hat{A}},\mathcal{D}_{\hat{B}}\} \;=\; T_{\hat{A}\hat{B}}{}^{\hat{C}}\mathcal{D}_{\hat{C}}+R_{\hat{A}\hat{B}}{}^{kl}J_{kl}+\frac{1}{2}R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}}+F_{\hat{A}\hat{B}}Z$$

  - $T_{\hat{A}\hat{B}}{}^{\hat{C}}$ are the torsions,
  - $R_{\hat{A}\hat{B}}{}^{kl}$ and $R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}$ SU(2)- and SO(4,1)-curvature,
  - $F_{\hat{A}\hat{B}}$ the central charge field strength

Geometry of 5D $\mathcal{N}=1$ off-shell Poincaré SUGRA (II)

Supergravity gauge group is generated by the local transformations:

$$\mathcal{D}_{\hat{A}} \rightarrow \mathcal{D}'_{\hat{A}} = \mathrm{e}^{K}\, \mathcal{D}_{\hat{A}}\, \mathrm{e}^{-K}\ , \quad K = K^{\hat{C}}(z)\mathcal{D}_{\hat{C}} + \frac{1}{2}K^{\hat{c}\hat{d}}(z)M_{\hat{c}\hat{d}} + K^{kl}(z)J_{kl} + \tau(z)Z$$

Given a tensor superfield $U(z)$, it transforms as follows:

$$U \rightarrow U' = \mathrm{e}^{K}\, U$$

- Now, following Howe, we use the SUGRA constraints

$$T_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{\hat{c}} = -2\mathrm{i}\,\varepsilon^{ij}(\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\ , \qquad F_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j} = -2\mathrm{i}\,\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}}\ , \qquad \text{(dimension-0)}$$

$$T_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{\hat{\gamma}}_{k} = T_{\hat{\alpha}}^{i}{}_{\hat{b}}{}^{\hat{c}} = F_{\hat{\alpha}}^{i}{}_{\hat{b}} = 0\ , \qquad \text{(dimension-1/2)}$$

$$T_{\hat{a}\hat{b}}{}^{\hat{c}} = T_{\hat{a}\hat{\beta}(j}{}^{\hat{\beta}}{}_{k)} = 0\ . \qquad \text{(dimension-1)}$$

Solving the superspace Bianchi identities we derive the 5D algebra of covariant derivatives (which was not given by Howe)

SUGRA covariant derivatives algebra

$$
\begin{aligned}
\{\mathcal{D}^i_{\hat\alpha},\mathcal{D}^j_{\hat\beta}\} &= -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta} - 2\mathrm{i}\,\varepsilon^{ij}\varepsilon_{\hat\alpha\hat\beta}Z \\
&\quad +3\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}S^{kl}J_{kl} - 2\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}\Big(F_{\hat a\hat b}+N_{\hat a\hat b}\Big)J^{ij} \\
&\quad -\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}F^{\hat c\hat d}M_{\hat c\hat d} + \frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon^{\hat a\hat b\hat c\hat d\hat e}N_{\hat a\hat b}(\Gamma_{\hat c})_{\hat\alpha\hat\beta}M_{\hat d\hat e} + 4\mathrm{i}\,S^{ij}M_{\hat\alpha\hat\beta}\,, \\
[\mathcal{D}_{\hat a},\mathcal{D}^j_{\hat\beta}] &= \frac{1}{2}(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\gamma}S^j{}_k\mathcal{D}^k_{\hat\gamma} - \frac{1}{2}F_{\hat a\hat b}(\Gamma^{\hat b})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma} - \frac{1}{8}\varepsilon_{\hat a\hat b\hat c\hat d\hat e}N^{\hat d\hat e}(\Sigma^{\hat b\hat c})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma} \\
&\quad +\Big(-3\varepsilon^{jk}\Xi_{\hat a\hat\beta}{}^l + \frac{5}{4}(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{F}_{\hat\alpha}{}^l - \frac{1}{4}(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{N}_{\hat\alpha}{}^l\Big)J_{kl} \\
&\quad +\Big(\frac{1}{2}(\Gamma_{\hat a})^{\hat\alpha\hat\gamma}\mathcal{D}^{\hat\delta j}F_{\hat\alpha\hat\beta} - \frac{1}{2}(\Gamma_{\hat a})_{\hat\beta\hat\alpha}\mathcal{D}^{\hat\gamma j}F^{\hat\delta\hat\alpha} - \frac{1}{2}(\Gamma_{\hat a})^{\hat\alpha\hat\gamma}\delta^{\hat\delta}_{\hat\beta}\mathcal{D}^{\hat\rho j}F_{\hat\alpha\hat\rho} \\
&\quad +\frac{1}{2}(\Gamma_{\hat a})_{\hat\rho\hat\alpha}\mathcal{D}^{\hat\rho j}F^{\hat\alpha\hat\gamma}\delta^{\hat\delta}_{\hat\beta} - \frac{1}{2}(\Gamma_{\hat a})_{\hat\beta\hat\rho}\mathcal{D}^{\hat\rho j}F^{\hat\gamma\hat\delta}\Big)M_{\hat\gamma\hat\delta}\,, \\
[\mathcal{D}_{\hat a},\mathcal{D}_{\hat b}] &= \frac{\mathrm{i}}{2}(\mathcal{D}^{\hat\gamma}_k F_{\hat a\hat b})\mathcal{D}^k_{\hat\gamma} - \frac{\mathrm{i}}{8}(\mathcal{D}^{\hat\gamma(k}\mathcal{D}^{l)}_{\hat\gamma}F_{\hat a\hat b})J_{kl} + F_{\hat a\hat b}Z \\
&\quad +\Big(\frac{1}{4}\varepsilon_{\hat m\hat n\hat d\hat e[\hat a}(\Sigma^{\hat m\hat n})_{\hat\gamma\hat\delta}\mathcal{D}_{\hat b]}N^{\hat d\hat e} - \frac{1}{2}(\Sigma_{\hat d[\hat a})_{\hat\gamma\hat\delta}N_{\hat b]\hat e}N^{\hat e\hat d} + \frac{1}{4}(\Sigma^{\hat e\hat d})_{\hat\gamma\hat\delta}N_{\hat d\hat a}N_{\hat b\hat e} \\
&\quad +\frac{1}{8}(\Sigma_{\hat a\hat b})_{\hat\gamma\hat\delta}N^{\hat d\hat e}N_{\hat d\hat e} + \frac{\mathrm{i}}{4}\mathcal{D}^k_{\hat\gamma}\mathcal{D}_{\hat\delta k}F_{\hat a\hat b} - (\Sigma^{\hat c\hat d})_{\hat\gamma\hat\delta}F_{\hat a\hat c}F_{\hat b\hat d} + \frac{1}{2}(\Sigma_{\hat a\hat b})_{\hat\gamma\hat\delta}S^{ij}S_{ij}\Big)M^{\hat\gamma\hat\delta}
\end{aligned}
$$

SUGRA covariant derivatives algebra

$$
\begin{aligned}
\{\mathcal{D}^i_{\hat\alpha},\mathcal{D}^j_{\hat\beta}\} &= -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta} - 2\mathrm{i}\,\varepsilon^{ij}\varepsilon_{\hat\alpha\hat\beta}Z \\
&\quad +3\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}S^{kl}J_{kl} - 2\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}\Big(F_{\hat a\hat b}+N_{\hat a\hat b}\Big)J^{ij} \\
&\quad -\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}F^{\hat c\hat d}M_{\hat c\hat d} + \frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon^{\hat a\hat b\hat c\hat d\hat e}N_{\hat a\hat b}(\Gamma_{\hat c})_{\hat\alpha\hat\beta}M_{\hat d\hat e} + 4\mathrm{i}\,S^{ij}M_{\hat\alpha\hat\beta}\ , \\
[\mathcal{D}_{\hat a},\mathcal{D}^j_{\hat\beta}] &= \frac{1}{2}(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\gamma}S^j{}_k\mathcal{D}^k_{\hat\gamma} - \frac12 F_{\hat a\hat b}(\Gamma^{\hat b})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma} - \frac18\varepsilon_{\hat a\hat b\hat c\hat d\hat e}N^{\hat d\hat e}(\Sigma^{\hat b\hat c})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma} \\
&\quad +\Big(-3\varepsilon^{jk}\Xi_{\hat a\hat\beta}{}^l + \frac54(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{F}_{\hat\alpha}{}^l - \frac14(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{N}_{\hat\alpha}{}^l\Big)J_{kl} \\
&\quad +\Big(\frac12(\Gamma_{\hat a})^{\hat\alpha\hat\gamma}\mathcal{D}^{\hat\delta j}F_{\hat\alpha\hat\beta} - \frac12(\Gamma_{\hat a})_{\hat\beta\hat\alpha}\mathcal{D}^{\hat\gamma j}F^{\hat\delta\hat\alpha} - \frac12(\Gamma_{\hat a})^{\hat\alpha\hat\gamma}\delta^{\hat\delta}_{\hat\beta}\mathcal{D}^{\hat\rho j}F_{\hat\alpha\hat\rho} \\
&\quad +\frac12(\Gamma_{\hat a})_{\hat\rho\hat\alpha}\mathcal{D}^{\hat\rho j}F^{\hat\alpha\hat\gamma}\delta^{\hat\delta}_{\hat\beta} - \frac12(\Gamma_{\hat a})_{\hat\beta\hat\rho}\mathcal{D}^{\hat\rho j}F^{\hat\gamma\hat\delta}\Big)M_{\hat\gamma\hat\delta}\ , \\
[\mathcal{D}_{\hat a},\mathcal{D}_{\hat b}] &= \frac{\mathrm{i}}{2}(\mathcal{D}^{\hat\gamma}_kF_{\hat a\hat b})\mathcal{D}^k_{\hat\gamma} - \frac{\mathrm{i}}{8}(\mathcal{D}^{\hat\gamma(k}\mathcal{D}^{l)}_{\hat\gamma}F_{\hat a\hat b})J_{kl} + F_{\hat a\hat b}Z \\
&\quad +\Big(\frac14\varepsilon_{\hat m\hat n\hat d\hat e[\hat a}(\Sigma^{\hat m\hat n})_{\hat\gamma\hat\delta}\mathcal{D}_{\hat b]}N^{\hat d\hat e} - \frac12(\Sigma_{\hat d[\hat a})_{\hat\gamma\hat\delta}N_{\hat b]\hat e}N^{\hat e\hat d} + \frac14(\Sigma^{\hat e\hat d})_{\hat\gamma\hat\delta}N_{\hat d\hat a}N_{\hat b\hat e} \\
&\quad +\frac18(\Sigma_{\hat a\hat b})_{\hat\gamma\hat\delta}N^{\hat d\hat e}N_{\hat d\hat e} + \frac{\mathrm{i}}{4}\mathcal{D}^k_{\hat\gamma}\mathcal{D}_{\hat\delta k}F_{\hat a\hat b} - (\Sigma^{\hat c\hat d})_{\hat\gamma\hat\delta}F_{\hat a\hat c}F_{\hat b\hat d} + \frac12(\Sigma_{\hat a\hat b})_{\hat\gamma\hat\delta}S^{ij}S_{ij}\Big)M^{\hat\gamma\hat\delta}
\end{aligned}
$$

You see the AdS$^{5|8}$ algebra: $S^{ij}=\mathrm{J}^{ij}$, U(1) $J\equiv\mathrm{J}^{kl}J_{kl}$

Some comments about the algebra

- Algebra is parametrized by 3 superfields: $S^{ij}=S^{ji}$, $N_{\hat{a}\hat{b}}=-N_{\hat{b}\hat{a}}$, $F_{\hat{a}\hat{b}}=-F_{\hat{b}\hat{a}}$ (the central charge field strength) and covariant derivatives of them
- $F_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$ and S^{ij} are constrained by the Bianchi identities.
- For example: a very important constraint is

$$\mathcal{D}^k_{\hat{\beta}} S^{jl} = \frac{1}{10}(\Sigma_{\hat{a}\hat{b}})_{\hat{\beta}}{}^{\hat{\delta}} \varepsilon^{k(j}\mathcal{D}^{l)}_{\hat{\delta}}\Big(3F^{\hat{a}\hat{b}}+N^{\hat{a}\hat{b}}\Big) \quad \Longrightarrow \quad \mathcal{D}^{(i}_{\hat{\alpha}} S^{jk)}=0$$

S^{ij} is a 5D $O(2)$ tensor multiplet

Note: introduced $u^{\pm}_i$, it follows $\mathcal{D}^+_{\hat{\alpha}} S^{++}=0$, $\mathcal{D}^-_{\hat{\alpha}} S^{--}=0$

Projective superspace

The construction works with few generalization of the AdS case.
First note

$$\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^+_{\hat\beta}\} = -4\mathrm{i}\left(F_{\hat\alpha\hat\beta} + N_{\hat\alpha\hat\beta}\right) J^{++} + 4\mathrm{i}\, S^{++} M_{\hat\alpha\hat\beta}$$

We define scalar projective superfields of weight-n as

- field over $\mathbb{C}P^1$

$$Q^{(n)}(z, c\, u^+) = c^n\, Q^{(n)}(z, u^+)\ , \qquad c \in \mathbb{C}^*$$

- infinitesimal gauge transformations

$$\delta Q^{(n)} = \delta Q^{(n)} = KQ^{(n)} = \left(K^{\hat C}\mathcal{D}_{\hat C} + K^{kl} J_{kl} + \tau Z\right) Q^{(n)}$$

$$J_{kl}\, Q^{(n)} = -\frac{1}{(u^+u^-)}\left(u^+_{(k} u^+_{l)} D^{--} - n\, u^+_{(k} u^-_{l)}\right) Q^{(n)}$$

- Analyticity condition:

$$\mathcal{D}^+_{\hat\alpha} Q^{(n)} = 0$$

which is consistent $(\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^+_{\hat\beta}\} Q^{(n)} = 0)$ due to $J^{++} Q^{(n)} = 0$

Projective action principle in Wess-Zumino gauge (I)

$$\mathcal{D}_{\hat{a}}| = \nabla_{\hat{a}} + \Psi_{\hat{a}}{}^{\hat{\gamma}}_{k}(x)\mathcal{D}^k_{\hat{\gamma}}| + \phi_{\hat{a}}{}^{kl}(x)J_{kl} + \mathcal{V}_{\hat{a}}(x)Z \ , \qquad \mathcal{D}^i_{\hat{\alpha}}| = \frac{\partial}{\partial\theta^{\hat{\alpha}}_i}$$

Here $\nabla_{\hat{a}}$ are space-time covariant derivatives,

$$\nabla_{\hat{a}} = e_{\hat{a}} + \omega_{\hat{a}} \ , \qquad e_{\hat{a}} = e_{\hat{a}}{}^{\hat{m}}(x)\,\partial_{\hat{m}} \ , \qquad \omega_{\hat{a}} = \frac{1}{2}\,\omega_{\hat{a}}{}^{\hat{b}\hat{c}}(x)\,M_{\hat{b}\hat{c}}$$

with $e_{\hat{a}}{}^{\hat{m}}$ the component inverse vielbein
$\omega_{\hat{a}}{}^{\hat{b}\hat{c}}$ the Lorentz connection
$\Psi_{\hat{a}}{}^{\hat{\gamma}}_{k}$ is the component gravitino
$\phi_{\hat{a}}{}^{kl} = \Phi_{\hat{a}}{}^{kl}|$ SU(2) connection
$\mathcal{V}_{\hat{a}} = V_{\hat{a}}|$ central charge gauge field (graviphoton)

Projective action principle in Wess-Zumino gauge (II)

Remember that in the AdS$^{5|8}$ case the action was **uniquely fixed by two independent requirements**

- Projective invariance (computationally much simpler)
- Isometry SU(2,2|1) invariance

Once implemented the consequences of the Wess-Zumino gauge, with the lagrangian $\mathcal{L}^{++}$ (for simplicity $Z\mathcal{L}^{++}=0$)

- an analytic $\mathcal{D}^+_{\hat{\alpha}}\mathcal{L}^{++}=0$
- weight two $\mathcal{L}^{++}(cu^+)=c^2\mathcal{L}^{++}(u^+)$ projective superfield

compute the projective variation $(\delta\mathcal{D}^-_{\hat{\alpha}}=b\mathcal{D}^+_{\hat{\alpha}})$ of

$$S = -\frac{1}{2\pi}\oint\frac{u^+_i \mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\, e\Big[(\mathcal{D}^-)^4\Big]\mathcal{L}^{++}\Big|$$

and iteratively add terms to impose projective invariance

Action in Wess-Zumino gauge

The projective invariant action

$$S=-\frac{1}{2\pi}\oint\frac{u_i^+\mathrm{d}u^{+i}}{(u^+u^-)^4}\int\mathrm{d}^5x\,e\Big[(\mathcal{D}^-)^4+\frac{\mathrm{i}}{4}\Psi^{\hat\alpha\hat\beta\hat\gamma-}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}-\frac{25}{24}\mathrm{i}\,S^{--}(\mathcal{D}^-)^2$$
$$-2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{\hat a}{}^{\hat\beta-}\Psi_{\hat b}{}^{\hat\delta-}\mathcal{D}^-_{[\hat\gamma}\mathcal{D}^-_{\hat\delta]}-\frac{\mathrm{i}}{4}\phi^{\hat\alpha\hat\beta--}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}+4(\Sigma^{\hat a\hat b})^{\hat\alpha}{}_{\hat\gamma}\phi_{[\hat a}{}^{--}\Psi_{\hat b]}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\alpha}$$
$$-10\,\Psi^{\hat\alpha-}S^{--}\mathcal{D}^-_{\hat\alpha}+2\mathrm{i}\,\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\Psi_{\hat c}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\gamma}+18S^{--}S^{--}$$
$$-6\mathrm{i}\,\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\phi_{\hat c}{}^{--}+18\mathrm{i}\,(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}S^{--}\Big]\mathcal{L}^{++}\Big|$$

expected to be Locally supersymmetric action.

Note that the existance of such action is highly non-trivial and is also a consistency check of the algebra

Action in Wess-Zumino gauge

The projective invariant action

$$S = -\frac{1}{2\pi} \oint \frac{u_i^+ \mathrm{d}u^{+i}}{(u^+ u^-)^4} \int \mathrm{d}^5 x\, e \Big[(\mathcal{D}^-)^4 + \frac{\mathrm{i}}{4} \Psi^{\hat\alpha\hat\beta\hat\gamma -} \mathcal{D}^-_{\hat\gamma} \mathcal{D}^-_{\hat\alpha} \mathcal{D}^-_{\hat\beta} - \frac{25}{24} \mathrm{i}\, S^{--} (\mathcal{D}^-)^2$$
$$-2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma} \Psi_{\hat a}{}^{\hat\beta -} \Psi_{\hat b}{}^{\hat\delta -} \mathcal{D}^-_{[\hat\gamma} \mathcal{D}^-_{\hat\delta]} - \frac{\mathrm{i}}{4} \phi^{\hat\alpha\hat\beta --} \mathcal{D}^-_{\hat\alpha} \mathcal{D}^-_{\hat\beta} + 4(\Sigma^{\hat a\hat b})^{\hat\alpha}{}_{\hat\gamma} \phi_{[\hat a}{}^{--} \Psi_{\hat b]}{}^{\hat\gamma -} \mathcal{D}^-_{\hat\alpha}$$
$$-10\, \Psi^{\hat\alpha -} S^{--} \mathcal{D}^-_{\hat\alpha} + 2\mathrm{i}\, \varepsilon^{\hat a\hat b\hat c\hat m\hat n} (\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta} \Psi_{\hat a}{}^{\hat\alpha -} \Psi_{\hat b}{}^{\hat\beta -} \Psi_{\hat c}{}^{\hat\gamma -} \mathcal{D}^-_{\hat\gamma} + 18 S^{--} S^{--}$$
$$-6\mathrm{i}\, \varepsilon^{\hat a\hat b\hat c\hat m\hat n} (\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta} \Psi_{\hat a}{}^{\hat\alpha -} \Psi_{\hat b}{}^{\hat\beta -} \phi_{\hat c}{}^{--} + 18\mathrm{i}\, (\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta} \Psi_{\hat a}{}^{\hat\alpha -} \Psi_{\hat b}{}^{\hat\beta -} S^{--} \Big] \mathcal{L}^{++} \Big|$$

expected to be Locally supersymmetric action.

Note that the existance of such action is highly non-trivial and is also a consistency check of the algebra

Note that the covariant terms are the one of the AdS action

Conclusions and Outlooks

- For the first time a formalism of curved projective superspace
  - 5D $\mathcal{N}=1$ AdS superspace
  - 5D $\mathcal{N}=1$ Poincaré SUGRA (to appear)
- Several open questions:
  - Wess-Zumino action useful because ready for a general component analysis
    besides WZ gauge: manifestly supersymmetric action principle?
  - What about 5D Weyl multiplet and conformal SUGRA?
  - Applications and $D \leq 6$?