Is a point-like ultraviolet finite theory of quantum gravity possible?

Valencia, Oct. 5, 2007

Zvi Bern, UCLA

Lecture 1

Lecture 1: Scattering amplitudes in quantum field theories. On-shell methods, unitarity and twistors.

Lecture 2: Ultraviolet properties of quantum gravity theories.

Based on following:
ZB, N.E.J. Bjerrum-Bohr, D. Dunbar, hep-th/0501137
ZB, L. Dixon , R. Roiban, hep-th/0611086
ZB, J.J. Carrasco, L. Dixon, H. Johansson, D. Kosower, R. Roiban, hep-th/0702112
ZB, J.J. Carrasco, D. Forde, H. Ita and H. Johansson, arXiv:0707.1035 [hep-th]

Plan

"A method is more important that a discovery, since the right method can lead to new and even more important discoveries" -- L.D. Landau

In these lectures we wish to discuss the general problem of computing scattering amplitudes in quantum field theory, before turning to the problem of UV properties of gravity theories..

The basic problem is the same in:

(a) Precision phenomenology at the LHC.

(b) Checking the AdS/CFT correspondence in scattering amplitudes.

(c) Ultraviolet finiteness of supergravity.

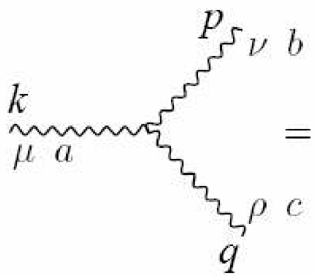
Need multi-leg or multi-loop scattering amplitudes.

Feynman diagrams develop severe difficulties when addressing these issues.

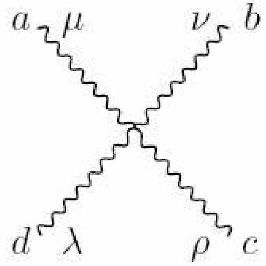
Topics

- Spinors twistors and amplitudes.
- MHV rules and on-shell recursion.
- Loop Amplitudes: Unitarity method.
- Kawai-Lewellen-Tye relations between gravity and gauge theory.
- Gravity. Surprising results at high energies.
- Multi-loop quantum gravity. Review of standard lore – nonrenormalizability.
- $N = 8$ supergravity. 3 loop computation and arguments that challenge standard lore.
- Proposal for origin of novel cancellations.

Gauge Theory Feynman Rules



$$= -gf^{abc}\Big(\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu\Big)$$



$$= \begin{cases} -ig^2[f^{abe}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\ \quad + f^{ade}f^{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\ \quad + f^{ace}f^{ebd}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})] \end{cases}$$

Also fermions and ghosts

Color and kinematics mixed together

Gravity Feynman Rules

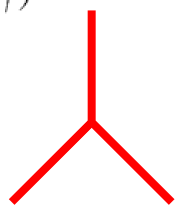
$$\mathcal{L}_{\text{gravity}} = \sqrt{g}\, R$$

Propagator in de Donder gauge:

$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\Big[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\Big]\frac{i}{k^2 + i\epsilon}$$

Three vertex:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\
&\quad \mathrm{sym}\big[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\big]
\end{aligned}$$



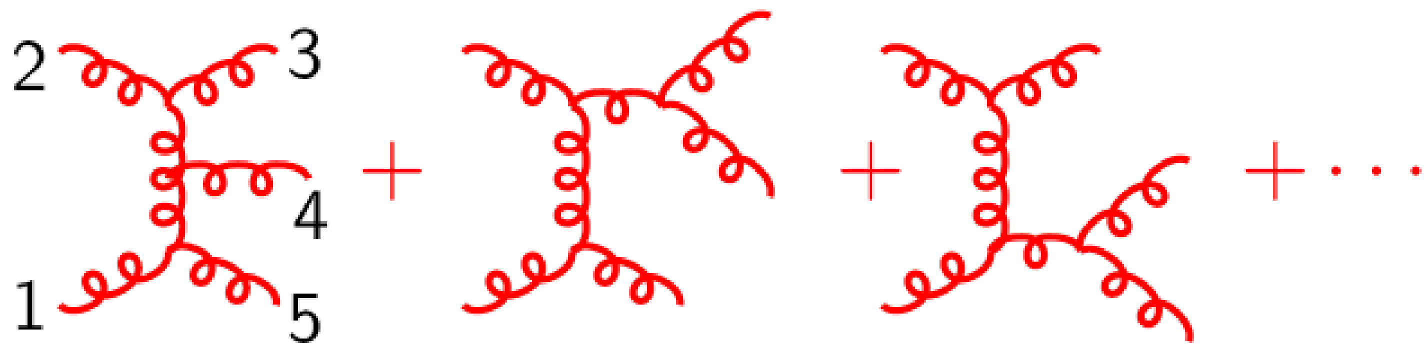
About 100 terms in three vertex

An infinite number of other messy vertices.

Naïve conclusion: Gravity is an unholy mess!

Tree-level example: Five gluons

Consider the five-gluon amplitude



If you evaluate this you find…

Result of evaluation (actually only a small part of it):

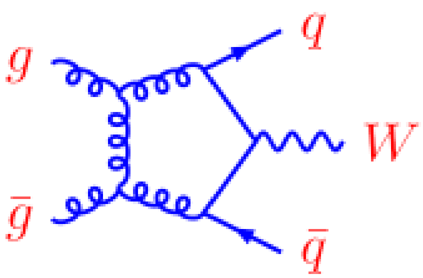
$$k_1 \cdot k_4\,\varepsilon_2 \cdot k_1\,\varepsilon_1 \cdot \varepsilon_3\,\varepsilon_4 \cdot \varepsilon_5$$

State-of-the-Art NLO QCD

Five point is *still* state-of-the art for QCD cross-sections:

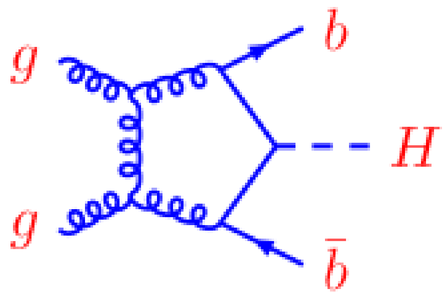
Typical examples:

$pp \to W, Z + 2$ jets



Bern, Dixon, Kosower
Dixon, Kunszt, and Signer
Campbell and Ellis: MCFM

$pp \to \bar{b}bH$ or $pp \to \bar{t}tH$



Reina, Dawson, Jackson and Wackeroth
Beenakker, Dittmaier, Kramer, Plumper, Spira

Brute force calculations give GB expressions – numerical stability?

Amusing numbers: 6g: 10,860 diagrams, 7g: 168,925 diagrams

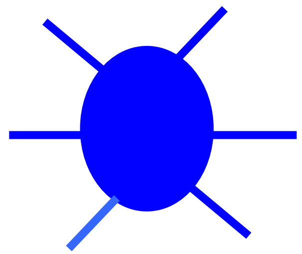
Much worse difficulty: integral reduction generates nasty dets.

$$\frac{1}{\det(k_i \cdot k_j)^n}$$ "Grim" determinant

Off-shell Formalisms

In graduate school you learned that scattering amplitudes need to be calculated using unphysical gauge dependent quantities: Off-shell Green functions

$$p^2 \neq m^2$$



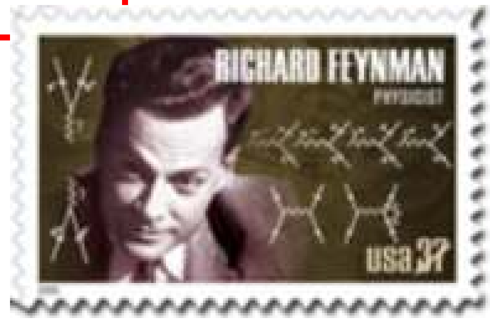
Standard machinery:

- **Fadeev-Popov procedure for gauge fixing.**
- **Taylor-Slavnov Identities.**
- **BRST.**
- **Gauge fixed Feynman rules.**
- **Batalin-Fradkin-Vilkovisky quantization for gravity.**
- **Off-shell constrained superspaces.**

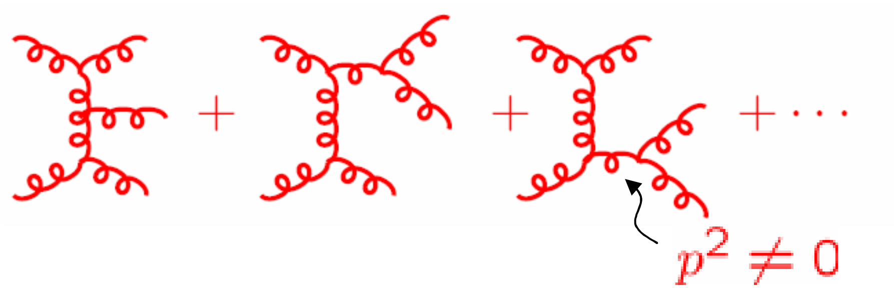
For all this machinery relatively few calculations in quantum gravity – very few checks of assertions on UV properties.

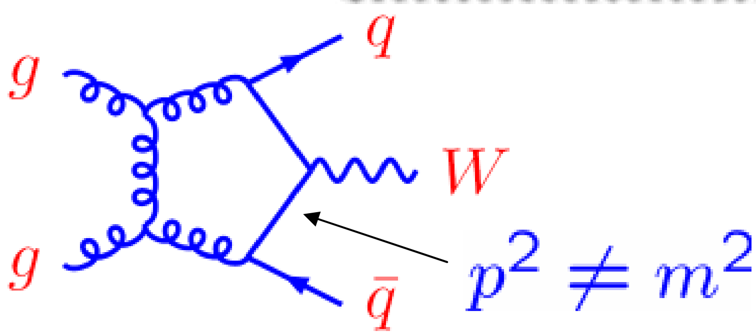
Explicit calculations from 't Hooft and Veltman;
Goroff and Sagnotti; van de Ven

Why are Feynman diagrams clumsy for high loop or multiplicity processes?



- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.

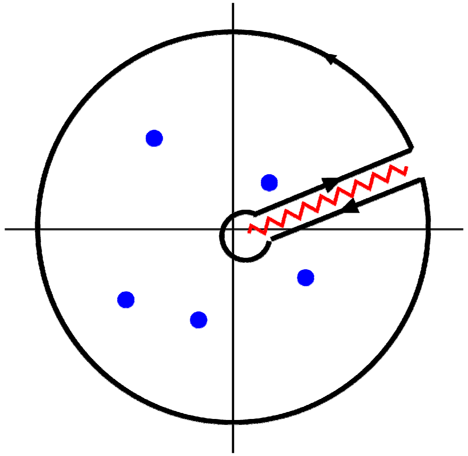


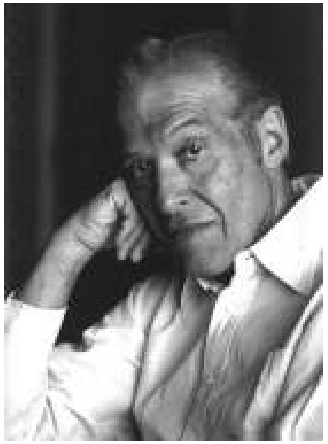


- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

• All steps should be in terms of gauge invariant on-shell states. $p^2 = m^2$ **On shell formalism.**

• Radical rewrite of gauge theory needed.





“One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.”

J. Schwinger in “Particles, Sources and Fields” Vol 1

On-shell methods reconstruct amplitudes from their poles and cuts. Each of these corresponds to propagation of particles. Automatically gauge independent.

Spinors expose simplicity

Xu, Zhang and Chang
Berends, Kleis and Causmaeker
Gastmans and Wu
Gunion and Kunszt
& many others

Spinor helicity for massless polarization vectors:

particle momentum

$$\varepsilon^{+}_{\mu}(k;q) = \frac{\langle q^{-}|\,\gamma_{\mu}\,|k^{-}\rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon^{-}_{\mu}(k,q) = \frac{\langle q^{+}|\,\gamma_{\mu}\,|k^{+}\rangle}{\sqrt{2}\,[k\,q]}$$

Reference momentum

Chinese magic

More sophisticated version of circular polarization: $\epsilon_{\mu} = (0, 1, \pm i, 0)$

All required properties of circular polarization satisfied:

$$\epsilon_i^2 = 0, \qquad k \cdot \epsilon_i = 0, \qquad \epsilon_i^{+} \epsilon_i^{-} = -1$$

Changes in reference momentum q equivalent to on-shell gauge transformations:

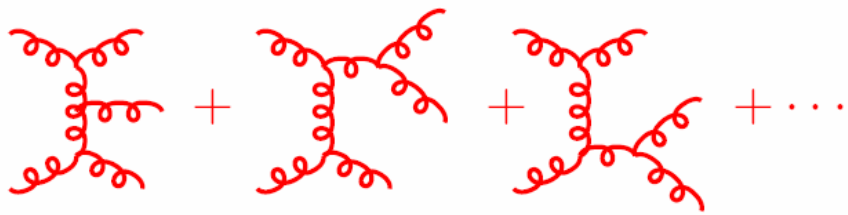
$$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\,l\rangle = \langle k_{j\,-}|k_{l+}\rangle = \sqrt{2k_j \cdot k_l}\; e^{i\phi} \;= \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l)$$

$$\epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_j^{\dot{a}}\tilde{\lambda}_l^{\dot{b}} \longleftrightarrow [j\,l] = \langle k_{j\,+}|k_{l\,-}\rangle = -\sqrt{2k_j \cdot k_l}\; e^{-i\phi} = \tfrac{1}{2}\bar{u}(k_j)(1-\gamma_5)u(k_l)$$

Graviton polarization tensors are squares of these:

$$\epsilon^{+}_{\mu\nu} = \epsilon^{+}_{\mu}\epsilon^{+}_{\nu} \qquad\qquad 2 = 1 + 1$$

Reconsider Five Gluon Tree



With a little Chinese magic:

$$A_5^{\text{tree}}(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$A_5^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^+) = i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

These are color stripped amplitudes:

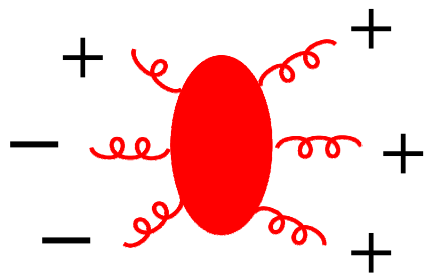
$$\mathcal{A}_5 = \sum_{\text{perms}} \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})\, A_5(1,2,3,4,5)$$

Parke and Taylor (1984)

MHV Amplitudes

At tree level Parke and Taylor conjectured a very simple form for *n*-gluon scattering.

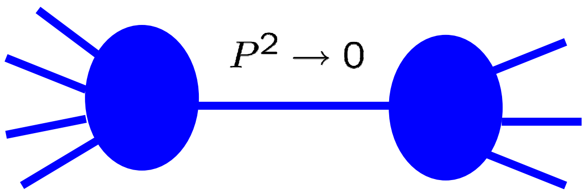
$$A(1^-,2^-,3^+,\ldots,n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$



$$\mathcal{A}(1^-,2^-,3^+,\ldots,n^+) = \sum_{\text{perms}} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A(1^-,2^-,3^+,\ldots,n^+)$$

This was guessed by calculating low points and then finding a formula with correct kinematic poles in all channels.

Proven by Berends and Giele



This simplicity has echoes for general helicities and at loop level.

ZB, Dixon, Dunbar, Kosower
Cachazo, Svrcek, Witten; ZB,Dixon, Kosower
Brandhuber, Spence and Travaglini

Twistors

In a remarkable paper Ed Witten demonstrated that twistor space reveals a hidden structure in scattering amplitudes.

Link is for $N = 4$ super-Yang-Mills theory, but at tree level hardly any difference from QCD.

Early work from Nair

Penrose twistor transform:

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j i\mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$$

Witten's remarkable twistor-space link:

Witten; Roiban, Spradlin and Volovich

QCD scattering amplitudes $\longleftrightarrow$ **Topological String Theory**

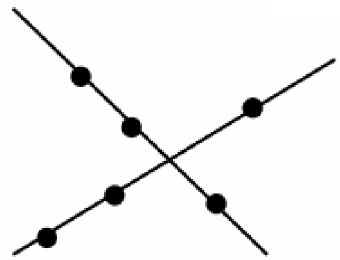
Here we will discuss only the field theory consequences

Amazing Simplicity

Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

$$d = q - 1 + L, \quad q = \#\ \text{negative helicities}, \quad L = \#\ \text{loops},$$

Connected picture



Disconnected picture

Structures imply an amazing simplicity in the scattering amplitudes. Massless gauge theory tree amplitudes are much much simpler than anyone imagined.

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena Bern and Kosower

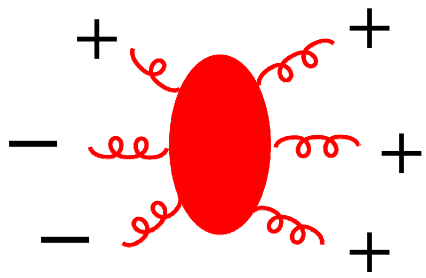
Remarkably, gravity is similar, except derivative of delta function support instead of delta-function support.

MHV Rules

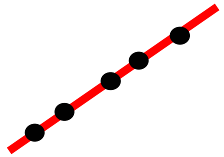
Cachazo, Svrcek and Witten

Consider MHV amplitudes.

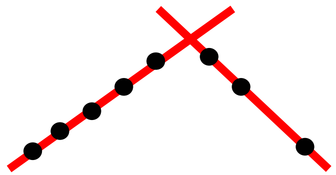
$$A(1^-, 2^-, 3^+, \ldots, n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}$$



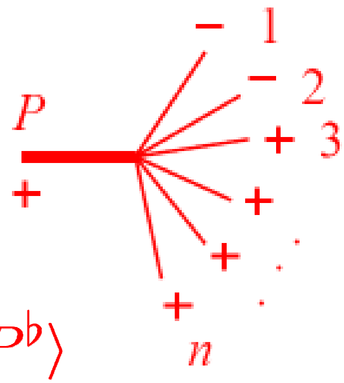
Supported on a straight line in twistor space

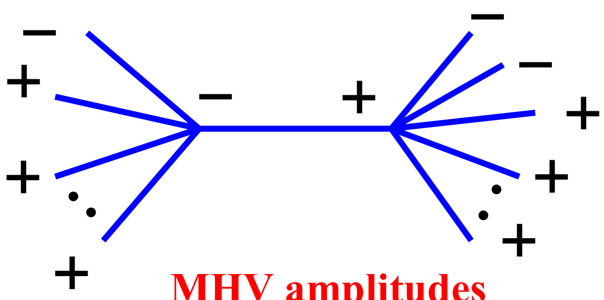


Non-MHV amplitudes supported on intersecting lines



In momentum space suggests MHV amplitudes are vertices for building new amplitudes.





MHV amplitudes at vertices

$$\langle aP\rangle \to \langle a|\not{P}|q\rangle \quad \text{or} \quad \langle aP\rangle \to \langle aP^\flat\rangle$$

$$q^2 = 0$$

Arbitrary null momentum

$$P^\flat \equiv P - \frac{P^2}{P\cdot q}\,q$$

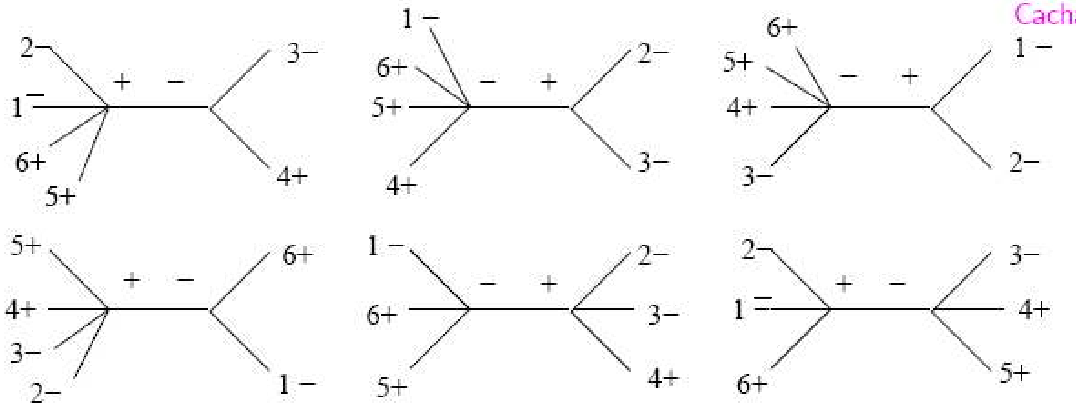
Massless

Six gluon example

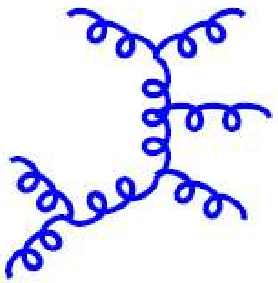
Cachazo, Svrcek and Witten

QCD gluon scattering amplitude

$- \; - \; - \; + \; + \; +$



$$
\begin{aligned}
A_6(1^-,2^-,3^-,4^+,5^+,6^+) = {} & \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle} \\
& + \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle} \\
& + \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle} \\
& + \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle} \\
& + \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle} \\
& + \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}
\end{aligned}
$$



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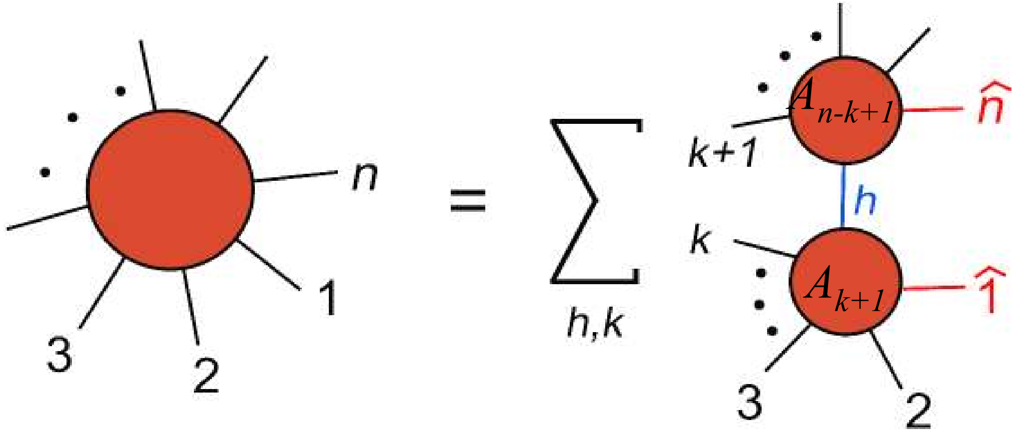
A “perfect” calculation

On-Shell Recursion

A very general machinery for constructing tree level scattering amplitudes are on-shell recursion relations.

Britto, Cachazo, Feng and Witten



Building blocks are on-shell amplitudes

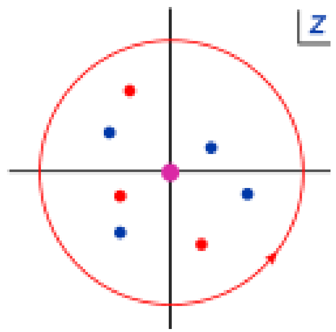
General replacement for tree-level Feynman diagrams

Contrast with Feynman diagram which are based on off-shell unphysical states with $p^2 \neq m^2$

Proof relies on so little. Power comes from generality

- **Cauchy's theorem**
- **Basic field theory factorization properties** Britto, Cachazo, Feng and Witten
- **Applies as well to massive theories.** Badger, Glover, Khoze and Svrcek
- **Applies as well to gravity theories.**

Large z behavior

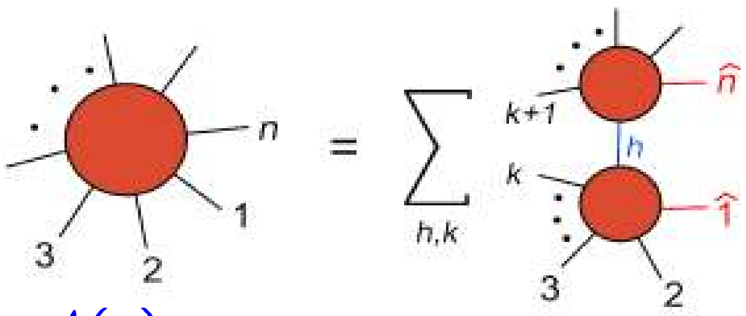


Consider amplitude under complex deformation of an amplitude. Proof of on-shell recursion relies on good behavior at large z.

$$p_1^\mu(z) = p_1^\mu - \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle \qquad p_n^\mu(z) = p_n^\mu + \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle \qquad (p_i^\mu(z))^2 = 0$$

complex momenta

A(z) is amplitude with shifted momenta



If $A(z) \to 0, \quad z \to \infty$

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \text{Res}_\alpha \frac{A(z)}{z}$$

$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

Poles in z come from kinematic poles in amplitude.

Sum over residues gives the on-shell recursion relation

Remarkably, gravity is as well behave at $z \longrightarrow \infty$ as gauge theory
This will play a crucial role in understanding how the high-energy behavior in quantum gravity can be better than people thought.

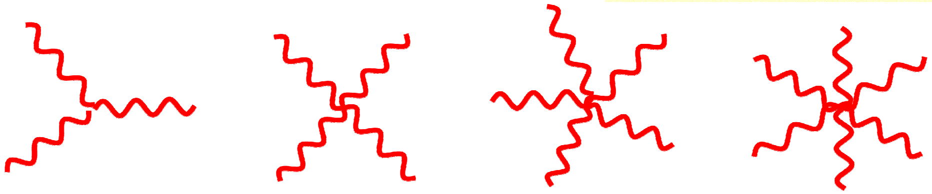
Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa}\sqrt{g}\,R \qquad \kappa^2 = 32\pi\, G_{\text{Newton}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

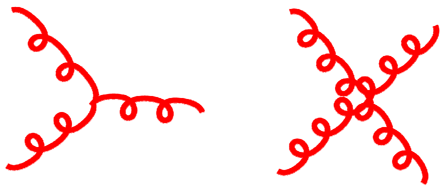
Infinite number of complicated interactions



$+ \cdots$

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$

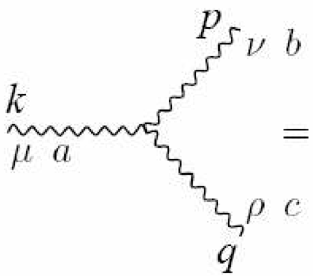


Only three and four point interactions

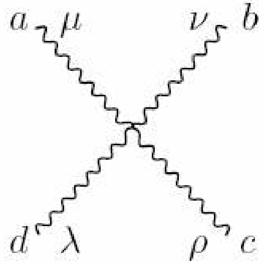
It seems completely implausible that the scattering amplitudes of gravity are directly obtainable from gauge theory ones.

However, such a simple relation does exist.

Gauge Theory Feynman Rules



$$= -gf^{abc}\Big(\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu\Big)$$



$$= \begin{cases} -ig^2[f^{abe}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\ \quad + f^{ade}f^{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\ \quad + f^{ace}f^{ebd}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})] \end{cases}$$

Also fermions and ghosts

Color and kinematics mixed together

Gravity Feynman Rules

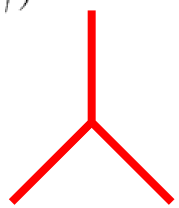
$$\mathcal{L}_{\text{gravity}} = \sqrt{g}\, R$$

Propagator in de Donder gauge:

$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\Big[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\Big]\frac{i}{k^2 + i\epsilon}$$

Three vertex:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\
&\quad \mathrm{sym}\Big[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big]
\end{aligned}$$



Does not look at all like Yang-Mills.

Gauge dependent and unphysical

KLT Relations

However, a remarkable relation between gauge and gravity amplitudes exist at tree level which we will exploit.

At *tree level* Kawai, Lewellen and Tye have derived a relationship between closed and open string amplitudes.

In field theory limit, relationship is between gravity and gauge theory

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Gravity amplitude

where we have stripped all coupling constants

Color stripped gauge theory amplitude

$$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\, A_4^{\rm tree}(1,2,3,4)$$

Full gauge theory amplitude

Holds for any external states.
See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories

Gravity and Gauge Theory Amplitudes

Berends, Giele, Kuijf; ZB, De Freitas, Wong

gravity

$$M_4^{\text{tree}}(1_h^-, 2_h^-, 3_h^+, 4_h^+) = \left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\text{tree}}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A_4^{\text{tree}}(1_g^-, 2_g^-, 4_g^+, 3_g^+)$$

$$= \left(\frac{\kappa}{2}\right)^2 s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} \times \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,4\rangle\,\langle 4\,3\rangle\,\langle 3\,1\rangle}$$

gauge theory

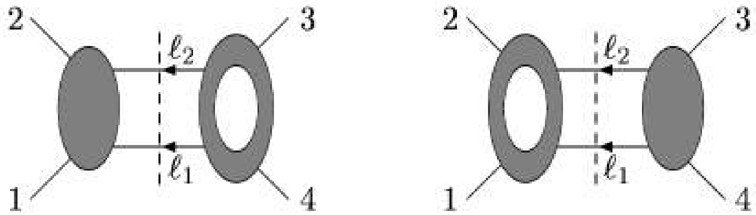
$$\langle jl\rangle = \langle k_j^- | k_l^+\rangle = \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l) = \sqrt{2k_j \cdot k_l}\, e^{i\phi}$$

- **Above result matches result from Feynman rules.**
- **Holds for all states appearing in a string theory.**
- **Holds for all states of $N = 8$ supergravity.**

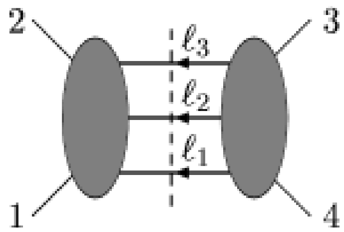
Bern, Dixon, Dunbar and Kosower

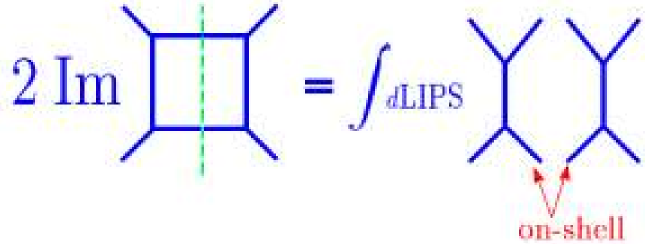
Onwards to Loops: Unitarity Method

Two-particle cut:



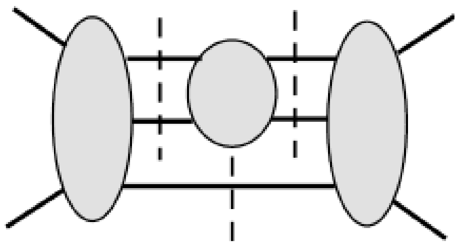
Three- particle cut:





Generalized unitarity:

Bern, Dixon and Kosower



Crucial for making high loop gravity Calculations feasible

Generalized cut interpreted as cut propagators not canceling.

A number of recent improvements to method

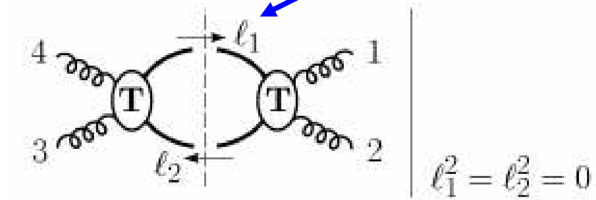
Britto, Buchbinder, Cachazo and Feng; Berger, Bern, Dixon, Forde and Kosower; Britto, Feng and Mastrolia

Example: $N = 4$ Loop Amplitudes

Bern, Rozowsky and Yan

Consider one-loop in $N = 4$ super-Yang-Mills

on-shell



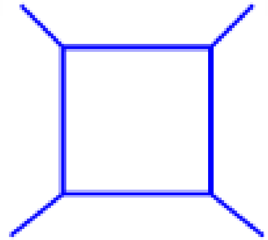
1 gluon, 4 gluinos, 6 real scalars

maximal susy

The basic two-particle sewing equation

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

Applying this at one-loop gives



$$\mathcal{A}_4^{\text{1-loop}}(1,2,3,4) = -st A_4^{\text{tree}} \mathcal{I}_4^{\text{1-loop}}(s,t)$$

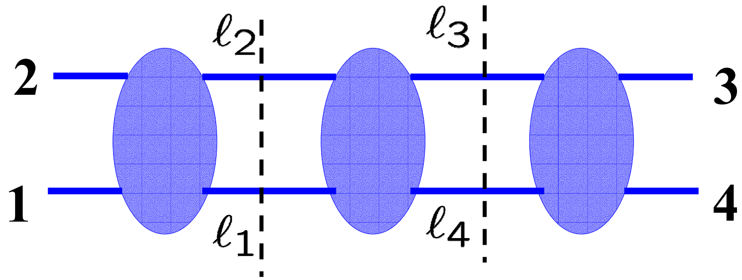
Agrees with known result of Green, Schwarz and Brink.

The two-particle cuts algebra recycles to all loop orders!

As I will explain in next lecture, this has important ramifications for $N = 8$ supergravity.

Example $N = 4$ Multi-loop Amplitude

Bern, Rozowsky and Yan



$$s = (k_1 + k_2)^2$$

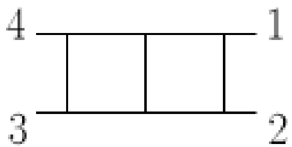
$$t = (k_1 + k_4)^2$$

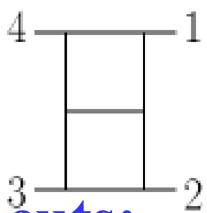
$$\sum_{N=4\text{states}} A_4^{\text{tree}}(1,2,\ell_2,\ell_1) \times A_4^{\text{tree}}(-\ell_1,-\ell_2,\ell_3,\ell_4) \times A_4^{\text{tree}}(-\ell_4,-\ell_3,3,4)$$

$$= -s^2 t \frac{A_4^{\text{tree}}(1,2,3,4)}{(\ell_1+k_1)^2(\ell_1-\ell_4)^2(\ell_4-k_4)^2}$$

Algebra same as at one loop

Get double box integrals:

$- st\, A_4^{\text{tree}} \Big\{ s$ 

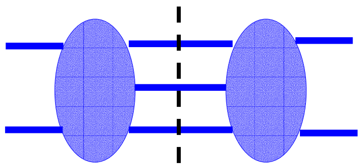
$+ \; t$ 

$\Big\}$

same integrals as in scalar φ^3 theory

Integrals known thanks to V. Smirnov

Also must evaluate three particle cuts:



No new contributions besides ones already found!

***N* = 8 supergravity is similar**

Anastasiou, Bern, Dixon, Kosower
Bern, Dixon and Smirnov
Alday and Maldacena

Loop Iteration of the $\mathcal{N}$=4 Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

Anastasiou, Bern, Dixon, Kosower

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{\text{1-loop}}(s,t)\Big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

where

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}}, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

$f(\epsilon)$ is universal function related to IR singularities $\qquad D = 4 - 2\epsilon$

Thus we have succeeded to express two-loop four-point planar amplitude as iteration of one-loop amplitude.

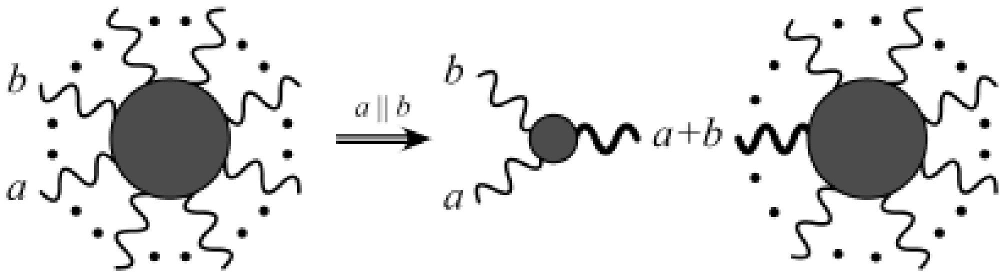
Recent confirmation directly on integrands. Cachazo, Spradlin and Volovich

Generalization to n Points

Anastasiou, Bern, Dixon, Kosower

Can we guess the n-point result? Expect simple structure.

Trick: use collinear behavior to guess



Bern, Dixon, Kosower

Have calculated two-loop splitting amplitudes.

Following ansatz satisfies all collinear constraints

$$M_n^{2\text{-loop}}(\epsilon) = \frac{1}{2}\left(M_n^{1\text{-loop}}(\epsilon)\right)^2 + f(\epsilon) M_n^{1\text{-loop}}(2\epsilon) - \frac{1}{2}\zeta_2^2$$

Valid for planar MHV amplitudes

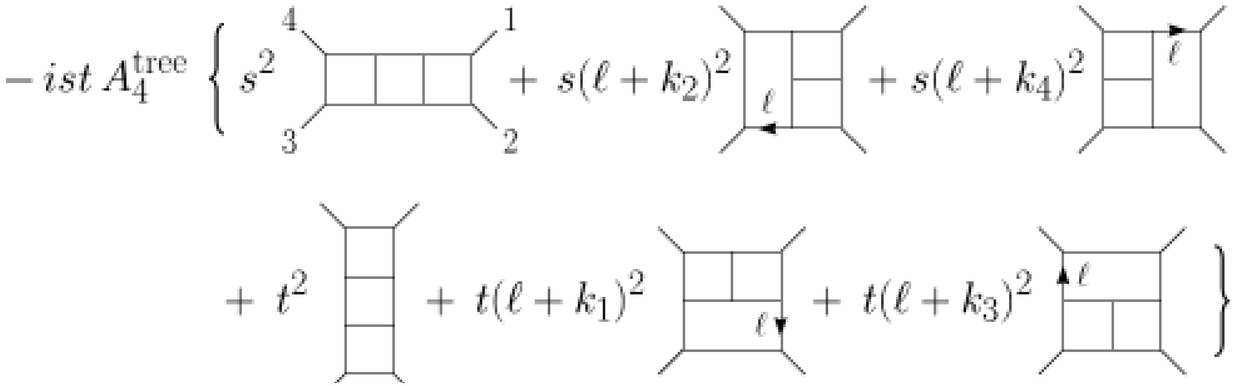
$$M_n^{\text{loop}} = A_n^{\text{loop}} / A_n^{\text{tree}}, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

$D = 4 - 2\epsilon$

Having correct factorization properties is crucial

Three-loop Generalization

From unitarity method we get three-loop planar integrand:



Bern, Rozowsky, Yan

Use Mellin-Barnes integration technology and apply hundreds of harmonic polylog identities:

V. Smirnov
Vermaseren and Remiddi

$$M_4^{\text{3-loop}}(\epsilon) = -\frac{1}{3}\Big[M_4^{\text{1-loop}}(\epsilon)\Big]^3 + M_4^{\text{1-loop}}(\epsilon)\, M_4^{\text{2-loop}}(\epsilon) + f^{\text{3-loop}}(\epsilon)\, M_4^{\text{1-loop}}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

Bern, Dixon, Smirnov

where

$$f^{\text{3-loop}}(\epsilon) = \frac{11}{2}\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2),$$

and

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2.$$

Answer actually does not actually depend on c_1 and c_2. Five-point calculation would determine these.

All-Leg All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop all legs (at least for MHV amplitudes)?

- **Remarkable formula from Magnea and Sterman tells us IR singularities to all loop orders. Confirms construction.**
- **Collinear limits gives us the key analytic information, at least for MHV amplitudes.**

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + \underbrace{C^{(l)}}_{\text{constant}} + \underbrace{E_n^{(l)}(\epsilon)}_{O(\epsilon)} \Big)\Big]$$

$$a = \frac{N_c \alpha_s}{2\pi} \qquad f^{(l)} = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} + \cdots$$

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}$$

- Soft anomalous dimension
- Or leading twist high spin anomalous dimension
- Or cusp anomalous dimension
- Or high moment limit of Altarelli-Parisi splitting kernel

All-loop Resummation in $N = 4$ Super-YM Theory

"BDS ansatz" for MHV amplitudes:

Anastasiou, ZB, Dixon, Kosower;
ZB, Dixon, Smirnov

$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude

constant independent of kinematics.

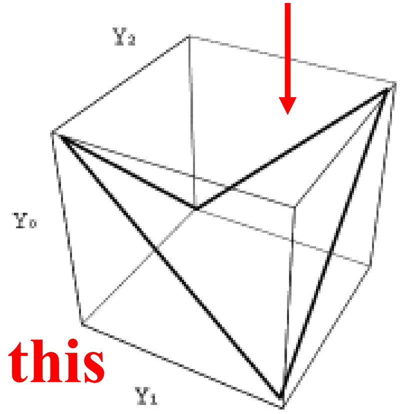
$$F_4^{\text{1-loop}} = \frac{1}{2}\ln^2(s/t) + \frac{2}{3}\pi^2$$

- The cusp anomalous dimension calculated through four loops.

ZB, Czakon, Dixon, Kosower, Smirnov

- Beisert, Eden, Staudacher integral equation gives a prediction of the cusp anomalous dimension for all values of the coupling.

Wilson loop



In a beautiful paper Alday and Maldacena confirmed this conjecture at strong coupling from AdS string computation.

See Alday's talk

New venue opened for studying AdS/CFT

Summary

In the next lecture we will apply the ideas discussed here to quantum gravity:

- Scattering amplitudes much simpler than anyone imagined
  -- on-shell simplicity exposed in twistor space.
- On-shell methods exploit the simplicity.
- KLT relations are simple tree-level relations between gravity and gauge theory.
- Unitarity method gives a means for exploiting tree-level simplicity to perform high-loop quantum calculations.
- Use the advances in our ability to compute in $N = 4$ super-Yang-Mill theory to compute in $N = 8$ supergravity to see if accepted wisdom on UV behavior is true or not.

Reading List: Techniques

- L. Dixon 1996 TASI lectures hep-ph/9601359 (trees basic ideas)
- Z. Bern, L. Dixon and D. Kosower, hep-ph/9602280 (early review on loops)
- Z. Bern, gr-qc/0206071 (applications to gravity)
- F. Cachazo and P. Svrcek, hep-th/0504194 (twistors)
- R. Britto, F. Cachazo, B. Feng and E. Witten, hep-th/0501052 (simple proof of on-shell recursion)
- L. Dixon, hep-ph/0512111 (recent overview)

Reading list – supergravity finiteness

Properties of gravity trees:

- J. Bedford, A. Brandhuber, B. Spence and G. Travaglini, hep-th/0502146. (Gravity recursion relations – large z ~ high energy)
- F. Cachazo, P. Svrcek, hep-th/0502160 (Gravity recursion relations)
- P. Benincasa, C. Boucher-Veronneau, F. Cachazo hep-th/0702032 (proof of large z behavior)

Loops. Studies of UV properties:

- Z. Bern, L. Dixon, R. Roiban, hep-th/0611086 (proposal of ultraviolet finiteness of $N = 8$ supergravity).
- Z. Bern, J.J. Carrasco, L. Dixon, D. Kosower, R. Roiban, hep-th/0702112 (three loop supergravity calculation).
- Z. Bern, J.J. Carrasco, D. Forde, H. Ita, H. Johansson, arXiv:0707.1035 [hep-th] (link to tree level properties).