Min-Max K-vehicles Windy Rural Postman Problem

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Abstract

In this paper the Min-Max version of the Windy Rural Postman Problem with several vehicles is introduced. For this problem, in which the objective is to minimize the length of the longest tour in order to find a set of balanced tours for the vehicles, we present here an ILP formulation and study its associated polyhedron. Based on its partial description, a branch-and-cut algorithm has been implemented and computational results on a large set of instances are finally presented.

Keywords: Rural Postman Problem, Windy Postman Problem, Windy Rural Postman Problem, Facets, multivehicles.

1   Introduction

The Windy Rural Postman Problem (WRPP) is an important Arc Routing Problem which generalizes most of the single-vehicle Arc Routing Problems, and can be defined as follows. Let $G = (V, E)$ be an undirected graph with two nonnegative costs c_{ij}, c_{ji} associated with each edge $e = (i, j)$, corresponding to the costs of traversing e from i to j and from j to i, respectively. Such a graph will be called a 'windy' graph in what follows. Let E_R be a subset of required edges (representing those edges requiring some service to be done along them). The WRPP consists of finding a tour of minimum cost, traversing each edge in E_R at least once.

The WRPP is interesting because some real-life situations can be modelled with windy graphs. For example, in Benavent et al. [8] the problem of finding the traversal of 3-dimensional structures such as bridges or building skeletons by a climbing robot is described. The problem is modelled as a WRPP because the cost (energy consumption) of traversing an edge (either inspecting a side of a beam or moving from it to another) in each of its two directions can be different. On the other hand, windy graphs include as special cases undirected, directed and mixed graphs, so all the theoretical results obtained from their study and the algorithms proposed for their resolution can also be applied to the knowledge and resolution of their special cases. Furthermore, the Rural Postman Problem also generalizes the famous Chinese Postman Problem (CPP), where all the edges of the graph are required.

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In this paper we deal with the more realistic situation where there exists a fleet of identical vehicles that will jointly serve the required edges. In the context of the above application, most real structures are so large that the robot is not capable of inspecting all the beams within the working period of time allowed by its on-board batteries. Then, several traversals for the robot need to be designed and the problem becomes a multi-vehicle problem.

Node Routing Problems with several vehicles, like the Capacitated Vehicle Routing Problem (CVRP) and its extensions, have been widely studied in the Literature. Its equivalent Arc Routing Problem is the Capacitated Arc Routing Problem (CARP) introduced by Golden and Wong in 1981 ([18]) that has recently received a lot of attention. Consider a fleet of identical vehicles of limited capacity based at a depot node and a subset of required edges with known demands to be serviced by the vehicles. The CARP consists of determining a set of vehicle tours, each one starting and ending at the depot, such that each required edge is serviced by only one vehicle, the demand associated with each tour does not exceed the vehicle capacity and the total cost is minimized. As far as we know, the only polyhedral study of the CARP appears in Belenguer and Benavent [7]. The situation for the case of the CVRP is similar. Despite of the huge number of papers on the CVRP, the literature on the polyhedral study of this problem is very scarce. The reason is that the dimension of the polyhedron, as well as the conditions for the inequalities to be facet-inducing, strongly depend on the demand values, thus leading to extremely involved conditions.

When the capacity of the vehicles is not restricted and we look for K routes starting and ending at the depot, in such a way that all the edges of the graph are serviced by exactly one vehicle and the total distance is minimized, we have the K-CPP which Assad, Pearn and Golden [5] showed to be solvable in polynomial time in the undirected and directed cases. Nevertheless, Pearn [20] proved that the K-CPP is NP-hard when it is defined on a mixed or windy graph. The Min-Max K-CPP, that was introduced in Frederickson, Hecht and Kim [14], is a more realistic problem, in which the objective is to minimize the length of the longest route among the K vehicles. This is a way of balancing the working load of the vehicles. Same authors proved that the Min-Max K-CPP is NP-hard and proposed a $(2-\frac{1}{K})$-approximation algorithm. More recently Ahr and Reinelt present several lower bounds and heuristics for this problem ([2]) and a Tabu Search procedure that produces very good solutions ([3]). Finally, in Ahr [1] some more results on the Min-Max K-CPP, including an exact solution method based on a branch-and-cut approach, are presented.

The problem we consider here is the Min-Max K-vehicles Windy Rural Postman Problem (MM K-WRPP) that can basically be defined as follows. Given a windy graph, a distinguished vertex (the depot), a subset of required edges and a fixed number K of vehicles, the problem consists of finding a set of K tours for the vehicles in such a way that each tour starts and ends at the depot and each required edge is serviced by exactly one vehicle. The objective is to minimize the length of the longest tour in order to find a set of balanced tours for the vehicles. Moreover, as noted in [3], "this kind of objective is preferable when the aim is to serve each customer as early as possible".

This paper is organized as follows. In Section 2 the definition and a formulation of the problem is presented. Also the notation that will be used through the paper and some valid inequalities are there introduced. Section 3 defines and partially describes the polyhedron associated with the MM K-WRPP solutions. A Branch-and-Cut algorithm based on the polyhedral description of the MM K-WRPP is presented in Section 4, while the computational results obtained on a large set of instances are shown in Section 5. Finally in Section 6 some

conclusions are drawn.

2 The Min-Max K-vehicles Windy RPP

Let $G = (V, E)$ be an undirected graph with two costs c_{ij}, c_{ji} associated with each edge $e = (i, j)$. Let us assume that node $1 \in V$ represents the depot. Let $E_R \subset E$ be the set of required edges. We have a given number K of vehicles and we have to find a tour (closed walk starting and ending at the depot) for each vehicle such that each required edge is traversed at least once by at least one vehicle. Let us call K-WRPP solution to a set of such tours (one per vehicle). The goal is to find a K-WRPP solution such that the cost of the maximum cost vehicle tour is minimum.

For the sake of simplicity we assume that each vertex in V is incident with at least one required edge. This is not a restriction as there exists a simple way to transform an instance not satisfying this assumption into an equivalent one which does (see, e.g., Christofides et al. [10] or Eiselt et al. [13]).

For each edge $e = (i, j) \in E$ we define $2K$ variables x_{ij}^k, x_{ji}^k representing the number of times edge e is traversed by vehicle k from i to j or from j to i, respectively. In addition, if edge e is required, we define K more variables y_{ij}^k which take the value 1 if edge e is serviced by the vehicle k and 0 otherwise. Finally, an artificial variable z is used to minimize the maximum tour cost.

Given two node subsets $S, S' \subseteq V$, $(S : S')$ denotes the edge set with one end-point in S and the other in S'. Given a node subset, $S \subseteq V$, let us denote $\delta(S) = (S : V \setminus S)$ and let $E(S) = \{(i, j) \in E : i, j \in S\}$ be the set of edges with both endpoints in S. The previous sets restricted to the required edges are denoted by $\delta_R(S)$, $E_R(S)$ and $(S : S')_R$. Finally, for any subset $F \subseteq E$, $x^k(F)$ denotes $\displaystyle\sum_{(i,j)\in F} (x_{ij}^k + x_{ji}^k)$.

We propose the following formulation for the Min-Max K-WRPP:

$$\text{Minimize} \quad z$$

s.t.:

$$\sum_{e \in E} \left(c_{ij} x_{ij}^k + c_{ji} x_{ji}^k \right) \leq z \qquad \forall k = 1, \ldots, K \tag{1}$$

$$\sum_{\forall k} y_e^k = 1, \qquad \forall e \in E_R \tag{2}$$

$$x_{ij}^k + x_{ji}^k \geq y_e^k \qquad \forall e \in E_R \text{ and } \forall k = 1, \ldots, K \tag{3}$$

$$\sum_{(i,j) \in \delta(i)} (x_{ij}^k - x_{ji}^k) = 0, \qquad \forall i \in V \text{ and } \forall k \tag{4}$$

$$x^k(\delta(S)) \geq 2y_e^k, \qquad \forall S \subset V \setminus \{1\}, \quad \forall e \in E_R(S), \quad \forall k = 1, \ldots, K \tag{5}$$

$$x_{ij}^k, x_{ji}^k \geq 0 \text{ and integer} \qquad \forall (i,j) \in E, \quad \forall k = 1, \ldots, K \tag{6}$$

$$y_e^k \in \{0, 1\} \qquad \forall e \in E_R \quad \forall k = 1, \ldots, K \tag{7}$$

3

Inequalities (1) imply that the maximum cost vehicle route is minimized. Equations (2) assure that each required edge is serviced by exactly one vehicle and inequalities (3), called *traversing inequalities*, force a vehicle to traverse the edges it serves. *Symmetry equations* (4) force each vehicle route to be symmetric, while *connectivity inequalities* (5) ensure that each vehicle route connects the edges it serves and the depot.

A solution for the K-WRPP on G is a vector $(x^1, y^1, x^2, y^2, \ldots, x^K, y^K)$ with $(2|E| + |E_R|)K$ components satisfying (2) to (7). We will call *route* associated with vehicle k to the pair (x^k, y^k) and *tour* associated with vehicle k to x^k. Note that this formulation allows solutions in which a vehicle tour x^k is formed by several disconnected subtours, one of them connecting all the edges it serves to the depot and the others traversing edges not serviced by this vehicle. In particular, if $(x^1, y^1, x^2, y^2, \ldots, x^K, y^K)$ is a K-WRPP solution and $B(e, k) \in \mathbb{Z}^{(2|E|+|E_R|)K}$ is a vector with the two components $x_{ij}^k = x_{ji}^k = 1$, for a given edge $e = (i, j) \in E$ and a vehicle k, and all the other components equal to zero, then $(x^1, y^1, x^2, y^2, \ldots, x^K, y^K) + MB(e, k)$ is also a K-WRPP solution, for any positive integer M. This property is needed for the convex hull of the K-WRPP solutions on G to be a polyhedron (see Belenguer and Benavent [7]). Note also that solutions where a given vehicle neither traverses nor serves any edges are also allowed ($x^k = y^k = 0$).

2.1   Other valid inequalities

In order to solve the problem with a Branch & Cut method, the above linear formulation should be strengthen. Consider the (1 vehicle) WRPP defined on graph G and let $F(x) \geq c_0$ be a valid inequality for this problem. Given a K-WRPP solution $(x^1, y^1, x^2, y^2, \ldots, x^K, y^K)$, adding all the vehicle tours we obtain that $x = \sum x^k$ is a WRPP tour on G. Then, x satisfies $F(x) \geq c_0$, or $\sum_k F(x^k) \geq c_0$, which turns to be valid for the K-WRPP. We will call aggregate inequalities to these inequalities. R-odd cut, K-C, Honeycomb and other inequalities for the WRPP described in Corberán et al. [11] are then valid inequalities for the MM K-WRPP. In particular, the aggregate R-odd cut inequalities are:

$$\sum_{k=1}^{K} x^k(\delta(S)) \geq |\delta_R(S)| + 1, \quad \forall S \subset V \quad \text{such that } |\delta_R(S)| \text{ is odd} \tag{8}$$

and are based on the fact that any K-WRPP solution must cross any given edge cutset an even number of times. On the other hand, when we consider a single vehicle k, all the edges $e \in E_R$ can not be considered as 'required' for vehicle k because it could either not serve nor traverse e. The actual required edges for a given vehicle k are those serviced by it, which are determined by the vector y^k. Then, given a route (x^k, y^k), the tour x^k must satisfy the conditions derived from the fact that edges $e \in E_R$ such that $y_e^k = 1$ are its required edges. This is the idea behind the disaggregate inequalities as the one we present in what follows. For example, let $\delta(S)$ be an edge cutset on G. If a given vehicle k serves a subset $F \subset \delta_R(S)$ of required edges and $|F|$ is odd, then this vehicle has to traverse the cutset $\delta(S)$ at least $|F| + 1$ times. Nevertheless, the inequality $x^k(\delta(S)) \geq |F| + 1$ is not valid for KWRPP(G) since vehicle k is not forced to serve the edges in F and therefore it does not have to necessarily traverse this cutset $|F| + 1$ times. A generalization of the R-odd inequalities involving variables y^k is then needed.

Let $\delta(S)$ be an edge cutset on G and let $F \subset \delta_R(S)$ be a subset of required edges with

$|F|$ odd. For each vehicle k, we call parity inequality to

$$x^k(\delta(S)) \geq 2y^k(F) - |F| + 1 \qquad (9)$$

Theorem 1 *Parity inequalities (9) are valid for the MM K-WRPP.*

Proof: If vehicle k serves all the edges in F then $y^k(F) = |F|$ and the RHS of the inequality is $|F| + 1$. In this case, since $|F|$ is odd, vehicle k has to traverse the cutset at least $|F| + 1$ times and $x^k(\delta(S)) \geq |F| + 1$ holds. If vehicle k serves all the edges in F but one then $y^k(F) = |F| - 1$, the RHS of the inequality is $|F| - 1$ and the inequality is trivially satisfied. Other cases are also trivial. ♦

Parity inequalities (9) are also known as cocircuit inequalities ([6]) and have also been used in other Arc Routing Problems ([7], [17]).

The argument supporting the validity of the above inequalities can be generalized from a single vehicle to several ones. For instance, if all the edges in $F \subset \delta_R(S)$ are serviced by a subset of vehicles then these vehicles have to jointly traverse the cutset at least $|F| + 1$ times. Let $\delta(S)$ be an edge cutset on G and let $F \subset \delta_R(S)$ be a subset of required edges with $|F|$ odd. For each subset of $P < K$ vehicles $\{k_1, k_2, \ldots, k_P\}$, we will call P-aggregate parity inequality to

$$x^{k_1}(\delta(S)) + \cdots + x^{k_P}(\delta(S)) \geq 2y^{k_1}(F) + \cdots + 2y^{k_P}(F) - |F| + 1 \qquad (10)$$

Theorem 2 *P-aggregate parity inequalities (10) are valid for the MM K-WRPP.*

Proof: The proof is similar to that of Theorem 1. Note that if all the edges in F are served by vehicles $k_1, k_2, \ldots, k_P$ then $y^{k_1}(F) + \cdots + y^{k_P}(F) = |F|$ and the RHS of the inequality is $|F| + 1$. In this case, vehicles $k_1, k_2, \ldots, k_P$ have to jointly traverse the cutset at least $|F| + 1$ times and the inequality holds. ♦

The above ideas about obtaining disaggregate and P-aggregate inequalities from valid inequalities for the (1 vehicle) WRPP can also be applied to K-C, Honeycomb, Path-Bridge and other inequalities. This will be the subject of future work.

3 The K-WRPP polyhedron

Let KWRPP(G) be the convex hull of all the K-WRPP solutions, i.e., of all vectors $(x^1, y^1, x^2, y^2, \ldots, x^K, y^K) \in \mathbb{Z}^{K\left(2|E|+|E_R|\right)}$ satisfying (2) to (7). It can be seen that KWRPP(G) is a polyhedron. Note that we include neither variable z nor inequalities (1) because this does not affect the set of feasible solutions of our problem. Moreover the polyhedral study done in this paper can be applied to similar problems with different objective functions such as minimizing the total cost of the K routes as in the K-CPP.

Theorem 3 *If G is connected then $dim(KWRPP(G)) = K\left(2|E| - |V| + 1\right) + (K-1)|E_R|$*

Proof: KWRPP(G) is a polyhedron in $\mathbb{Z}^{K\left(2|E|+|E_R|\right)}$. All its points satisfy the $|E_R|$ linearly independent equations (2) and the $K|V|$ equations (4), $K|V-1|$ of which are linearly independent. Hence,

$$\dim(\text{KWRPP}(G)) \leq K\big(2|E|-|V|+1\big)+(K-1)|E_R|$$

Let us prove that $\dim(\text{KWRPP}(G)) \geq K\big(2|E|-|V|+1\big)+(K-1)|E_R|$ by finding such a number plus one affinely independent (or linearly independent, because $0 \notin \text{aff}(\text{KWRPP}(G))$) K-WRPP solutions.

Consider the (1 vehicle) WRPP defined on G and let WRPP(G) denote its associated polyhedron. It is known ([11]) that, if G is connected, $\dim(\text{WRPP}(G)) = 2|E|-|V|+1$. Let us call $m = \dim(\text{WRPP}(G))$. Since $0 \in \text{aff}(\text{WRPP}(G))$, there exist m linearly independent WRPP tours $z_1, z_2, \ldots, z_m$ on G which traverse all the edges in E_R at least once. Note that we can build K-WRPP solutions in the following way. One vehicle performs any WRPP tour z_j above and serves all the required edges while the other vehicles do nothing. In a more general way, we can select some vehicles to perform WRPP tours z_j above while the other vehicles do not leave the depot. The required edges are assigned to be serviced by the former vehicles in an arbitrary way. For the sake of simplicity, let us suppose we have $K = 3$ vehicles. We can define K-WRPP solutions arranged as rows of the matrix shown in Figure 1, where a block with a large 0 (or 1) represents a submatrix with all its entries 0 (or 1), I represents the identity matrix and a block with a large Z_1 means a submatrix with all its rows equal to the WRPP tour z_1. These $K\big(2|E|-|V|+1\big)+(K-1)|E_R|+1$ K-WRPP solutions are linearly independent because the matrix has full rank. Note that, from the structure of the matrix, we can assert that the result is also true for any value of $K \geq 2$. ◆

In what follows we study if the inequalities in the formulation induce facets of KWRPP(G). As in Theorem 3 above, in the proofs of the theorems in this section we will represent the K-WRPP solutions we define only for $K = 3$ vehicles, although they can be extended to any value of K. Let us begin with the trivial inequalities.

3.1 Trivial inequalities

Theorem 4 *If edge $e = (i,j) \in E$ is not a bridge of G, then inequalities $x_{ij}^k \geq 0$ and $x_{ji}^k \geq 0$ are facet-inducing for KWRPP(G) for all k.*

Proof: Let us consider, for example, $x_{ij}^1 \geq 0$. We define $d = \dim(\text{KWRPP}(G))$ affinely independent K-WRPP solutions satisfying $x_{ij}^1 = 0$. Since edge (i,j) is not a bridge of G then inequality $x_{ij} \geq 0$ is facet-inducing for WRPP(G) (see [11]) and there are $m = \dim(\text{WRPP}(G)) = 2|E|-|V|+1$ affinely independent WRPP tours on G satisfying $x_{ij} = 0$, say $w_1, w_2, \ldots, w_m$. Let us suppose that $w_1, w_2, \ldots, w_{m-1}$ are linearly independent. As in the proof of theorem 3, let $z_1, z_2, \ldots, z_m$ be m linearly independent WRPP tours on G. The K-WRPP solutions arranged as the rows of the matrix in Figure 2 proves the result for $K = 3$. ◆

X^1	X^2	X^3	Y^1	Y^2	Y^3
z_1 z_2 ... z_m	0	0	1	0	0
0	z_1 z_2 ... z_m	0	0	1	0
0	0	z_1 z_2 ... z_m	0	0	1
Z_1	Z_1	0	I	$1-I$	0
0	Z_1	Z_1	0	I	$1-I$
z_1	z_1	z_1	$1\ 0\ ...\ 0$	$0\ 1\ ...\ 1$	$0\ 0\ ...\ 0$

Figure 1: K-WRPP solutions to prove dimension

X^1	X^2	X^3	Y^1	Y^2	Y^3
w_1 w_2 ... w_{m-1}	0	0	1	0	0
0	z_1 z_2 ... z_m	0	0	1	0
0	0	z_1 z_2 ... z_m	0	0	1
W_1	Z_1	0	I	$1-I$	0
0	Z_1	Z_1	0	I	$1-I$
w_1	z_1	z_1	$1\ 0\ ...\ 0$	$0\ 1\ ...\ 1$	$0\ 0\ ...\ 0$

Figure 2: K-WRPP solutions satisfying $x^1_{ij} = 0$

Theorem 5 *If edge $e = (i, j) \in E$ is not a bridge of G, then inequality $y_e^k \geq 0$ is facet-inducing for KWRPP(G) for all k.*

Proof: Let us consider, for example, $y_e^1 \geq 0$. Again, let $z_1, z_2, \ldots, z_m$ be m linearly independent WRPP tours on G. The K-WRPP solutions arranged as the rows of the matrix in Figure 3 satisfy $y_e^1 = 0$. Note that there are $\dim(\text{KWRPP}(G))$ rows, because blocks in (d) have only $|E_R| - 1$ rows). It is easy to see that this matrix has full rank if we do as follows. Subtract the first row of block (d) and the first row of block (c) from the last row. Subtract the first row of block (b) and the first row of block (c) from the rows in block (e). Subtract the first row of block (a) from the rows in block (d). ⬧

	X^1	X^2	X^3	$e \quad Y^1$	$e \quad Y^2$	Y^3
(a)	z_1 z_2 ... z_m	z_1 z_2 ... z_m	0	0 0 $\vdots$ 1 0	1 1 $\vdots$ 0 1	0
(b)	0	z_1 z_2 ... z_m	0	0	1	0
(c)	0	0	z_1 z_2 ... z_m	0	0	1
(d)	Z_1	Z_1	0	0 $\vdots$ I 0	1 $\vdots$ 1-I 1	0
(e)	0	Z_1	Z_1	0	I	1-I
(f)	z_1	z_1	z_1	$0\ 1\ 0\ ...\ 0$	$1\ 0\ 1\ ...\ 1$	$0\ 0\ ...\ 0$

Figure 3: K-WRPP solutions satisfying $y_e^1 = 0$

Note: Trivial inequalities $y_e^k \leq 1$ for $e \in E_R$ are implied by $y_e^k \geq 0$ and the system equations (2) $\sum_{k=1}^{K} y_e^k = 1$.

3.2 Traversing inequalities

Theorem 6 *If edge $e = (i, j) \in E_R$ is not a bridge of G, then the traversing inequality (3), $x_{ij}^k + x_{ji}^k \geq y_e^k$, is facet-inducing for KWRPP(G) for all k.*

Proof: The proof is similar to that of Theorem 5 and is omitted here for the sake of brevity.

3.3   Connectivity inequalities

Theorem 7 *Connectivity inequalities (5), $x^k(\delta(S)) \geq 2y_e^k$, for all $S \subset V \setminus \{1\}$ and $e \in E_R(S)$ and for all k are facet-inducing for KWRPP(G) if subgraphs $G(S)$ and $G(V \setminus S)$ are connected.*

Proof: Let $S \subset V \setminus \{1\}$ and $e \in E_R(S)$. We define a WRPP instance G' in which all the edges in E are required except those in $\delta(S)$. Given that subgraphs $G(S)$ and $G(V \setminus S)$ are connected, inequality $x(\delta(S)) \geq 2$ is facet-inducing for WRPP(G') (see [11]) and there exist m linearly independent tours $w_1, w_2, \ldots, w_m$ for the WRPP on G' satisfying $x(\delta(S)) = 2$. Note that these tours traverse only one or two edges in $\delta(S)$ and then they are not able to serve all the edges in $\delta_R(S)$. Nevertheless, it can be seen that given any edge $f \in E_R \setminus \{e\}$, there is a tour w_i above that traverses edges e and f (rows (d) of the matrix in Figure 4). These d vectors represented as rows are linearly independent feasible solutions for the K-WRPP and satisfy $x^1(\delta(S)) = 2y_e^1$. $\blacklozenge$

	X^1	X^2	X^3	e Y^1	e Y^2	Y^3
(a)	w_1 w_2 ... w_m	Z_1	0	1 1 $\vdots$ 0 1	0 0 $\vdots$ 1 0	0
(b)	0	z_1 z_2 ... z_m	0	0	1	0
(c)	0	0	z_1 z_2 ... z_m	0	0	1
(d)	w_i ... w_j	Z_1	0	1 $\vdots$ I 1	0 $\vdots$ 1-I 0	0
(e)	0	Z_1	Z_1	0	I	1-I
(f)	w_i	z_1	z_1	1 1 0 ... 0	0 0 1 ... 1	0 0 ... 0

Figure 4: K-WRPP solutions satisfying $x^1(\delta(S)) = 2y_e^1$

Note: Aggregate connectivity inequalities $\sum_k x^k(\delta(S)) \geq 2$ are not facet-inducing because they can be obtained by adding the k facet-inducing connectivity inequalities (5) $x^k(\delta(S)) \geq 2y_e^k$ associated with a required edge e, where $e \in E_R(V \setminus S)$ ($e \in E_R(S)$) if the depot $1 \in S$ ($1 \in V \setminus S$).

3.4 R-odd cut and parity inequalities

Theorem 8 *Aggregate R-odd cut inequalities (8), $\sum_{k=1}^{K} x^k(\delta(S)) \geq |\delta_R(S)|+1$, for all $S \subset V$ such that $|\delta_R(S)|$ is odd, are facet-inducing for KWRPP(G) if $|\delta_R(S)| \geq 3$ and subgraphs $G(S)$, $G(V \setminus S)$ are connected.*

Proof: Given that $G(S)$ and $G(V \setminus S)$ are connected, the R-odd cut inequality $x(\delta(S)) \geq |\delta_R(S)|+1$ is facet-inducing for WRPP(G) ([11]) and there are m linearly independent tours $w_1, w_2, \ldots, w_m$ for the WRPP on G satisfying $x(\delta(S)) = |\delta_R(S)| + 1$. Hence, the vectors represented in the rows of the 3 first blocks of matrix shown in Figure 5 are feasible K-WRPP solutions. We also need feasible K-WRPP solutions satisfying $\sum_{k=1}^{K} x^k(\delta(S)) = |\delta_R(S)| + 1$

	X^1	X^2	X^3	Y^1	Y^2	Y^3
(a)	w_1 w_2 ... w_m	0	0	1	0	0
(b)	0	w_1 w_2 ... w_m	0	0	1	0
(c)	0	0	w_1 w_2 ... w_m	0	0	1
(d)	w_i^* ... w_r^*	w_j^* ... w_s^*	0	B	1-B	0
(e)	0	w_i^* ... w_r^*	w_j^* ... w_s^*	0	B	1-B

Figure 5: *K-WRPP solutions satisfying $\sum_{k=1}^{K} x^k(\delta(S)) = |\delta_R(S)| + 1$*

and using two vehicles simultaneously. This is possible since $|\delta_R(S)| \geq 3$. Let $p \geq 2$ be the integer such that $|\delta_R(S)| = 2p-1$. The R-odd cut inequality (8) is then $\sum_{k=1}^{K} x^k(\delta(S)) \geq 2p$. For each edge $e \in E_R$ we define a feasible K-WRPP solution using vehicles 1 and 2 in the following way. Let us suppose that $1 \in S$.

- If $e \in \delta_R(S)$ then vehicle 1 serves only edge e and traverses the cutset $\delta(S)$ twice. Vehicle 2 serves all the edges in $E_R \setminus \{e\}$ and traverses $\delta(S)$ $2p-2$ times.

- If $e \in E_R(S)$ then vehicle 1 serves only edge e and does not traverse the cutset $\delta(S)$. Vehicle 2 serves all the edges in $E_R \setminus \{e\}$ and traverses the cutset $\delta(S)$ $2p$ times.

- If $e \in E_R(V \setminus S)$ then vehicle 1 serves edge e and another required edge $\bar{e} \in \delta(S)$ and traverses the cutset twice. Vehicle 2 serves all the edges in $E_R \setminus \{e, \bar{e}\}$ and traverses the cutset $\delta(S)$ $2p-2$ times.

We repeat the above construction for vehicles 2 and 3, 3 and 4 and so on. Hence, the vectors represented in the rows of the blocks (d) and (e) of matrix shown in Figure 5 are also feasible K-WRPP solutions satisfying $\sum_{k=1}^{K} x^k(\delta(S)) = 2p$, where matrix block B is:

$$
B =
\begin{array}{c|c|c|c}
 & \delta_R(S) & E_R(S) & E_R(V\setminus S) \\
\hline
\begin{matrix}\bar{e}\end{matrix} & I & 0 & 0 \\
\hline
 & 0 & I & 0 \\
\hline
\begin{matrix}1\\1\\\vdots\\1\end{matrix} & 0 & 0 & I
\end{array}
$$

In order to see that the matrix shown in Figure 5 has full rank we proceed as follows. Given that the m vectors $w_1, w_2, \ldots, w_m$ satisfy $x(\delta(S)) = 2p$, if we add the columns corresponding to the variables X^1 in the cutset $\delta(S)$ and divide this sum by $2p$ we obtain a column with all its entries in the block (a) equal to 1. By subtracting this column from those columns corresponding to Y^1 we obtain a 0 block in the rows in (a) and columns corresponding to Y^1. We proceed in the same way with the columns corresponding to vehicle 2, and with the columns corresponding to vehicle 3 and we transform the '1' blocks in (b) and (c) into '0' blocks. Obviously, blocks B in (d) and (e) described above also change. Note that the proof will be complete if we prove that the matrix in which B is transformed is non singular.

From each column of B in (d) we have subtracted the sum of the columns of (d) corresponding to the variables X^1 in the cutset $\delta(S)$ divided by $2p$. From the definition of the K-WRPP solutions corresponding to (d), what we have subtracted is 0 for the edges in $E_R(S)$ and $\frac{2}{2p} = \frac{1}{p}$ for the edges in $\delta_R(S) \cup E_R(V\setminus S)$. Then B has been transformed into:

$$
\begin{array}{c|c|c|c}
 & \delta_R(S) & E_R(S) & E_R(V\setminus S) \\
\hline
\begin{matrix}\bar{e}\end{matrix} & I-\frac{1}{p}\mathbf{1} & -\frac{1}{p}\mathbf{1} & -\frac{1}{p}\mathbf{1} \\
\hline
 & 0 & I & 0 \\
\hline
\begin{matrix}\frac{p-1}{p}\\\frac{p-1}{p}\\\vdots\\\frac{p-1}{p}\end{matrix} & -\frac{1}{p}\mathbf{1} & -\frac{1}{p}\mathbf{1} & I-\frac{1}{p}\mathbf{1}
\end{array}
$$

11

Subtracting the first row from all the rows in the bottom blocks, a non singular matrix is obtained because the $(2p-1)\times(2p-1)$ submatrix in the top-left block is non singular. ♦

Note: If $\delta_R(S) = \{e\}$ the aggregate R-odd cut inequality (8), $\sum_k x^k(\delta(S)) \geq 2$, is not facet-inducing because it is the sum of K parity inequalities (9), $x^k(\delta(S)) \geq 2y_e^k$ (taking $F = \{e\}$), that are showed to be facet-inducing in what follows.

Theorem 9 *Parity inequalities (9), $x^k(\delta(S)) \geq 2y^k(F) - |F| + 1$, for all $S \subset V$, for all $F \subset \delta_R(S)$ such that $|F|$ is odd and for all k are facet-inducing for KWRPP(G) if subgraphs $G(S)$ and $G(V \setminus S)$ are connected.*

Proof: Let G' be the WRPP instance in which all the edges in G are required except those in $\delta_R(S) \setminus F$. Given that $G(S)$ and $G(V \setminus S)$ are connected, inequality $x(\delta(S)) \geq |F| + 1$ is facet-inducing for WRPP(G') ([11]) and therefore there are m linearly independent tours $w_1, w_2, \ldots, w_m$ for the WRPP on G' satisfying $x(\delta(S)) = |F| + 1$ and traversing the edges in F. Let $z_1, z_2, \ldots, z_m$ be as in the proof of Theorem 3. Furthermore, for each $e \in F$ let w_i^* be a tour on G satisfying $x(\delta(S)) = |F| - 1$ which traverses all the required edges in F except e. Finally, for each edge $e \in E_R \setminus F$ let w_i be the tour on set $\{w_1, w_2, \ldots, w_m\}$ which traverses all the edges in F exactly once plus edge e. It can be seen that the rows of matrix shown in Figure 6 represent feasible K-WRPP solutions on G and that they are linearly independent. ♦

Theorem 10 *P-aggregate parity inequalities (10), $x^{k_1}(\delta(S)) + \cdots + x^{k_P}(\delta(S)) \geq 2y^{k_1}(F) + \cdots + 2y^{k_P}(F) - |F| + 1$, for all $S \subset V$, for all $F \subset \delta_R(S)$ such that $|F|$ is odd and for every set $\{k_1, k_2, \ldots, k_P\}$ of $P < K$ vehicles are facet-inducing for KWRPP(G) if subgraphs $G(S)$ and $G(V \setminus S)$ are connected and $|F| \geq 3$.*

Proof: Let $p \geq 2$ be the integer such that $|F| = 2p - 1$. As in the proof of Theorem 9, if G' is the WRPP instance in which all the edges in G are required except those in $\delta_R(S) \setminus F$ then there exist m linearly independent tours $w_1, w_2, \ldots, w_m$ for the WRPP on G' satisfying $x(\delta(S)) = 2p$. Let $z_1, z_2, \ldots, z_m$ be as in the proof of Theorem 3. For each $e \in F$ let w_i^* be a tour on G satisfying $x(\delta(S)) = 2p - 2$ which traverses all the required edges in F except e and let $\overline{w}_i$ be a tour on G satisfying $x(\delta(S)) = 2$ that traverses edge e. Finally, for each edge $e \in E_R \setminus F$ let w_i be the tour on set $\{w_1, w_2, \ldots, w_m\}$ traversing all the edges in $F \cup \{e\}$ exactly once. For the sake of simplicity, in what follows we will assume $K = 3$ and $P = 2$ ($k_1 = 1$, $k_2 = 2$). It can be seen that the rows of matrix shown in Figure 7 represent feasible K-WRPP solutions on G satisfying the inequality (10), $x^1(\delta(S)) + x^2(\delta(S)) \geq 2y^1(F) + 2y^2(F) - |F| + 1$ as an equality. To show they are linearly independent we will work with the matrix columns as in Theorem 8 in order to transform the two 1 blocks in the columns associated with Y_F^2 into zeros. Adding the columns corresponding to the variables X^2 in the cutset $\delta(S)$ and dividing this sum by $2p$ we obtain a column with all its entries corresponding to the vectors $w_1, w_2, \ldots, w_m$ equal to 1 (since they satisfy $x(\delta(S)) = 2p$) and with all its entries corresponding to the vectors $\overline{w}_i, \ldots, \overline{w}_j$ equal to $\frac{1}{p}$ (since they satisfy $x(\delta(S)) = 2$). By subtracting this column from those corresponding to Y_F^2, all its blocks become 0 except for the block corresponding to vectors $\overline{w}_i$ which becomes I$-\frac{1}{p}$1 (that is non singular since its size is $(2p-1)\times(2p-1)$). It is easy to see that the submatrix formed by the 9 top left blocks has full rank and hence the complete matrix has also full rank. ♦

X^1	X^2	X^3	Y_F^1	Y_F^2	Y_F^3	$Y_{\overline{F}}^1$	$Y_{\overline{F}}^2$	$Y_{\overline{F}}^3$
w_1 w_2 ... w_m	Z_1	0	1	0	0	0	1	0
W_1	$2z_1$ z_2 ... z_m	0	1	0	0	0	1	0
W_1	Z_1	z_1 z_2 ... z_m	1	0	0	0	0	1
w_i^* ... w_j^*	Z_1	0	1-I	I	0	0	1	0
w_i^* ... w_j^*	0	Z_1	1-I	0	I	0	0	1
w_i ... w_j	Z_1	0	1	0	0	I	1-I	0
W_1	Z_1	Z_1	1	0	0	0	I	1-I

Figure 6: K-WRPP solutions satisfying $x^1(\delta(S)) = 2y^1(F) - |F| + 1$

Note: If $F = \{e\}$ then the P-aggregate parity inequality (10) is not facet-inducing because it is the sum of the parity inequalities (9) corresponding to each vehicle k_i, $x^{k_i}(\delta(S)) \geq 2y_e^{k_i}$.

4 The Branch & Cut algorithm

In this section an exact algorithm for the MM K-WRPP is described. The Branch & Cut method presented here is based on a cutting-plane procedure that identifies violated inequalities of several classes.

4.1 Initial relaxation

The initial LP relaxation contains equations (2), assuring that each required edge is serviced by exactly one vehicle, symmetry equations (4), min-max inequalities (1), traversing inequalities (3) and a new family of constraints whose aim is to reduce the symmetry of polyhedron KWRPP(G). Note that if $(x^1, y^1, x^2, y^2, \ldots, x^K, y^K) \in$ KWRPP(G) then any arrangement of the single routes, such as $(x^2, y^2, x^1, y^1, \ldots, x^K, y^K)$ also belongs to KWRPP(G). In other words, given a point in KWRPP(G), if the routes associated with two different vehicles

X^1	X^2	X^3	Y^1_F	Y^2_F	Y^3_F	$Y^1_{\overline{F}}$	$Y^2_{\overline{F}}$	$Y^3_{\overline{F}}$
w_1 w_2 ... w_m	0	Z_1	1	0	0	0	0	1
0	w_1 w_2 ... w_m	Z_1	0	1	0	0	0	1
W_1	0	$2z_1$ z_2 ... z_m	1	0	0	0	0	1
w_i^* ... w_j^*	$\overline{w}_i$ ... $\overline{w}_j$	Z_1	$1\text{-}I$	I	0	0	0	1
w_i^* ... w_j^*	0	Z_1	$1\text{-}I$	0	I	0	0	1
w_i ... w_j	0	Z_1	1	0	0	I	0	$1\text{-}I$
0	w_i ... w_j	Z_1	0	1	0	0	I	$1\text{-}I$

Figure 7: K-WRPP solutions satisfying $x^1(\delta(S)) + x^2(\delta(S)) = 2y^1(F) + 2y^2(F) - |F| + 1$

are switched, another feasible point in KWRPP(G) is obtained. Therefore we have a very large number of equivalent solutions. This property is specially bad for the behavior of cutting-plane algorithms, and in order to avoid this problem, we introduce the following set of additional constraints as suggested by Gendreau ([15]) and Ghiani et al. ([16]). Let $(e_1, e_2, \ldots, e_m)$ be any ordering of the required edges . The idea is to force the numbering of vehicles to follow the numbering of the smallest index edge that they service. This can be done as follows ([16]):

$$y^1_{e_1} = 1 \tag{11}$$

$$y^k_{e_i} \leq \sum_{j=1,\ldots,i-1} y^{k-1}_{e_j} \qquad k=3,\ldots,K, \quad i \geq 2 \tag{12}$$

$$y^k_{e_i} = 0 \qquad k=i+1,\ldots,K, \quad i=1,\ldots,m-1 \tag{13}$$

Vehicle 1 will serve edge e_1. Constraints (12) state that if a required edge e_i is serviced by vehicle k then at least one 'previous' edge $e_j, j = 1, \ldots, i-1$, has to be serviced by the vehicle $k-1$, while constraints (13) prevent edges $e_i, i = 1, \ldots, m-1$ from being serviced by vehicles with indices larger than i.

We have noted that the strength of the above inequalities depends to a great extent on the ordering chosen for the required edges. After some computational testing, we decided to

14

order the required edges following a simple rule: the first required edge is the farthest one from the depot, the second edge is the farthest one from both, the depot and the first edge, and so on.

4.2 Separation Algorithms

In this section, we present the separation algorithms that have been used to identify the inequalities that are violated by the current LP solution at any iteration of the cutting plane algorithm.

Given an LP solution $(\overline{x}^1, \overline{y}^1, \overline{x}^2, \overline{y}^2, \ldots, \overline{x}^K, \overline{y}^K) \in \mathbb{R}^{(2|E|+|E_R|)K}$ we define the 'aggregate' vector $\overline{x}$ as $\overline{x} = \sum_{k=1}^{K} \overline{x}^k$ and its associated weighted graph $\overline{G} = (V, E, E_R, \overline{x})$, where V, E and E_R are the sets of vertices, edges and required edges of the original graph G and $\overline{x}_e = \overline{x}_{ij} + \overline{x}_{ji}$ is the weight associated with each edge $e \in E$. This 'aggregate' graph is the input to the separation procedures described in Corberán et al. [12] to find connectivity, R-odd cut, KC, KC_{02}, HC and HC_{02} inequalities violated by $\overline{x}$. If such an inequality $F(x) \geq c_0$ is found then the corresponding aggregate inequality $\sum_{k=1}^{K} F(x^k) \geq c_0$ is violated by $(\overline{x}^1, \overline{y}^1, \overline{x}^2, \overline{y}^2, \ldots, \overline{x}^K, \overline{y}^K)$. We refer to that paper for a detailed description of these aggregate separation procedures.

Concerning the disaggregate inequalities, we proceed as follows. For each vehicle k, the part of the LP solution that corresponds to this vehicle can be represented by the weighted graph $\overline{G}^k = (\overline{V}^k, \overline{E}^k, \overline{x}^k, \overline{y}^k)$, where $\overline{V}^k, \overline{E}^k$ are the sets of vertices and edges of the subgraph of G induced by the edges $e \in E$ such that $\overline{x}_e^k = \overline{x}_{ij}^k + \overline{x}_{ji}^k > 0$, plus the depot node, if necessary.

4.2.1 Separation of disaggregate connectivity inequalities

Connectivity inequalities (5) can be exactly separated, for each vehicle k, with the following polynomial algorithm. For each edge $e \in E_R$, such that $\overline{y}_e^k > 0$, compute the minimum cut in graph $\overline{G}^k$ separating edge e from the depot. If the weight of this cut is less than $2\overline{y}_e^k$, then the corresponding inequality (5) is violated.

Note that, although polynomial, the above exact algorithm is quite time consuming, so we have also used a simple heuristic that consists of computing the connected components of a graph constructed as $\overline{G}^k$, but including only those edges $e \in E$ such that $\overline{x}_e^k > 0$. For each connected component of this graph that does not include the depot node, inequality (5) is checked for the set S of nodes in the component and the required edge $e \in E_R(S)$ with maximum value of $\overline{y}_e^k$.

4.2.2 Separation of disaggregate parity inequalities

Disaggregate parity inequalities (9) have been separated using two heuristic and one exact polynomial procedures. The heuristic algorithms are based on the following simple procedure. Given a cutset $\delta(S)$, it can be easily checked if there exists a subset $F \subseteq \delta_R(S)$ for which (9) is violated. Note that (9) can be written as follows:

$$\sum_{e\in\delta(S)\setminus\delta_R(S)} x_e^k + \sum_{e\in\delta_R(S)\setminus F} x_e^k + \sum_{e\in F}(x_e^k - 2y_e^k + 1) \geq 1$$

Then, as in Ghiani and Laporte [17], we can determine the subset $F \subseteq \delta_R(S)$ of odd cardinality for which the LHS of the inequality gets its maximum value for the current LP solution as follows. For each $e \in \delta_R(S)$, include e in the subset F if, and only if, $\overline{x}_e^k \geq \overline{x}_e^k - 2\overline{y}_e^k + 1$, that is, if $\overline{y}_e^k \geq 0.5$. If $|F|$ is odd, we are done, otherwise, a simple check can determine the required edge to be removed from or added to F, in such a way that the resulting subset minimizes the LHS. If the LHS is less than 1 the inequality (9) is then violated. Otherwise there exists no $F \subseteq \delta_R(S)$ for which (9) is violated for the given cutset $\delta(S)$.

This procedure has been applied to several subsets $S \subseteq V \setminus \{1\}$. In particular for $S = \{v\}$, for every $v \in V$. Furthermore, we construct the graph induced by those required edges e for which $\overline{y}_e^k \geq 0.5$ and $\overline{x}_e^k - \overline{y}_e^k > 0$, and those non required edges e for which $\overline{x}_e^k > 0$. Then, the connected components of this graph determine cutsets for which the above procedure is applied.

Disaggregate parity inequalities can also be exactly separated with a polynomial algorithm similar to the one proposed by Padberg & Rao [19]. It is based on the computation of a minimum weight odd cut on a graph G_k' which is constructed as follows. Initially, G_k' contains the set of nodes $\overline{V}^k$, the edges $e \in E \setminus E_R$ such that $\overline{x}_e^k > 0$, and the edges $e \in E_R$ such that $\overline{x}_e^k - \overline{y}_e^k > 0$. All the nodes are initially labelled as *even*. Then, for each edge $e = (i,j) \in E_R$ such that $\overline{y}_e^k > 0$, a new node i_e is created and two edges are added: (i, i_e) with weight $1 - \overline{y}_e^k$, and (i_e, j) with weight $\overline{y}_e^k$; the new node i_e is labelled *odd*, and the label of node i is changed, from *even* to *odd* (or viceversa). Let S' be the set of nodes in the shore of the depot defining the odd minimum cut in this graph and let $S = S' \cap \overline{V}^k$. Let F be the subset of required edges $e \in \delta_R(S)$ such that the edge with weight $1 - \overline{y}_e^k$ is in the odd minimum cut $\delta(S')$. It can be shown that $|F|$ is odd and the weight of the minimum cut $\delta(S')$ in graph G_k' can be written as:

$$\overline{x}^k(\delta(S)) - 2\overline{y}^k(F) + |F|$$

Therefore, a violated parity inequality (9) exists if, and only if, the weight of this cut is less than 1.

4.3   The cutting plane algorithm and branching strategies

At each iteration of the cutting plane algorithm the exact and heuristic separation algorithms are called in a specific order and a number of violated inequalities are added to the LP relaxation, which is then solved again. The algorithm has been implemented as follows:

1. Aggregate inequalities:

   (a) R-odd cut and connectivity separation heuristics.
   (b) Exact connectivity separation if the corresponding heuristics have failed.
   (c) Exact R-odd cut separation if no violated inequalities have been found by the corresponding heuristics.

(d) If the total number of violated inequalities found is less than 10, run heuristic algorithms for separating K-C and K-C$_{02}$ inequalities.

(e) If the total number of violated inequalities found is less than 10, run heuristic algorithms for separating HC and HC$_{02}$ inequalities.

2. Disaggregate inequalities:

(a) Parity and connectivity separation heuristics.

(b) Exact connectivity separation if no violated disaggregate connectivity inequalities have been found so far.

(c) Exact parity separation for every vertex. If no violated disaggregate parity inequalities have been found so far execute the exact parity separation algorithm.

The above cutting plane procedure is applied at each node of the tree until no new violated inequalities are found. When this happens we branch using the Strong Branching strategy [4] implemented in Cplex. This strategy branches on variables and allows to assign different priorities to them. Variables with higher priority are the first ones used for branching. We have assigned decreasing priorities to the variables y_e^k associated with the service of the required edges according to the ordering determined to implement the inequalities (11) to (13) avoiding symmetry.

5   Computational results

We present here the computational results obtained on a large set of instances previously published in the literature. The B&C procedure has been coded in C/C++ using the Cplex 9.0 MIP Solver with Concert Technology 2.0. All the tests were run on an Intel Pentium IV 2.80GHz and 2GB RAM with a time limit of 30 minutes. In what follows we describe the characteristics of the instances.

The set of WRPP instances was presented in Benavent et al. [8] and generated from the 24 RPP instances proposed in Christofides et al. [9]. They can be found in http://www.uv.es/corberan. These RPP instances have up to 50 vertices, 184 edges, 78 required edges and 8 R-sets (vertex sets of the connected components induced by the required edges). From each of these 24 RPP instances 6 different WRPP instances were generated. The computational results obtained on this set of 24×6 instances for 2 and 3 vehicles are shown in Tables 1 and 2, respectively.

Table 1 shows the characteristics of the instances as well as the computational results for the case with 2 vehicles. First column shows the number of the set of WRPP instances corresponding to the original RPP from Christofides et al. [9]. Columns 2,3,4 and 5 present the number of vertices, edges, required edges and R-sets for all the instances in each set, respectively. Column 7 shows the average gap between the lower bound at the root node and the best feasible solution found, while column 8 reports the average final gap for the unsolved instances. Finally, in column 9 the average time in seconds for the 6 instances is presented.

With 2 vehicles, all the small size instances have been solved in a few seconds. We have also solved 25 out of 30 medium size instances (C20 to C24) in less than 30 minutes. For the 5 unsolved instances, the gap between the lower bound and the best feasible solution found is only 2.71% on average. Note that very few instances have been solved at the root node (before

	$|V|$	$|E|$	$|E_R|$	# of R-sets	# of opt.	Root node gap	Final gap	Time (scs.)
C01	11	13	7	4	6	0.88	-	0.07
C02	14	33	12	4	6	5.59	-	0.19
C03	28	57	26	4	6	2.53	-	1.03
C04	17	35	22	3	6	5.15	-	0.56
C05	20	35	16	5	6	5.04	-	0.24
C06	24	46	20	7	6	2.24	-	0.68
C07	23	47	24	3	6	3.03	-	1.35
C08	17	40	24	2	6	4.02	-	0.62
C09	14	26	14	3	6	4.93	-	0.13
C10	12	20	10	4	6	6.78	-	0.09
C11	9	14	7	3	6	5.78	-	0.05
C12	7	18	5	3	6	4.59	-	0.03
C13	7	10	4	3	6	0.00	-	0.02
C14	28	79	31	6	6	3.07	-	3.35
C15	26	37	19	8	6	3.05	-	0.35
C16	31	94	34	7	6	3.54	-	14.10
C17	19	44	17	5	6	3.24	-	0.32
C18	23	37	16	8	6	4.40	-	0.32
C19	33	54	29	7	6	2.28	-	1.52
C20	50	98	63	7	6	2.83	-	173.20
C21	49	110	67	6	6	1.68	-	214.23
C22	50	184	74	6	4	1.31	0.72	85.18
C23	50	158	78	6	4	2.82	5.64	359.69
C24	41	125	55	7	5	2.15	0.84	35.84
Global	25.13	58.92	28.08	5.04		3.37	2.71	31.9

Table 1: Computational results on the MM 2-WRPP instances

branching), not even the smallest ones. The average gap obtained at the root node is 3.37%. This could mean that, besides the MM K-WRPP being a very hard problem, the polyhedral description must be improved, specially with regard to the disaggregate inequalities. This is more evident for the instances with 3 vehicles, where the average gap at the root node is 6.05% (see Table 2). Even so, all the instances of small size but one have been solved in a few seconds and also 7 out of the 30 medium size instances in less than 30 minutes. We want to point out that although the graphs are of medium size the corresponding LP's are large.

	# of opt.	Root node gap	Final gap	Time (scs.)
C01	6	0.00	-	0.1
C02	6	10.56	-	0.6
C03	6	7.06	-	6.2
C04	6	7.27	-	7.3
C05	6	7.63	-	0.7
C06	6	7.29	-	8.9
C07	6	5.81	-	12.2
C08	6	7.47	-	40.0
C09	6	7.85	-	0.4
C10	6	5.20	-	0.1
C11	6	3.58	-	0.1
C12	6	3.50	-	0.0
C13	6	0.00	-	0.0
C14	6	7.08	-	61.3
C15	6	4.61	-	4.5
C16	5	7.63	4,92	198.8
C17	6	6.03	-	2.3
C18	6	4.66	-	1.6
C19	6	4.67	-	28.0
C20	3	6.20	5.22	270.2
C21	1	11.27	11.59	1329.7
C22	0	5.78	5.56	1800.0
C23	0	5.47	5.32	1800.0
C24	3	8.60	6.65	1687.8
Global		6.05	6.82	220.9

Table 2: Computational results on the MM 3-WRPP instances

6 Conclusions and future research directions

In this paper we have presented the Min-Max version of the Windy Rural Postman Problem with multiple vehicles, in which the objective is to minimize the length of the longest tour. An ILP formulation has been proposed and some families of valid inequalities have been presented. We have also studied its associated polyhedron and we have proved that several large families of inequalities are facet-inducing. Finally, a branch-and-cut algorithm has been implemented with separation procedures for the families above. Our algorithm has been tested on a large set of small and medium size instances from the Literature. The results are encouraging since the algorithm has been capable of solving all of the small and some of the medium size instances in only a few minutes. We think these results could be improved by using separation procedures for some other families of disaggregate inequalities. The implementation of such procedures as well as a better description of the MM K-WRPP polyhedron will be the subject of future work.

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