An Introduction to
Objective Bayesian Statistics

José M. Bernardo

Universitat de València, Spain

<jose.m.bernardo@uv.es>

http://www.uv.es/bernardo

Université de Neuchâtel, Switzerland

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Summary

1. *Concept of Probability*

   Introduction. Notation. Statistical models.
   Intrinsic discrepancy. Intrinsic convergence of distributions.
   Foundations. Probability as a rational degree of belief.

2. *Basics of Bayesian Analysis*

   Parametric inference. The learning process.
   Reference analysis. No relevant initial information.
   Inference summaries. Point and region estimation.
   Prediction. Regression.
   Hierarchical models. Exchangeability.

3. *Decision Making*

   Structure of a decision problem. Intrinsic loss functions.
   Point and region estimation. Intrinsic estimators and credible regions.
   Hypothesis testing. Bayesian reference criterion (BRC).

1. Concept of Probability

1.1. Introduction

- Tentatively accept a *formal* statistical model
  - Typically suggested by informal descriptive evaluation
  - Conclusions conditional on the assumption that model is correct
- Bayesian approach firmly based on *axiomatic foundations*
  - Mathematical need to describe by probabilities all uncertainties
  - Parameters *must* have a (*prior*) distribution describing available information about their values
  - *Not* a description of their variability (*fixed unknown* quantities), but a description of the *uncertainty* about their true values.
- Important particular case: no relevant (or subjective) initial information:
  - scientific and industrial reporting, public decision making, ...
  - Prior *exclusively* based on model assumptions and available, well-documented data: *Objective Bayesian Statistics*

- *Notation*

  - Under conditions C, $p(\boldsymbol{x} \mid C)$, $\pi(\boldsymbol{\theta} \mid C)$ are, respectively, *probability* densities (or mass) functions of *observables* $\boldsymbol{x}$ and *parameters* $\boldsymbol{\theta}$
    $p(\boldsymbol{x} \mid C) \geq 0$, $\int_{\mathcal{X}} p(\boldsymbol{x} \mid C)\, \mathrm{d}\boldsymbol{x} = 1$, $\mathrm{E}[\boldsymbol{x} \mid C] = \int_{\mathcal{X}} \boldsymbol{x}\, p(\boldsymbol{x} \mid C)\, \mathrm{d}\boldsymbol{x}$,
    $\pi(\boldsymbol{\theta} \mid C) \geq 0$, $\int_{\Theta} \pi(\boldsymbol{\theta} \mid C)\, \mathrm{d}\boldsymbol{\theta} = 1$, $\mathrm{E}[\boldsymbol{\theta} \mid C] = \int_{\Theta} \boldsymbol{\theta}\, \pi(\boldsymbol{\theta} \mid C)\, \mathrm{d}\boldsymbol{\theta}$.
  - Special densities (or mass) functions use specific notation, as $\mathrm{N}(x \mid \mu, \sigma)$, $\mathrm{Bi}(x \mid n, \theta)$, or $\mathrm{Pn}(x \mid \lambda)$. Other examples:

Beta $\quad \{\mathrm{Be}(x \mid \alpha, \beta), \quad 0 < x < 1, \quad \alpha > 0, \beta > 0\}$

$$\mathrm{Be}(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}$$

Gamma $\quad \{\mathrm{Ga}(x \mid \alpha, \beta), \quad x > 0, \quad \alpha > 0, \beta > 0\}$

$$\mathrm{Ga}(x \mid \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\beta x}$$

Student $\quad \{\mathrm{St}(x \mid \mu, \sigma, \alpha), \quad x \in \Re, \quad \mu \in \Re, \sigma > 0, \alpha > 0\}$

$$\mathrm{St}(x \mid \mu, \sigma, \alpha) = \frac{\Gamma\{(\alpha+1)/2)\}}{\Gamma(\alpha/2)} \frac{1}{\sigma\sqrt{\alpha\pi}} \left[1 + \frac{1}{\alpha}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-(\alpha+1)/2}$$

- *Statistical Models*

  - *Statistical model* generating $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, $\{p(\boldsymbol{x} \mid \boldsymbol{\theta}), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}, \boldsymbol{\theta} \in \Theta\}$

    Parameter vector $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\} \in \Theta$. *Parameter space* $\Theta \subset \Re^k$.
    Data set $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$. *Sampling (Outcome) space* $\boldsymbol{\mathcal{X}}$, of arbitrary structure.

  - *Likelihood function* of $\boldsymbol{x}$, $l(\boldsymbol{\theta} \mid \boldsymbol{x})$.

    $l(\boldsymbol{\theta} \mid \boldsymbol{x}) = p(\boldsymbol{x} \mid \boldsymbol{\theta})$, as a function of $\boldsymbol{\theta} \in \Theta$.

  - *Maximum likelihood estimator (mle)* of $\boldsymbol{\theta}$

    $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\boldsymbol{x}) = \arg\sup_{\boldsymbol{\theta} \in \Theta} \; l(\boldsymbol{\theta} \mid \boldsymbol{x})$

  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ *random sample* (iid) from model if

    $p(\boldsymbol{x} \mid \boldsymbol{\theta}) = \prod_{j=1}^{n} p(x_j \mid \boldsymbol{\theta}), \; x_j \in \mathcal{X}, \quad \boldsymbol{\mathcal{X}} = \mathcal{X}^n$

  - Behaviour under repeated sampling (general, not iid data)

    Considering $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\}$, a (possibly infinite) sequence
    of possible replications of the *complete* data set $\boldsymbol{x}$.

    Denote by $\boldsymbol{x}^{(m)} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$ a finite set of m such replications.

  - Asymptotic results obtained as $m \to \infty$

1.2. Intrinsic Divergence

- *Logarithmic divergences*

  - The logarithmic divergence (Kullback-Leibler) $k\{\hat{p}\,|\,p\}$ of a density $\hat{p}(\boldsymbol{x})$, $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$ from its true density $p(\boldsymbol{x})$, is

    $$\kappa\{\hat{p}\,|\,p\} = \int_{\mathcal{X}} p(\boldsymbol{x}) \log \frac{p(\boldsymbol{x})}{\hat{p}(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x}, \text{ (provided this exists)}$$

    The functional $\kappa\{\hat{p}\,|\,p\}$ is non-negative, (zero iff, $\hat{p}(\boldsymbol{x}) = p(\boldsymbol{x})$ a.e.) and *invariant* under one-to-one transformations of $\boldsymbol{x}$.

  - But $\kappa\{p_1\,|\,p_2\}$ is *not symmetric* and diverges if, strictly, $\boldsymbol{\mathcal{X}}_2 \subset \boldsymbol{\mathcal{X}}_1$.

- *Intrinsic discrepancy between distributions*

  - $\delta\{p_1, p_2\} = \min\left\{ \int_{\mathcal{X}_1} p_1(\boldsymbol{x}) \log \frac{p_1(\boldsymbol{x})}{p_2(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x}, \int_{\mathcal{X}_2} p_2(\boldsymbol{x}) \log \frac{p_2(\boldsymbol{x})}{p_1(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x} \right\}$

    The *intrinsic discrepancy* $\delta\{p_1, p_2\}$ is non-negative (zero iff, $p_1 = p_2$ a.e.), and *invariant* under one-to-one transformations of $\boldsymbol{x}$,

  - Defined if $\boldsymbol{\mathcal{X}}_2 \subset \boldsymbol{\mathcal{X}}_1$ or $\boldsymbol{\mathcal{X}}_1 \subset \boldsymbol{\mathcal{X}}_2$, operative interpretation as the minimum amount of information (in *nits*) required to discriminate.

- *Interpretation and calibration of the intrinsic discrepancy*

  - Let $\{p_1(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1), \boldsymbol{\theta}_1 \in \Theta_1\}$ or $\{p_2(\boldsymbol{x}\,|\,\boldsymbol{\theta}_2), \boldsymbol{\theta}_2 \in \Theta_2\}$ be two alternative statistical models for $\boldsymbol{x} \in X$, one of which is assumed to be true. The intrinsic divergence $\delta\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\} = \delta\{p_1, p_2\}$ is then *minimum expected log-likelihood ratio in favour of the true model*.

    Indeed, if $p_1(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1)$ true model, the expected log-likelihood ratio in its favour is $\mathrm{E}_1[\log\{p_1(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1)/p_2(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1)\}] = \kappa\{p_2\,|\,p_1\}$. If the true model is $p_2(\boldsymbol{x}\,|\,\boldsymbol{\theta}_2)$, the expected log-likelihood ratio in favour of the true model is $\kappa\{p_2\,|\,p_1\}$. But $\delta\{p_2\,|\,p_1\} = \min[\kappa\{p_2\,|\,p_1\}, \kappa\{p_1\,|\,p_2\}]$.

  - *Calibration*. $\delta = \log[100] \approx 4.6$ *nits*, likelihood ratios for the true model larger than 100 making *discrimination very easy*.

    $\delta = \log(1+\varepsilon) \approx \varepsilon$ *nits*, likelihood ratios for the true model may about $1+\epsilon$ making *discrimination very hard*.

Intrinsic Discrepancy δ	0.01	0.69	2.3	4.6	6.9
Average Likelihood Ratio for **true** model $\exp[\delta]$	1.01	2	10	100	1000

- *Example.* Conventional Poisson approximation $\text{Pn}(r \mid n\theta)$ of Binomial probabilities $\text{Bi}(r \mid n, \theta)$

  Intrinsic discrepancy between Binomial and Poisson distributions

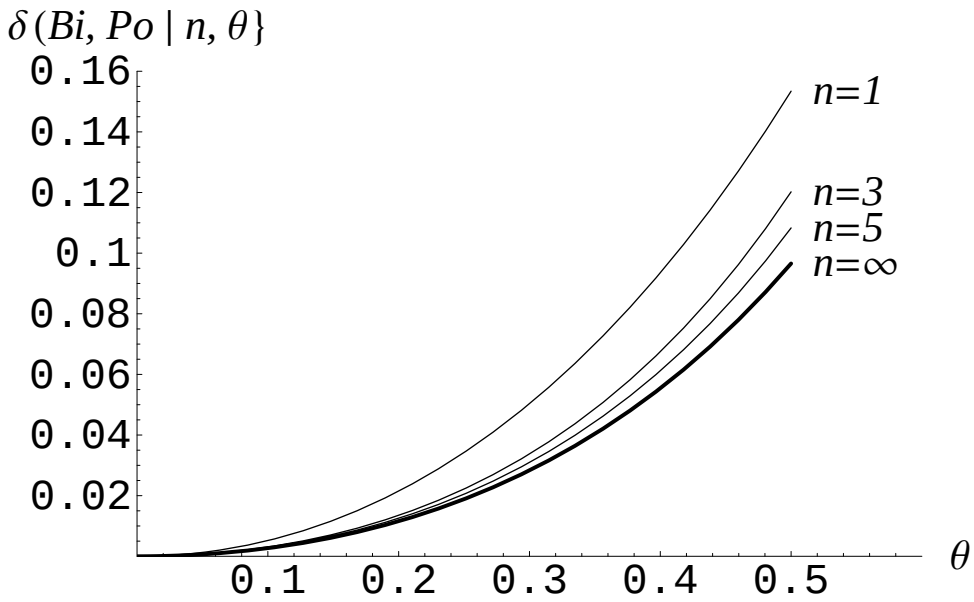
  $$\delta\{\text{Bi}(r \mid n, \theta), \text{Po}(r \mid n\theta\} = \min[k\{\text{Bi} \mid \text{Po}\}, k\{\text{Po} \mid \text{Bi}\}] = k\{\text{Bi} \mid \text{Po}\}$$
  $$= \sum_{r=0}^{n} \text{Bi}(r \mid n,\ \theta) \log[\text{Bi}(r \mid n,\ \theta)/\text{Po}(r \mid n\theta)] = \delta\{n, \theta\}$$

$\delta\{3, 0.05\} = 0.00074$

$\delta\{5000, 0.05\} = 0.00065$

$\delta\{\infty, \theta\} = \frac{1}{2}[-\theta - \log(1 - \theta)]$

Good Poisson approximations are *impossible* if θ is not small, however large n might be.



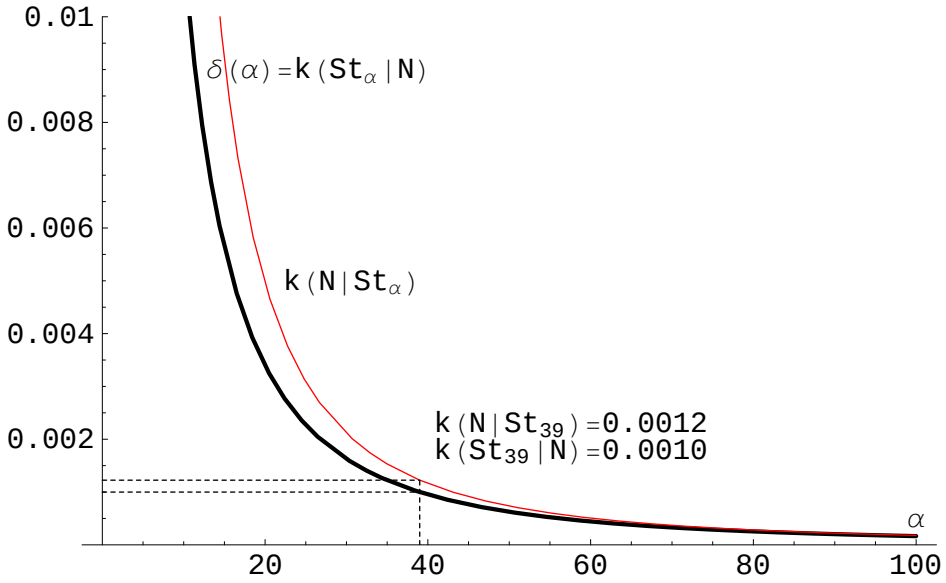
- *Intrinsic Convergence of Distributions*

  - *Intrinsic convergence.* A sequence of probability densities (or mass) functions $\{p_i(\boldsymbol{x})\}_{i=1}^{\infty}$ converges *intrinsically* to $p(\boldsymbol{x})$ if (and only if) the intrinsic divergence between $p_i(x)$ and $p(x)$ converges to zero. *i.e.*, iff $\lim_{i\to\infty} \delta(p_i, p) = 0$.

  - *Example.* Normal approximation to a Student distribution.

$$\delta(\alpha) = \delta\{\mathrm{St}(x \,|\, \mu, \sigma, \alpha), \mathrm{N}(x \,|\, \mu, \sigma)\} = \min[k\{\mathrm{St}_\alpha \,|\, \mathrm{N}\}, k\{\mathrm{N} \,|\, \mathrm{St}_\alpha\}]$$

$$= k\{\mathrm{St}_\alpha \,|\, \mathrm{N}\} = \int_{\Re} \mathrm{N}(x \,|\, 0, 1) \log \frac{\mathrm{N}(x \,|\, 0, 1)}{\mathrm{St}(x \,|\, 0, 1, \alpha)} \,\mathrm{d}x \approx \frac{7}{\alpha(22 + 4\alpha)}$$



$k\{\mathrm{N} \,|\, \mathrm{St}_\alpha\}$ diverges for $\alpha \leq 2$

$k\{\mathrm{St}_\alpha \,|\, \mathrm{N}\}$ is finite for all $\alpha > 0$.

$\delta(18) \approx 0.04 \quad \delta(25) \approx 0.02$

Expected log-density ratios at least 0.001 when $\alpha < 40$.

1.3. Foundations

- *Foundations of Statistics*

  - Axiomatic foundations on rational description of uncertainty imply that the uncertainty about all unknown quantities should be measured with *probability* distributions $\{\pi(\boldsymbol{\theta} \,|\, C), \boldsymbol{\theta} \in \Theta\}$ describing the plausibility of their given available conditions C.
  - Axioms have a strong intuitive appeal; examples include
    - *Transitivity of plausibility*.
      If $E_1 \succ E_2 \,|\, C$, and $E_2 \succ E_3 \,|\, C$, then $E_1 \succ E_3 \,|\, C$
    - *The sure-thing principle*.
      If $E_1 \succ E_2 \,|\, A, C$ and $E_1 \succ E_2 \,|\, \overline{A}, C$, then $E_1 \succ E_2 \,|\, C$).
  - Axioms are not a *description* of actual human activity, but a *normative* set of principles for those aspiring to rational behaviour.
  - "Absolute" probabilities do not exist. Typical applications produce $\Pr(E \,|\, \boldsymbol{x}, A, K)$, a measure of rational belief in the occurrence of the *event* E, given data $\boldsymbol{x}$, assumptions A and available knowledge K.

- *Probability as a Measure of Conditional Uncertainty*

  - Axiomatic foundations imply that $\Pr(E \mid C)$, the *probability* of an event E given C is *always* a conditional measure of the (presumably rational) uncertainty, on a $[0, 1]$ scale, about the occurrence of E in conditions C.

    - *Probabilistic diagnosis.*V is the event that a person carries a virus and $+$ a positive test result. *All* related probabilities, *e.g.*,
      $\Pr(+ \mid V) = 0.98$, $\Pr(+ \mid \overline{V}) = 0.01$, $\Pr(V \mid K) = 0.002$,
      $\Pr(+ \mid K) = \Pr(+ \mid V)\Pr(V \mid K) + \Pr(+ \mid \overline{V})\Pr(\overline{V} \mid K) = 0.012$
      $\Pr(V \mid +, A, K) = \dfrac{\Pr(+ \mid V)\Pr(V \mid K)}{\Pr(+ \mid K)} = 0.164$ (Bayes' Theorem)
      are conditional uncertainty measures (and proportion estimates).

    - *Estimation of a proportion.*Survey conducted to estimate the proportion θ of positive individuals in a population.
      Random sample of size n with r positive.
      $\Pr(a < \theta < b \mid r, n, A, K)$, a conditional measure of the uncertainty about the event that θ belongs to $[a, b]$ *given* assumptions A, initial knowledge K and data $\{r, n\}$.

- *Measurement of a physical constant*.Measuring the unknown value of physical constant μ, with data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, considered to be measurements of μ subject to error. Desired to find $\Pr(a < \mu < b \,|\, x_1, \ldots, x_n, A, K)$, the *probability* that the unknown value of μ (fixed in nature, but unknown to the scientists) belongs to $[a, b]$ given the information provided by the data $\boldsymbol{x}$, assumptions A made, and available knowledge K.

- The statistical model may include *nuisance* parameters, unknown quantities , which have to be eliminated in the statement of the final results.

  For instance, the precision of the measurements described by unknown standard deviation σ in a $\mathrm{N}(x \,|\, \mu, \sigma)$ normal model

- Relevant scientific information may impose *restrictions* on the admissible values of the quantities of interest. These must be taken into account.

  For instance, in measuring the value of the gravitational field g in a laboratory, it is known that it must lie between 9.7803 m/sec^2 (average value at the Equator) and 9.8322 m/sec^2 (average value at the poles).

- *Future discrete observations*.Experiment counting the number r of times that an event E takes place in each of n replications. Desired to forecast the number of times r that E will take place in a future, similar situation, $\Pr(r \mid r_1, \ldots, r_n, A, K)$. For instance, no accidents in each of $n = 10$ consecutive months may yield $\Pr(r = 0 \mid \boldsymbol{x}, A, K) = 0.953$.
- *Future continuous observations*.Data $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$. Desired to forecast the value of a future observation $\boldsymbol{y}$, $p(\boldsymbol{y} \mid \boldsymbol{x}, A, K)$. For instance, from breaking strengths $\boldsymbol{x} = \{y_1, \ldots, y_n\}$ of n randomly chosen safety belt webbings, the engineer may find $\Pr(y > y^* \mid \boldsymbol{x}, A, K) = 0.9987$.
- *Regression*.Data set consists of pairs $\boldsymbol{x} = \{(\boldsymbol{y}_1, \boldsymbol{v}_1), \ldots, (\boldsymbol{y}_n, \boldsymbol{v}_n)\}$ of quantity $\boldsymbol{y}_j$ observed in conditions $\boldsymbol{v}_j$. Desired to forecast the value of $\boldsymbol{y}$ in conditions $\boldsymbol{v}$, $p(\boldsymbol{y} \mid \boldsymbol{v}, \boldsymbol{x}, A, K)$. For instance, y contamination levels, v wind speed from source; environment authorities interested in $\Pr(y > y^* \mid v, \boldsymbol{x}, A, K)$

2. Basics of Bayesian Analysis

2.1. Parametric Inference

- *Bayes Theorem*

  - Let $\mathcal{M} = \{p(\boldsymbol{x} \mid \boldsymbol{\theta}), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}, \boldsymbol{\theta} \in \Theta\}$ be an statistical model, let $\pi(\boldsymbol{\theta} \mid K)$ be a probability density for $\boldsymbol{\theta}$ given prior knowledge K and let $\boldsymbol{x}$ be some available data.

$$\pi(\boldsymbol{\theta} \mid \boldsymbol{x}, \mathcal{M}, K) = \frac{p(\boldsymbol{x} \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid K)}{\int_{\Theta} p(\boldsymbol{x} \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid K)\, d\boldsymbol{\theta}} \,,$$

    encapsulates all information about $\boldsymbol{\theta}$ given data and prior knowledge.

  - Simplifying notation, Bayes' theorem may be expressed as

$$\pi(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto p(\boldsymbol{x} \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta}) :$$

    The posterior is proportional to the likelihood times the prior. The missing proportionality constant $[\int_{\Theta} p(\boldsymbol{x} \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}]^{-1}$ may be deduced from the fact that $\pi(\boldsymbol{\theta} \mid \boldsymbol{x})$ must integrate to one. To identify a posterior distribution it suffices to identify a *kernel* $k(\boldsymbol{\theta}, \boldsymbol{x})$ such that $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}) = c(\boldsymbol{x})\, k(\boldsymbol{\theta}, \boldsymbol{x})$. This is a very common technique.

- *Bayesian Inference with a Finite Parameter Space*

  - Model $\{p(\boldsymbol{x} \mid \theta_i), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}, \theta_i \in \Theta\}$, with $\Theta = \{\theta_1, \ldots, \theta_m\}$, so that θ may only take a *finite* number m of different values. Using the finite form of Bayes' theorem,

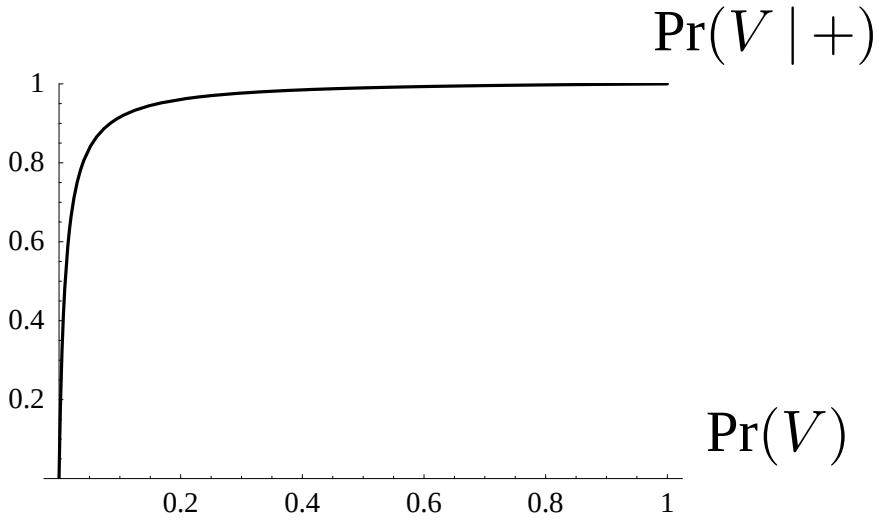
$$\Pr(\theta_i \mid \boldsymbol{x}) = \frac{p(\boldsymbol{x} \mid \theta_i)\Pr(\theta_i)}{\sum_{j=1}^{m} p(\boldsymbol{x} \mid \theta_j)\Pr(\theta_j)}, \quad i = 1, \ldots, m.$$

  - *Example:* *Probabilistic diagnosis*. A test to detect a virus, is known from laboratory research to give a positive result in 98% of the infected people and in 1% of the non-infected. The posterior probability that a person who tested positive is infected is

$$\Pr(V \mid +) = \frac{0.98\,p}{0.98\,p + 0.01\,(1-p)}$$

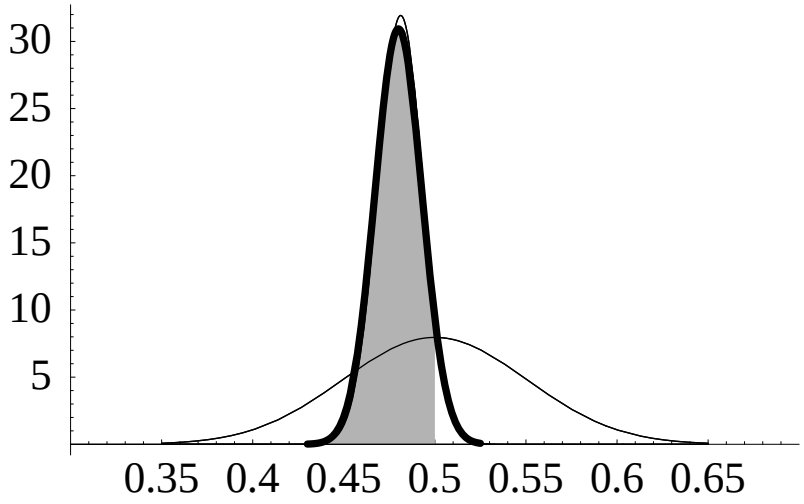
    as a function of $p = \Pr(V)$.

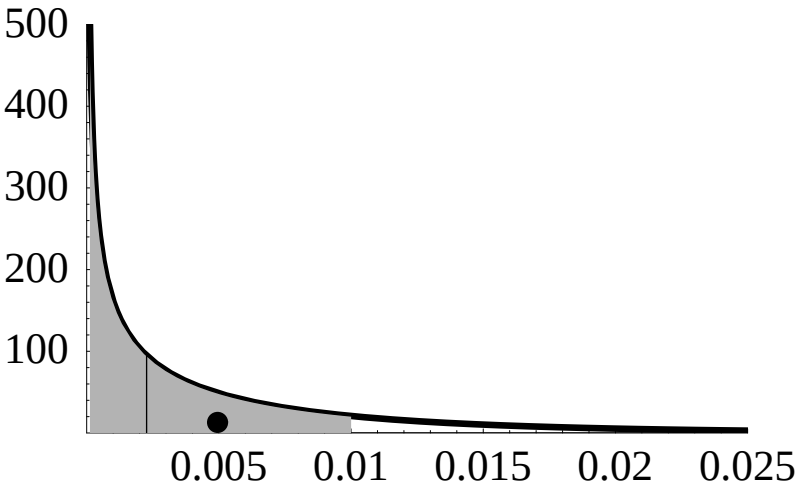
  - Notice sensitivity of posterior $\Pr(V \mid +)$ to changes in the prior $p = \Pr(V)$.



- *Example: Inference about a binomial parameter*

  - Let data $\boldsymbol{x}$ be n Bernoulli observations with parameter θ which contain r positives, so that $p(\boldsymbol{x} \mid \theta, n) = \theta^r (1-\theta)^{n-r}$.
  - If $\pi(\theta) = \text{Be}(\theta \mid \alpha, \beta)$, then $\pi(\theta \mid \boldsymbol{x}) \propto \theta^{r+\alpha-1}(1-\theta)^{n-r+\beta-1}$ kernel of $\text{Be}(\theta \mid r+\alpha, n-r+\beta)$.
  - Prior information (K) $P(0.4 < \theta < 0.6) = 0.95$, and symmetric, yields $\alpha = \beta = 47$;
  - No prior information $\alpha = \beta = 1/2$
  - $n = 1500, r = 720$ $P(\theta < 0.5 \mid \boldsymbol{x}, K) = 0.933$ $P(\theta < 0.5 \mid \boldsymbol{x}) = 0.934$
  - $n = 100, r = 0$ $P(\theta < 0.01 \mid \boldsymbol{x}) = 0.844$ Notice: $\hat{\theta} = 0$, but $\text{Me}[\theta \mid \boldsymbol{x}] = 0.0023$





- *Sufficiency*

  - Given a model $p(\boldsymbol{x} \mid \boldsymbol{\theta})$, a function of the data $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x})$, is a *sufficient* statistic if it encapsulates all information about $\boldsymbol{\theta}$ available in $\boldsymbol{x}$.
  - Formally, $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x})$ is *sufficient* if (and only if), for any prior $\pi(\boldsymbol{\theta})$ $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}) = \pi(\boldsymbol{\theta} \mid \boldsymbol{t})$. Hence, $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}) = \pi(\boldsymbol{\theta} \mid \boldsymbol{t}) \propto p(\boldsymbol{t} \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})$.
  - This is equivalent to the frequentist definition; thus $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x})$ is sufficient iff $p(\boldsymbol{x} \mid \boldsymbol{\theta}) = f(\boldsymbol{\theta}, \boldsymbol{t}) g(\boldsymbol{x})$.
  - A sufficient statistic always exists, for $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{x}$ is obviously sufficient

    A much simpler sufficient statistic, with fixed dimensionality independent of the sample size, often exists.

    This is case whenever the statistical model belongs to the *generalized exponential family*, which includes many of the more frequently used statistical models.
  - In contrast to frequentist statistics, Bayesian methods are independent on the possible existence of a sufficient statistic of fixed dimensionality.

    For instance, if data come from an Student distribution, there is *no sufficient statistic* of fixed dimensionality: *all data are needed*.

- *Example: Inference from Cauchy observations*

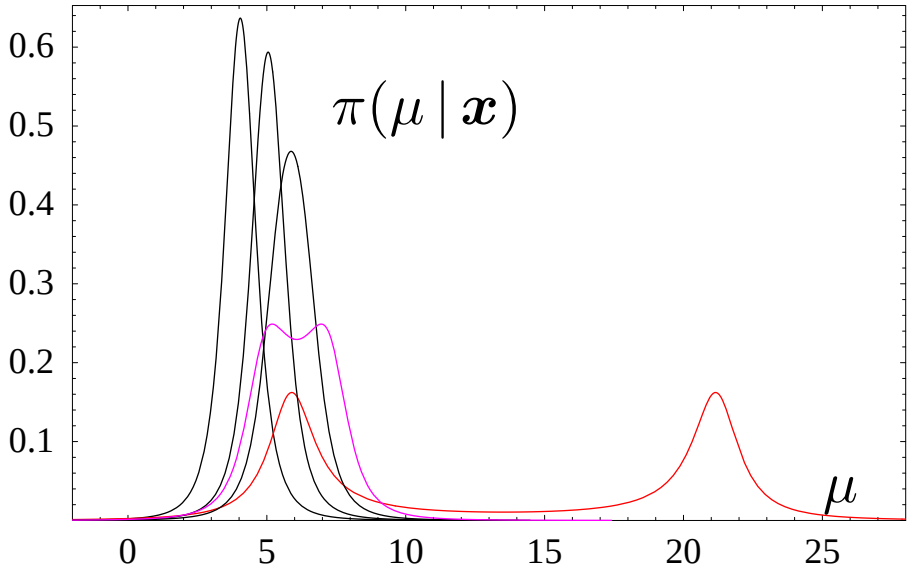
  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ random from $\text{Ca}(x \mid \mu, 1) = \text{St}(x \mid \mu, 1, 1)$.
  - Objective reference prior for the location parameter μ is $\pi(\mu) = 1$.
  - By Bayes' theorem,

$$\pi(\mu \mid \boldsymbol{x}) \propto \prod_{j=1}^{n} \text{Ca}(x_j \mid \mu, 1)\pi(\mu) \propto \prod_{j=1}^{n} \frac{1}{1 + (x_j - \mu)^2} \, .$$

    Proportionality constant easily obtained by numerical integration.

  - Five samples of size $n = 2$ simulated from $\text{Ca}(x \mid 5, 1)$.

x_1	x_2
4.034	4.054
21.220	5.831
5.272	6.475
4.776	5.317
7.409	4.743



- *Improper prior functions*

  - Objective Bayesian methods often use functions which play the role of prior distributions but are *not* probability distributions.
  - An *improper prior function* is an non-negative function $\pi(\boldsymbol{\theta})$ such that $\int_\Theta \pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}$ is not finite.

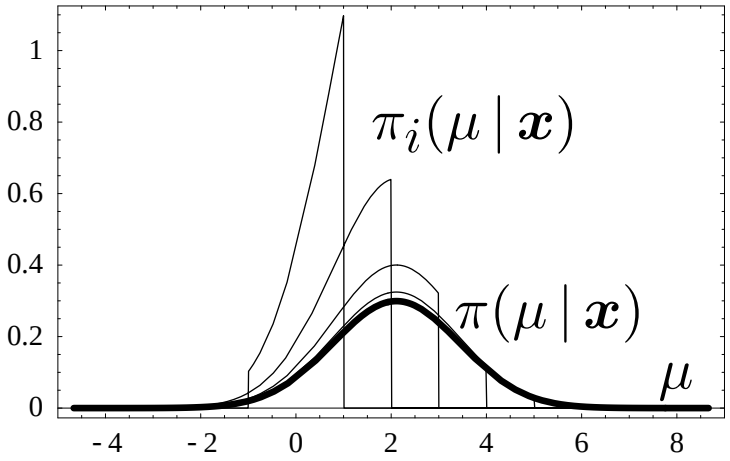
    The Cauchy example uses the improper prior function $\pi(\mu) = 1$, $\mu \in \Re$.
  - $\pi(\boldsymbol{\theta})$ is an improper prior function, $\{\Theta_i\}_{i=1}^{\infty}$ an increasing sequence approximating Θ, such that $\int_{\Theta_i} \pi(\boldsymbol{\theta}) < \infty$, and $\{\pi_i(\boldsymbol{\theta})\}_{i=1}^{\infty}$ the proper priors obtained by *renormalizing* $\pi(\boldsymbol{\theta})$ within the Θ_i's.
  - For any data $\boldsymbol{x}$ with likelihood $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$, the sequence of posteriors $\pi_i(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ converges intrinsically to $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})$.
  - Normal data, σ known, $\pi(\mu) = 1$.

    $\pi(\mu \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \mu, \sigma)\pi(\mu)$

    $\propto \exp[-\frac{n}{2\sigma^2}(\overline{x} - \mu)^2]$

    $\pi(\mu \,|\, \boldsymbol{x}) = \mathrm{N}(\mu \,|\, \overline{x}, \sigma/\sqrt{n})$

    Example: $n = 9,\ \overline{x} = 2.11,\ \sigma = 4$



- *Sequential updating*

  - Prior and posterior are terms *relative* to a set of data.
  - If data $\boldsymbol{x} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ are sequentially presented, the final result will be the same whether data are globally or sequentially processed.

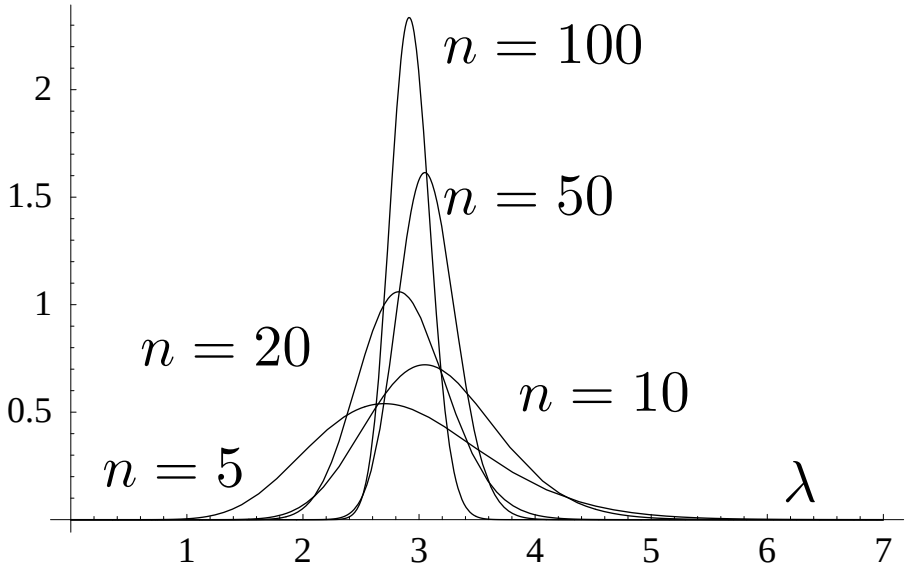
$$\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i+1}) \propto p(\boldsymbol{x}_{i+1} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_i).$$

    The "posterior" at a given stage becomes the "prior" at the next.

  - Typically (but not always), the new posterior, $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i+1})$, is more concentrated around the true value than $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_i)$.
  - Posteriors $\pi(\lambda \,|\, x_1, \ldots, x_i)$ from increasingly large simulated data from Poisson $\mathrm{Pn}(x \,|\, \lambda)$, with $\lambda = 3$

$$\pi(\lambda \,|\, x_1, \ldots, x_i) = \mathrm{Ga}(\lambda \,|\, r_i + 1/2, i)$$

$$r_i = \sum_{j=1}^{i} x_j$$



- *Nuisance parameters*

  - In general the *vector of interest* is not the whole parameter vector $\boldsymbol{\theta}$, but some function $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta})$ of possibly lower dimension.
  - By Bayes' theorem $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})$. Let $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{\theta}) \in \Omega$ be another function of $\boldsymbol{\theta}$ such that $\boldsymbol{\psi} = \{\boldsymbol{\phi}, \boldsymbol{\omega}\}$ is a bijection of $\boldsymbol{\theta}$, and let $J(\boldsymbol{\psi}) = (\partial\boldsymbol{\theta}/\partial\boldsymbol{\psi})$ be the Jacobian of the inverse function $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{\theta})$.

    From probability theory, $\pi(\boldsymbol{\psi} \,|\, \boldsymbol{x}) = |J(\boldsymbol{\psi})| [\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})]_{\boldsymbol{\theta}=\boldsymbol{\theta}(\boldsymbol{\psi})}$

    and $\pi(\boldsymbol{\phi} \,|\, \boldsymbol{x}) = \int_\Omega \pi(\boldsymbol{\phi}, \boldsymbol{\omega} \,|\, \boldsymbol{x})\, d\boldsymbol{\omega}$.
  - Any valid conclusion on $\boldsymbol{\phi}$ will be contained in $\pi(\boldsymbol{\phi} \,|\, \boldsymbol{x})$.
  - Particular case: *marginal posteriors*

    Often model directly expressed in terms of vector of interest $\boldsymbol{\phi}$, and vector of nuisance parameters $\boldsymbol{\omega}$, $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) = p(\boldsymbol{x} \,|\, \boldsymbol{\phi}, \boldsymbol{\omega})$.

    Specify the prior $\quad \pi(\boldsymbol{\theta}) = \pi(\boldsymbol{\phi})\, \pi(\boldsymbol{\omega} \,|\, \boldsymbol{\phi})$

    Get the joint posterior $\quad \pi(\boldsymbol{\phi}, \boldsymbol{\omega} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\phi}, \boldsymbol{\omega})\, \pi(\boldsymbol{\omega} \,|\, \boldsymbol{\phi})\, \pi(\boldsymbol{\phi})$

    Integrate out ω, $\quad \pi(\boldsymbol{\phi} \,|\, \boldsymbol{x}) \propto \pi(\boldsymbol{\phi}) \int_\Omega p(\boldsymbol{x} \,|\, \boldsymbol{\phi}, \boldsymbol{\omega})\, \pi(\boldsymbol{\omega} \,|\, \boldsymbol{\phi})\, \mathrm{d}\boldsymbol{\omega}$

- *Example: Inferences about a Normal mean*

  - Data $\boldsymbol{x} = \{x_1, \ldots x_n\}$ random from $\mathrm{N}(x \,|\, \mu, \sigma)$. Likelihood function $p(\boldsymbol{x} \,|\, \mu, \sigma) \propto \sigma^{-n} \exp[-n\{s^2 + (\overline{x} - \mu)^2\}/(2\sigma^2)]$, with $n\overline{x} = \sum_i x_i$, and $ns^2 = \sum_i (x_i - \overline{x})^2$.

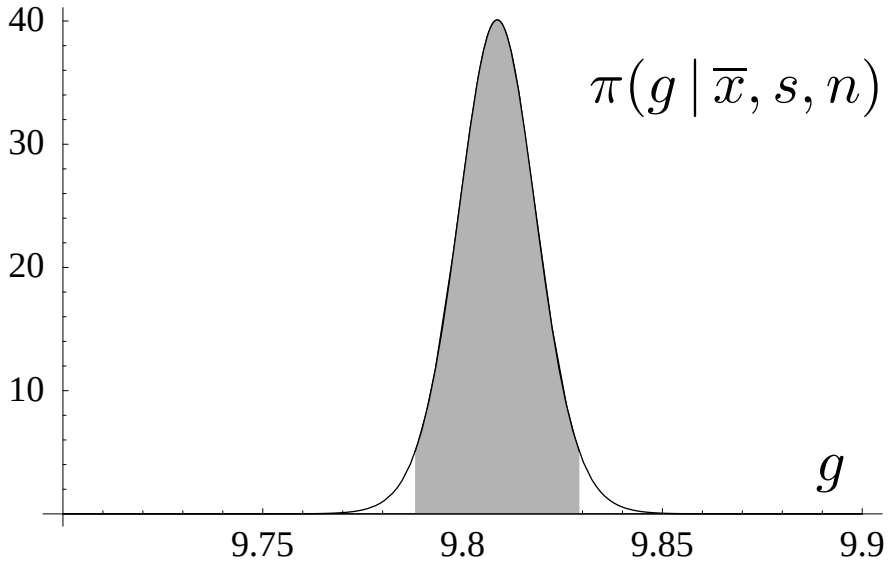
  - Objective prior is uniform in both μ and $\log(\sigma)$, *i.e.*, $\pi(\mu, \sigma) = \sigma^{-1}$. Joint posterior $\pi(\mu, \sigma \,|\, \boldsymbol{x}) \propto \sigma^{-(n+1)} \exp[-n\{s^2 + (\overline{x} - \mu)^2\}/(2\sigma^2)]$.

  - Marginal posterior $\pi(\mu \,|\, \boldsymbol{x}) \propto \int_0^\infty \pi(\mu, \sigma \,|\, \boldsymbol{x})\, \mathrm{d}\sigma \propto [s^2 + (\overline{x} - \mu)^2]^{-n/2}$, kernel of the Student density $\mathrm{St}(\mu \,|\, \overline{x}, s/\sqrt{n-1}, n-1)$

  - Classroom experiment to measure gravity g yields $\overline{x} = 9.8087$, $s = 0.0428$ with $n = 20$ measures.

    $\pi(g \,|\, \overline{x}, s, n)$
    $= \mathrm{St}(g \,|\, 9.8087, 0.0098, 19)$

    $\Pr(9.788 < g < 9.829 \,|\, \boldsymbol{x})$
    $= 0.95$ (shaded area)



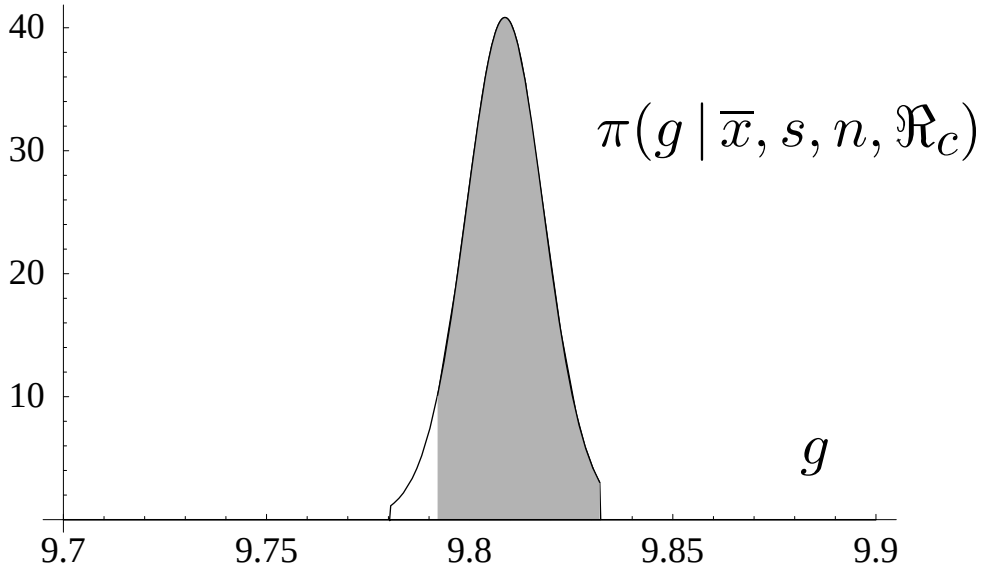
- *Restricted parameter space*

  - Range of values of $\boldsymbol{\theta}$ restricted by contextual considerations.
    If $\boldsymbol{\theta}$ known to belong to $\Theta_c \subset \Theta$, $\pi(\boldsymbol{\theta}) > 0$ iff $\boldsymbol{\theta} \in \Theta_c$
    By Bayes' theorem,

$$\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}, \boldsymbol{\theta} \in \Theta_c) = \begin{cases} \dfrac{\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})}{\int_{\Theta_c} \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta}}, & \text{if} \quad \boldsymbol{\theta} \in \Theta_c \\ 0 & \text{otherwise} \end{cases}$$

  - To incorporate a restriction, it suffices to *renormalize* the unrestricted posterior distribution to the set $\Theta_c \subset \Theta$ of admissible parameter values.

  - Classroom experiment to measure gravity g with restriction to lie between
    $g_0 = 9.7803$ (equator)
    $g_1 = 9.8322$ (poles).
    $\Pr(9.7921 < g < 9.8322 \,|\, \boldsymbol{x}) = 0.95$ (shaded area)



- *Asymptotic behaviour, discrete case*

  - If the parameter space $\Theta = \{\theta_1, \theta_2, \ldots\}$ is *countable* and
    The true parameter value θ_t is *distinguishable* from the others,*i.e.*,
    $\delta\{p(\boldsymbol{x} \mid \boldsymbol{\theta}_t), p(\boldsymbol{x} \mid \boldsymbol{\theta}_i)) > 0$, $i \neq t$,

$$\lim_{n\to\infty} \pi(\theta_t \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1$$
$$\lim_{n\to\infty} \pi(\theta_i \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 0, \quad i \neq t$$

  - To prove this, take logarithms is Bayes' theorem,
    define $z_i = \log[p(\boldsymbol{x} \mid \boldsymbol{\theta}_i)/p(\boldsymbol{x} \mid \boldsymbol{\theta}_t)]$,
    and use the strong law of large numbers on the n
    i.i.d. random variables $z_1, \ldots, z_n$.
  - For instance, in probabilistic diagnosis the posterior probability of the true disease converges to one as new relevant information accumulates, *provided* the model distinguishes the probabilistic behaviour of data under the true disease from its behaviour under the other alternatives.

- *Asymptotic behaviour, continuous case*

  - If the parameter θ is *one-dimensional and continuous*, so that $\Theta \subset \Re$, and the model $\{p(\boldsymbol{x} \,|\, \theta),\ \boldsymbol{x} \in \mathcal{X}\}$ is *regular*: basically,
    $\mathcal{X}$ does not depend on θ,
    $p(\boldsymbol{x} \,|\, \theta)$ is twice differentiable with respect to θ
  - Then, as $n \to \infty$, $\pi(\theta \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ converges intrinsically to a *normal* distribution with mean at the mle estimator $\hat{\theta}$, and with variance $v(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\theta})$, where
    $v^{-1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\theta}) = -\sum_{j=1}^{n} \frac{\partial^2}{\partial \theta^2} \log[p(\boldsymbol{x}_j \,|\, \theta]$
  - To prove this, express is Bayes' theorem as
    $\pi(\theta \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \propto \exp[\log \pi(\theta) + \sum_{j=1}^{n} \log p(\boldsymbol{x}_j \,|\, \theta)]$,
    and expand $\sum_{j=1}^{n} \log p(\boldsymbol{x}_j \,|\, \theta)]$ about its maximum, the mle $\hat{\theta}$
  - The result is easily extended to the multivariate case $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\}$, to obtain a limiting k-variate normal centered at $\hat{\boldsymbol{\theta}}$, and with a dispersion matrix $\boldsymbol{V}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\boldsymbol{\theta}})$ which generalizes $v(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\theta})$.

- *Asymptotic behaviour, continuous case. Simpler form*

  - Using the strong law of large numbers on the sums above a simpler, less precise approximation is obtained:
  - If the parameter $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\}$ is continuous, so that $\Theta \subset \Re^k$ and the model $\{p(\boldsymbol{x} \mid \theta),\ \boldsymbol{x} \in \mathcal{X}\}$ is *regular*, so that $\boldsymbol{\mathcal{X}}$ does not depend on $\boldsymbol{\theta}$ and $p(\boldsymbol{x} \mid \boldsymbol{\theta})$ is twice differentiable with respect to each of the θ_i's, then, as $n \to \infty$, $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ converges intrinsically to a *multivariate normal* distribution with mean the mle $\hat{\boldsymbol{\theta}}$ and precision matrix (inverse of the dispersion or variance-covariance matrix) $n\, \boldsymbol{F}(\hat{\boldsymbol{\theta}})$, where $\boldsymbol{F}(\boldsymbol{\theta})$ is Fisher's matrix, of general element
    $$\boldsymbol{F}_{ij}(\boldsymbol{\theta}) = -\mathrm{E}_{\boldsymbol{x} \mid \boldsymbol{\theta}}[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log p(\boldsymbol{x} \mid \boldsymbol{\theta})]$$
  - The properties of the multivariate normal yield from this result the asymptotic forms for the *marginal* and the *conditional* posterior distributions of any subgroup of the θ_j's.
  - In one dimension, $\pi(\theta \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \approx \mathrm{N}(\theta \mid \hat{\theta},\ (nF(\theta)^{-1/2})$, where $F(\theta) = -\mathrm{E}_{\boldsymbol{x} \mid \theta}[\partial^2 \log p(\boldsymbol{x} \mid \theta)/\partial\theta^2]$

- *Example: Asymptotic approximation with Poisson data*

  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ random from $\text{Pn}(x \,|\, \lambda) \propto e^{-\lambda}\lambda^x/x!$
    hence, $p(\boldsymbol{x} \,|\, \lambda) \propto e^{-n\lambda}\lambda^r$, $r = \Sigma_j\, x_j$, and $\hat{\lambda} = r/n$.

    Fisher's function is $F(\lambda) = -\text{E}_{x\,|\,\lambda}\left[\dfrac{\partial^2}{\partial\lambda^2}\log \text{Pn}(x \,|\, \lambda)\right] = \dfrac{1}{\lambda}$

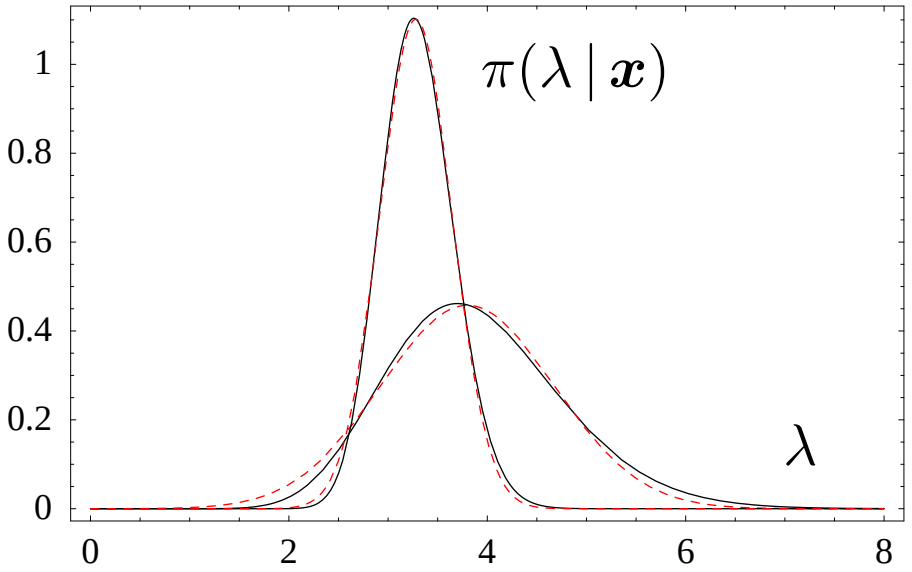
  - The objective prior function is $\pi(\lambda) = F(\lambda)^{1/2} = \lambda^{-1/2}$

    Hence $\pi(\lambda \,|\, \boldsymbol{x}) \propto e^{-n\lambda}\lambda^{r-1/2}$

    the kernel of $\text{Ga}(\lambda \,|\, r + \frac{1}{2}, n)$

  - The Normal approximation is
    $\pi(\lambda \,|\, \boldsymbol{x}) \approx \text{N}\{\lambda \,|\, \hat{\lambda}, (n\, F(\hat{\lambda}))^{-1/2}\}$
    $= \text{N}\{\lambda \,|\, r/n,\, \sqrt{r}/n\}$

  - Samples $n = 5$ and $n = 25$
    simulated from Poisson $\lambda = 3$
    yielded $r = 19$ and $r = 82$



2.2. Reference Analysis

- *No Relevant Initial Information*

  - Identify the mathematical form of a "noninformative" prior. One with *minimal effect, relative to the data, on the posterior distribution of the quantity of interest*.
  - Intuitive basis:

    Use *information theory* to measure the amount on information about the quantity of interest to be expected from data. This depends on prior knowledge: the more it is known, the less the amount of information the data may be expected to provide.

    Define the *missing information* about the quantity of interest as that which infinite independent replications of the experiment could possible provide.

    Define the *reference prior* as that which *maximizes the missing information about the quantity if interest*.

- *Expected information from the data*

  - Given model $\{p(\boldsymbol{x}\,|\,\theta), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}, \theta \in \Theta\}$, the *amount of information* $I^{\theta}\{\boldsymbol{\mathcal{X}}, \pi(\theta)\}$ which may be expected to be provided by $\boldsymbol{x}$, about the value of θ is defined by
    $I^{\theta}\{\boldsymbol{\mathcal{X}}, \pi(\theta)\} = \delta\{p(\boldsymbol{x}, \theta), p(\boldsymbol{x})\pi(\theta)\}$,
    the intrinsic discrepancy between the joint distribution $p(\boldsymbol{x}, \theta)$ and the product of their marginals $p(\boldsymbol{x})\pi(\theta)$, which is the *instrinsic association* between the random quantities $\boldsymbol{x}$ and θ.

  - Consider $I^{\theta}\{\boldsymbol{\mathcal{X}}^k, \pi(\theta)\}$ the information about θ which may be expected from k conditionally independent replications of the original setup.
    As $k \to \infty$, this would provide any *missing information* about θ. Hence, as $k \to \infty$, the functional $I^{\theta}\{\boldsymbol{\mathcal{X}}^k, \pi(\theta)\}$ will approach the missing information about θ associated with the prior $\pi(\theta)$.

  - Let $\pi_k(\theta)$ be the prior which maximizes $I^{\theta}\{\boldsymbol{\mathcal{X}}^k, \pi(\theta)\}$ in the class $\mathcal{P}$ of strictly positive prior distributions compatible with accepted assumptions on the value of θ (which be the class of *all* strictly positive priors).

    The *reference prior* $\pi^*(\theta)$ is the limit as $k \to \infty$ (in a sense to be made precise) of the sequence of priors $\{\pi_k(\theta), k = 1, 2, \ldots\}$.

- *Reference priors in the finite case*

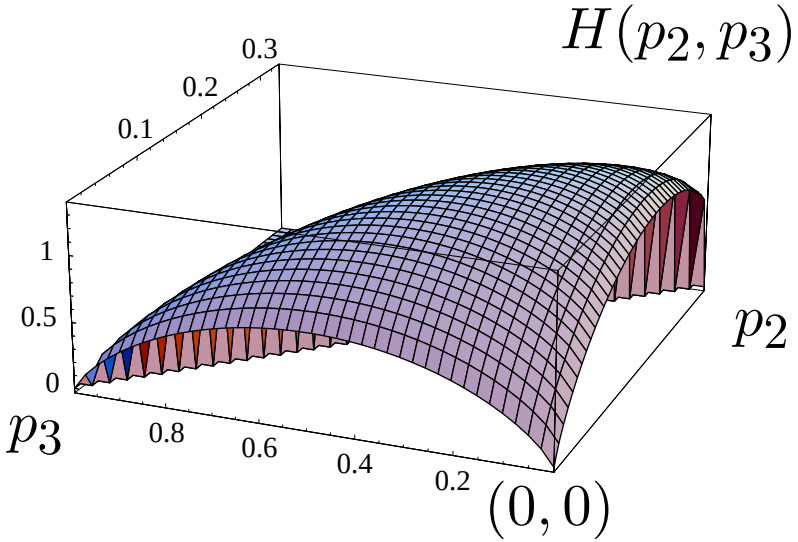
  - If θ may only take a *finite* number m of different values $\{\theta_1, \ldots, \theta_m\}$ and $\pi(\theta) = \{p_1, \ldots, p_m\}$, with $p_i = \Pr(\theta = \theta_i)$, then $\lim_{k\to\infty} I^{\theta}\{\boldsymbol{\mathcal{X}}^k, \pi(\theta)\} = H(p_1, \ldots, p_m) = -\sum_{i=1}^{m} p_i \log(p_i)$, that is, the *entropy* of the prior distribution $\{p_1, \ldots, p_m\}$.

  - In the finite case, the reference prior is that with *maximum entropy* within the class $\mathcal{P}$ of priors compatible with accepted assumptions.
    (cf. Statistical Physics)

  - If, in particular, $\mathcal{P}$ contains *all* priors over $\{\theta_1, \ldots, \theta_m\}$, the reference prior is the *uniform* prior, $\pi(\theta) = \{1/m, \ldots, 1/m\}$.
    (cf. Bayes-Laplace postulate of insufficient reason)

  - Prior $\{p_1, p_2, p_3, p_4\}$ in genetics problem where $p_1 = 2p_2$.

    Reference prior is $\{0.324, 0.162, 0.257, 0.257\}$



- *Reference priors in one-dimensional continuous case*

  - Let $\pi_k(\theta)$ be the prior which maximizes $I^{\theta}\{\boldsymbol{\mathcal{X}}^k, \pi(\theta)\}$ in the class $\mathcal{P}$ of acceptable priors.

    For any data $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, let $\pi_k(\theta \mid \boldsymbol{x}) \propto p(\boldsymbol{x} \mid \theta)\, \pi_k(\theta)$ be the corresponding posterior.

  - The *reference posterior density* $\pi^*(\theta \mid \boldsymbol{x})$ is defined to be the intrinsic limit of the sequence $\{\pi_k(\theta \mid \boldsymbol{x}), k = 1, 2, \ldots\}$

    A *reference prior function* $\pi^*(\theta)$ is any positive function such that, for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, $\pi^*(\theta \mid \boldsymbol{x}) \propto p(\boldsymbol{x} \mid \theta)\, \pi^*(\theta)$.
    This is defined up to an (irrelevant) arbitrary constant.

  - Let $\boldsymbol{x}^{(k)} \in \boldsymbol{\mathcal{X}}^k$ be the result of k independent replications of $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$.
    The exact expression for $\pi_k(\theta)$ (which may be obtained with calculus of variations) is

    $$\pi_k(\theta) = \exp\left[\mathrm{E}_{\boldsymbol{x}^{(k)} \mid \theta}\{\log \pi_k(\theta \mid \boldsymbol{x}^{(k)})\}\right]$$

  - This formula may be used, by repeated simulation from $p(\boldsymbol{x} \mid \theta)$ for different θ values, to obtain a *numerical approximation* to the reference prior.

- *Reference priors under regularity conditions*

  - Let $\tilde{\theta}_k = \tilde{\theta}(x^{(k)})$ be a consistent, asymptotically sufficient estimator of θ. In regular problems this is often the case with the mle estimator $\hat{\theta}$.

    The exact expression for $\pi_k(\theta)$ then becomes, for large k,

  - $\pi_k(\theta) \approx \exp[\mathrm{E}_{\tilde{\theta}_k \,|\, \theta}\{\log \pi_k(\theta \,|\, \tilde{\theta}_k)\}]$

    As $k \to \infty$ this converges to $\pi_k(\theta \,|\, \tilde{\theta}_k)|_{\tilde{\theta}_k = \theta}$

  - Let $\tilde{\theta}_k = \tilde{\theta}(x^{(k)})$ be a consistent, asymptotically sufficient estimator of θ. Let $\pi(\theta \,|\, \tilde{\theta}_k)$ be any asymptotic approximation to $\pi(\theta \,|\, x^{(k)})$, the posterior distribution of θ.

    Hence, $\pi^*(\theta) = \pi(\theta \,|\, \tilde{\theta}_k)|_{\tilde{\theta}_k = \theta}$

  - Under regularity conditions, the posterior distribution of θ is asymptotically Normal, with mean$\hat{\theta}$ and precision $n\, F(\hat{\theta})$, where $F(\theta) = -\mathrm{E}_{\boldsymbol{x} \,|\, \theta}[\partial^2 \log p(\boldsymbol{x} \,|\, \theta)/\partial\theta^2]$ is Fisher's information function.

    Hence, $\pi^*(\theta) = F(\theta)^{1/2}$ (Jeffreys' rule).

- *One nuisance parameter*

  - *Two parameters*: reduce the problem to a *sequential* application of the one parameter case. Probability model is $\{p(\boldsymbol{x} \mid \theta, \lambda, \theta \in \Theta, \lambda \in \Lambda\}$ and a θ-reference prior $\pi^*_\theta(\theta, \lambda)$ is required. Two steps:

    (i) Conditional on θ, $p(\boldsymbol{x} \mid \theta, \lambda)$ only depends on λ, and it is possible to obtain the *conditional* reference prior $\pi^*(\lambda \mid \theta)$.

    (ii) If $\pi^*(\lambda \mid \theta)$ is proper, integrate out λ to get the one-parameter model $p(\boldsymbol{x} \mid \theta) = \int_\Lambda p(\boldsymbol{x} \mid \theta, \lambda)\, \pi^*(\lambda \mid \theta)\, d\lambda$, and use the one-parameter solution to obtain $\pi^*(\theta)$.

    The θ-reference prior is then $\pi^*_\theta(\theta, \lambda) = \pi^*(\lambda \mid \theta)\, \pi^*(\theta)$.
    The required reference posterior is $\pi^*(\theta \mid \boldsymbol{x}) \propto p(\boldsymbol{x} \mid \theta) \pi^*(\theta)$.

  - If $\pi^*(\lambda \mid \theta)$ is an *improper* prior function, proceed within an increasing sequence $\{\Lambda_i\}$ over which $\pi^*(\lambda \mid \theta)$ is integrable and, for given data $\boldsymbol{x}$, obtain the corresponding sequence of reference posteriors $\{\pi^*_i(\theta \mid \boldsymbol{x}\}$.

    The required reference posterior $\pi^*(\theta \mid \boldsymbol{x})$ is their intrinsic limit.

    A θ-reference prior is any positive function such that, for any data $\boldsymbol{x}$, $\pi^*(\theta \mid \boldsymbol{x}) \propto \int_\Lambda p(\boldsymbol{x} \mid \theta, \lambda)\, \pi^*_\theta(\theta, \lambda)\, d\lambda$.

- *The regular two-parameter continuous case*

  - Model $p(\boldsymbol{x} \,|\, \theta, \lambda)$. If the joint posterior of (θ, λ) is asymptotically normal, the θ-reference prior may be derived in terms of the corresponding Fisher's information matrix, $\boldsymbol{F}(\theta, \lambda)$.

    $$\boldsymbol{F}(\theta, \lambda) = \begin{pmatrix} F_{\theta\theta}(\theta, \lambda) & F_{\theta\lambda}(\theta, \lambda) \\ F_{\theta\lambda}(\theta, \lambda) & F_{\lambda\lambda}(\theta, \lambda) \end{pmatrix}, \quad \boldsymbol{S}(\theta, \lambda) = \boldsymbol{F}^{-1}(\theta, \lambda),$$

    The θ-reference prior is $\pi^*_\theta(\theta, \lambda) = \pi^*(\lambda \,|\, \theta)\, \pi^*(\theta)$, where

    $\pi^*(\lambda \,|\, \theta) \propto F_{\lambda\lambda}^{1/2}(\theta, \lambda),\ \lambda \in \Lambda$, and, if $\pi^*(\lambda \,|\, \theta)$ is proper,

    $\pi^*(\theta) \propto \exp\{ \int_\Lambda \pi^*(\lambda \,|\, \theta)\, \log[S_{\theta\theta}^{-1/2}(\theta, \lambda)]\, d\lambda \},\ \theta \in \Theta.$

  - If $\pi^*(\lambda \,|\, \theta)$ is not proper, integrations are performed within an approximating sequence $\{\Lambda_i\}$ to obtain a sequence $\{\pi^*_i(\lambda \,|\, \theta)\, \pi^*_i(\theta)\}$, and the θ-reference prior $\pi^*_\theta(\theta, \lambda)$ is defined as its intrinsic limit.

  - Even if $\pi^*(\lambda \,|\, \theta)$ is improper, if θ and λ are variation independent, $S_{\theta\theta}^{-1/2}(\theta, \lambda) \propto f_\theta(\theta)\, g_\theta(\lambda)$, and $F_{\lambda\lambda}^{1/2}(\theta, \lambda) \propto f_\lambda(\theta)\, g_\lambda(\lambda)$,

    Then $\pi^*_\theta(\theta, \lambda) = f_\theta(\theta)\, g_\lambda(\lambda)$.

- *Examples: Inference on normal parameters*

  - The information matrix for the normal model $\mathrm{N}(x \mid \mu, \sigma)$ is
    $\boldsymbol{F}(\mu,\sigma) = \begin{pmatrix} \sigma^{-2} & 0 \\ 0 & 2\sigma^{-2} \end{pmatrix}$, $\quad \boldsymbol{S}(\mu,\sigma) = \boldsymbol{F}^{-1}(\mu,\sigma) = \begin{pmatrix} \sigma^{2} & 0 \\ 0 & \sigma^{2}/2 \end{pmatrix}$;
    Since μ and σ are variation independent, and both $F_{\sigma\sigma}$ and $S_{\mu\mu}$ factorize,
    $\pi^*(\sigma \mid \mu) \propto F_{\sigma\sigma}^{1/2} \propto \sigma^{-1}$, $\pi^*(\mu) \propto S_{\mu\mu}^{-1/2} \propto 1$.
    The μ-reference prior, as anticipated, is
    $\pi^*_\mu(\mu,\sigma) = \pi^*(\sigma \mid \mu)\, \pi^*(\mu) = \sigma^{-1}$,
    i.e., uniform on both μ and $\log\sigma$

  - Since $\boldsymbol{F}(\mu,\sigma)$ is diagonal the σ-reference prior is
    $\pi^*_\sigma(\mu,\sigma) = \pi^*(\mu \mid \sigma)\pi^*(\sigma) = \sigma^{-1}$, the same as $\pi^*_\mu(\mu,\sigma) = \pi^*_\sigma(\mu,\sigma)$.

  - In fact, it may be shown that, for location-scale models,
    $p(x \mid \mu, \sigma) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$,
    the reference prior for the location and scale parameters are always
    $\pi^*_\mu(\mu,\sigma) = \pi^*_\sigma(\mu,\sigma) = \sigma^{-1}$.

- Within any given model $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$ the ϕ-reference prior $\pi^*_\phi(\boldsymbol{\theta})$ maximizes the missing information about $\phi = \phi(\boldsymbol{\theta})$ and, in multiparameter problems, that prior *may change with the quantity of interest* ϕ.

- For instance, within a normal $\mathrm{N}(x \,|\, \mu, \sigma)$ model, let the *standardized mean* $\phi = \mu/\sigma$. be the quantity of interest.

  Fisher's information matrix in terms of the parameters ϕ and σ is $\boldsymbol{F}(\phi, \sigma) = J^t\, \boldsymbol{F}(\mu, \sigma)\, J$, where $J = (\partial(\mu, \sigma)/\partial(\phi, \sigma))$ is the Jacobian of the inverse transformation; this yields

$$\boldsymbol{F}(\phi, \sigma) = \begin{pmatrix} 1 & \phi/\sigma \\ \phi/\sigma & (2+\phi^2)/\sigma^2 \end{pmatrix}, \quad \boldsymbol{S}(\phi, \sigma) = \begin{pmatrix} 1+\phi^2/2 & -\phi\,\sigma/2 \\ -\phi\,\sigma/2 & \sigma^2/2 \end{pmatrix},$$

  with $F^{1/2}_{\sigma\sigma} \propto \sigma^{-1}$, and $S^{-1/2}_{\phi\phi} \propto (1+\phi^2/2)^{-1/2}$.

- The ϕ-reference prior is, $\pi^*_\phi(\phi, \sigma) = (1+\phi^2/2)^{-1/2}\sigma^{-1}$.

  In the original parametrization, $\pi^*_\phi(\mu, \sigma) = (1+(\mu/\sigma)^2/2)^{-1/2}\sigma^{-2}$, which is different from $\pi^*_\mu(\mu, \sigma) = \pi^*_\sigma(\mu, \sigma)$.

  This prior is shown to lead to a reference posterior for ϕ with *consistent marginalization properties*.

- *Many parameters*

  - The reference algorithm generalizes to any number of parameters. If the model is $p(\boldsymbol{x} \mid \boldsymbol{\theta}) = p(\boldsymbol{x} \mid \theta_1, \ldots, \theta_m)$, a joint reference prior $\pi^*(\phi_m \mid \phi_{m-1}, \ldots, \phi_1) \times \ldots \times \pi^*(\phi_2 \mid \phi_1) \times \pi^*(\phi_1)$ may sequentially be obtained for each *ordered parametrization*, $\{\phi_1(\boldsymbol{\theta}), \ldots, \phi_m(\boldsymbol{\theta})\}$.

    Reference priors are *invariant* under reparametrization of the $\phi_i(\boldsymbol{\theta})$'s.

  - The choice of the ordered parametrization $\{\phi_1, \ldots, \phi_m\}$ describes the particular prior required, namely that which *sequentially* maximizes the missing information about each of the ϕ_i's, conditional on $\{\phi_1, \ldots, \phi_{i-1}\}$, for $i = m, m-1, \ldots, 1$.

  - Example: *Stein's paradox*. Data random from a m-variate normal $\mathrm{N}_m(\boldsymbol{x} \mid \boldsymbol{\mu}, \boldsymbol{I})$. The reference prior function for any permutation of the μ_i's is uniform, and leads to appropriate posterior distributions for any of the μ_i's, but cannot be used if the quantity of interest is $\theta = \sum_i \mu_i^2$, the distance of $\boldsymbol{\mu}$ to the origin.

    The reference prior for $\{\theta, \lambda_1, \ldots, \lambda_{m-1}\}$ produces, for any choice of the λ_i's, an appropriate the reference posterior for θ.

2.3. Inference Summaries

- *Summarizing the posterior distribution*

  - *The* Bayesian final *outcome* of a problem of inference about any unknown quantity $\boldsymbol{\theta}$ *is* precisely the *posterior density* $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}, C)$.
  - Bayesian inference may be described as the problem of stating a probability distribution for the quantity of interest encapsulating all available information about its value.
  - In one or two dimensions, a *graph of the posterior probability density* of the quantity of interest conveys an intuitive summary of the main conclusions. This is greatly appreciated by users, and is an important asset of Bayesian methods.
  - However, graphical methods not easily extend to more than two dimensions and elementary *quantitative* conclusions are often required.

    The simplest forms to *summarize* the information contained in the posterior distribution are closely related to the conventional concepts of point estimation and interval estimation.

- *Point Estimation: Posterior mean and posterior mode*

  - It is often required to provide point estimates of relevant quantities. Bayesian point estimation is best described as a *decision problem* where one has to *choose* a particular value $\tilde{\boldsymbol{\theta}}$ as an approximate proxy for the actual, unknown value of $\boldsymbol{\theta}$.

  - Intuitively, any location measure of the posterior density $\pi(\boldsymbol{\theta} \mid \boldsymbol{x})$ may be used as a point estimator. When they exist, either
    $\mathrm{E}[\boldsymbol{\theta} \mid \boldsymbol{x}] = \int_{\Theta} \boldsymbol{\theta}\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta}$ *(posterior mean)*, or
    $\mathrm{Mo}[\boldsymbol{\theta} \mid \boldsymbol{x}] = \arg\sup_{\boldsymbol{\theta} \in \Theta} \pi(\boldsymbol{\theta} \mid \boldsymbol{x})$ *(posterior mode)*
    are often regarded as natural choices.

  - *Lack of invariance*. Neither the posterior mean not the posterior mode are invariant under reparametrization. The point estimator $\tilde{\boldsymbol{\psi}}$ of a bijection $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$ will generally not be equal to $\boldsymbol{\psi}(\tilde{\boldsymbol{\theta}})$.

    In pure "inferential" applications, where one is requested to provide a point estimate of the vector of interest without an specific application in mind, it is difficult to justify a non-invariant solution:
    The best estimate of, say, $\phi = \log(\theta)$ should be $\phi^* = \log(\theta^*)$.

- *Point Estimation: Posterior median*

  - A summary of a multivariate density $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$, where $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\}$, should contain summaries of:
    (i) each of the marginal densities $\pi(\theta_i \,|\, \boldsymbol{x})$,
    (ii) the densities $\pi(\phi \,|\, \boldsymbol{x})$ of other functions of interest $\phi = \phi(\boldsymbol{\theta})$.
  - In *one-dimensional continuous* problems the *posterior median*, is easily defined and computed as
    $\text{Me}[\theta \,|\, \boldsymbol{x}] = q\,; \quad \Pr[\theta \leq q \,|\, \boldsymbol{x}] = \int_{\{\theta \leq q\}} \pi(\theta \,|\, \boldsymbol{x})\, \text{d}\theta = 1/2$

    The one-dimensional posterior median has many attractive properties:
    (i) it is *invariant* under bijections, $\text{Me}[\phi(\theta) \,|\, \boldsymbol{x}] = \phi(\text{Me}[\theta \,|\, \boldsymbol{x}])$.
    (ii) it *exists* and it is *unique* under very wide conditions
    (iii) it is rather *robust* under moderate perturbations of the data.
  - The posterior median is often considered to be the best 'automatic' Bayesian point estimator in one-dimensional continuous problems.
  - The posterior median is not easily used to a multivariate setting. The natural extension of its definition produces *surfaces* (not points).

    General invariant multivariate definitions of point estimators is possible using Bayesian *decision theory*

- *General Credible Regions*

  - To describe $\pi(\boldsymbol{\theta} \mid \boldsymbol{x})$ it is often convenient to quote regions $\Theta_p \subset \Theta$ of given probability content p under $\pi(\boldsymbol{\theta} \mid \boldsymbol{x})$. This is the intuitive basis of graphical representations like boxplots.
  - A subset Θ_p of the parameter space Θ such that
    $\int_{\Theta_p} \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta} = p, \quad$ so that $\Pr(\boldsymbol{\theta} \in \Theta_p \mid \boldsymbol{x}) = p,$
    is a *posterior p-credible region* for $\boldsymbol{\theta}$.
  - A credible region is invariant under reparametrization:
    If Θ_p is p-credible for $\boldsymbol{\theta}$, $\boldsymbol{\phi}(\Theta_p)$ is a p-credible for $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta})$.
  - For any given p there are generally infinitely many credible regions.
    Credible regions may be selected to have minimum size (length, area, volume), resulting in *highest probability density* (HPD) regions, where all points in the region have larger probability density than all points outside.
  - HPD regions are *not invariant* : the image $\boldsymbol{\phi}(\Theta_p)$ of an HPD region Θ_p will be a credible region for $\boldsymbol{\phi}$, but will not generally be HPD.
    There is no reason to restrict attention to HPD credible regions.

- *Credible Intervals*

  - In *one-dimensional continuous* problems, posterior quantiles are often used to derive credible intervals.
  - If $\theta_q = Q_q[\theta \,|\, \boldsymbol{x}]$ is the q-quantile of the posterior distribution of θ, the interval $\Theta_p = \{\theta;\ \theta \leq \theta_p\}$ is a p-credible region, and it is invariant under reparametrization.
  - *Equal-tailed* p-credible intervals of the form $\Theta_p = \{\theta;\ \theta_{(1-p)/2} \leq \theta \leq \theta_{(1+p)/2}\}$ are typically unique, and they invariant under reparametrization.
  - Example: Model $\mathrm{N}(x \,|\, \mu, \sigma)$. *Credible intervals for the normal mean*. The reference posterior for μ is $\pi(\mu \,|\, \boldsymbol{x}) = \mathrm{St}(\mu \,|\, \overline{x}, s/\sqrt{n-1}, n-1)$. Hence the reference *posterior* distribution of $\tau = \sqrt{n-1}(\mu - \overline{x})/s$, *a function of* μ, is $\pi(\tau \,|\, \overline{x}, s, n) = \mathrm{St}(\tau \,|\, 0, 1, n-1)$.

    Thus, the equal-tailed p-credible intervals for μ are $\{\mu;\ \mu \in \overline{x} \pm q_{n-1}^{(1-p)/2}\, s/\sqrt{n-1}\}$, where $q_{n-1}^{(1-p)/2}$ is the $(1-p)/2$ quantile of a standard Student density with $n-1$ degrees of freedom.

- *Calibration*

  - In the normal example above , the expression $t = \sqrt{n-1}(\mu - \overline{x})/s$ may *also* be analyzed, for fixed μ, as a *function of the data*.

    The fact that the *sampling* distribution of the statistic $t = t(\overline{x}, s \,|\, \mu, n)$ is *also* an standard Student $p(t \,|\, \mu, n) = \mathrm{St}(t \,|\, 0, 1, n-1)$ with the same degrees of freedom implies that, in this example, objective Bayesian credible intervals are *also* be *exact* frequentist confidence intervals.

  - *Exact numerical agreement* between Bayesian credible intervals and frequentist confidence intervals is the *exception, not the norm*.

  - For *large samples*, convergence to normality implies *approximate numerical agreement*. This provides a frequentist *calibration* to objective Bayesian methods.

  - Exact numerical *agreement* is obviously *impossible when the data are discrete*: Precise (non randomized) frequentist confidence intervals do not exist in that case for most confidence levels.

    The computation of Bayesian credible regions for continuous parameters is however *precisely the same* whether the data are *discrete or continuous*.

2.4. Prediction

- *Posterior predictive distributions*

  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, $x_i \in \mathcal{X}$, set of "homogeneous" observations. Desired to predict the value of a future observation $x \in \mathcal{X}$ generated by the same mechanism.
  - From the foundations arguments the solution *must* be a probability distribution $p(x \mid \boldsymbol{x}, K)$ describing the uncertainty on the value that x will take, given data $\boldsymbol{x}$ and any other available knowledge K. This is called the (posterior) *predictive density* of x.
  - To derive $p(x \mid \boldsymbol{x}, K)$ it is necessary to specify the *precise sense* in which the x_i's are judged to be *homogeneous*.
  - It is often directly assumed that the data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ consist of a *random sample* from some specified model, $\{p(x \mid \boldsymbol{\theta}), x \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$, so that $p(\boldsymbol{x} \mid \boldsymbol{\theta}) = p(x_1, \ldots, x_n \mid \boldsymbol{\theta}) = \prod_{j=1}^{n} p(x_j \mid \boldsymbol{\theta})$.

    If this is the case, the solution to the prediction problem is immediate once a prior distribution $\pi(\boldsymbol{\theta})$ has been specified.

- *Posterior predictive distributions from random samples*

  - Let $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, $x_i \in \mathcal{X}$ a random sample of size n from the statistical model $\{p(x \mid \boldsymbol{\theta}), x \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$
    Let $\pi(\boldsymbol{\theta})$ a prior distribution describing available knowledge (in any) about the value of the parameter vector $\boldsymbol{\theta}$.
    The *posterior predictive distribution* is

    $$p(x \mid \boldsymbol{x}) = p(x \mid x_1, \ldots, x_n) = \int_\Theta p(x \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, d\boldsymbol{\theta}$$

    This encapsulates all available information about the outcome of any future observation $x \in \mathcal{X}$ from the same model.

  - To prove this, make use the total probability theorem, to have
    $p(x \mid \boldsymbol{x}) = \int_\Theta p(x \mid \boldsymbol{\theta}, \boldsymbol{x})\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta}$
    and notice the new observation x has been assumed to be conditionally independent of the observed data $\boldsymbol{x}$, so that $p(x \mid \boldsymbol{\theta}, \boldsymbol{x}) = p(x \mid \boldsymbol{\theta})$.

  - The observable values $x \in \mathcal{X}$ may be either *discrete* or *continuous* random quantities. In the discrete case, the predictive distribution will be described by its probability *mass* function; in the continuous case, by its probability *density* function. Both are denoted $p(x \mid \boldsymbol{x})$.

- *Prediction in a Poisson process*

  - Data $\boldsymbol{x} = \{r_1, \ldots, r_n\}$ random from $\text{Pn}(r \,|\, \lambda)$. The reference posterior density of λ is $\pi^*(\lambda \,|\, \boldsymbol{x}) = \text{Ga}(\lambda \,|\, t + 1/2, n)$, where $t = \Sigma_j \, r_j$.

    The (reference) posterior predictive distribution is

$$p(r \,|\, \boldsymbol{x}) = \Pr[r \,|\, t, n] = \int_0^\infty \text{Pn}(r \,|\, \lambda) \, \text{Ga}(\lambda \,|\, t + \tfrac{1}{2}, n) \, d\lambda$$

$$= \frac{n^{t+1/2}}{\Gamma(t + 1/2)} \, \frac{1}{r!} \, \frac{\Gamma(r + t + 1/2)}{(1 + n)^{r+t+1/2}} \,,$$

    an example of a Poisson-Gamma probability mass function.

  - For example, no flash floods have been recorded on a particular location in 10 consecutive years. Local authorities are interested in forecasting possible future flash floods. Using a Poisson model, and assuming that meteorological conditions remain similar, the probabilities that r flash floods will occur next year in that location are given by the Poisson-Gamma mass function above, with $t = 0$ and $n = 10$. This yields, $\Pr[0 \,|\, t, n] = 0.953$, $\Pr[1 \,|\, t, n] = 0.043$, and $\Pr[2 \,|\, t, n] = 0.003$.

    Many other situations may be described with the same model.

- *Prediction of Normal measurements*

  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ random from $N(x \,|\, \mu, \sigma)$. Reference prior $\pi^*(\mu, \sigma) = \sigma^{-1}$ or, in terms of the precision $\lambda = \sigma^{-2}$, $\pi^*(\mu, \lambda) = \lambda^{-1}$.

    The *joint* reference posterior, $\pi^*(\mu, \lambda \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \mu, \lambda)\, \pi^*(\mu, \lambda)$, is $\pi^*(\mu, \lambda \,|\, \boldsymbol{x}) = N(\mu \,|\, \overline{x}, (n\lambda)^{-1/2})\, \mathrm{Ga}(\lambda \,|\, (n-1)/2, ns^2/2)$.

  - The predictive distribution is

$$\pi^*(x \,|\, \boldsymbol{x}) = \int_0^\infty \int_{-\infty}^\infty \mathrm{N}(x \,|\, \mu, \lambda^{-1/2})\, \pi^*(\mu, \lambda \,|\, \boldsymbol{x})\, \mathrm{d}\mu\, \mathrm{d}\lambda$$
$$\propto \{(1+n)s^2 + (\mu - \overline{x})^2\}^{-n/2},$$

    a kernel of the *Student* density $\pi^*(x \,|\, \boldsymbol{x}) = \mathrm{St}(x \,|\, \overline{x}, s\sqrt{\frac{n+1}{n-1}}, n-1)$.

  - *Example*. Production of safety belts. Observed breaking strengths of 10 randomly chosen webbings have mean $\overline{x} = 28.011$ kN and standard deviation $s = 0.443$ kN. Specification requires $x > 26$ kN.

    Reference posterior predictive $p(x \,|\, \boldsymbol{x}) = \mathrm{St}(x \,|\, 28.011, 0.490, 9)$.

    $\Pr(x > 26 \,|\, \boldsymbol{x}) = \int_{26}^\infty \mathrm{St}(x \,|\, 28.011, 0.490, 9)\, \mathrm{d}x = 0.9987$.

- *Regression*

  - Often *additional information* from relevant covariates. Data structure, set of pairs $\boldsymbol{x} = \{(\boldsymbol{y}_1, \boldsymbol{v}_1), \ldots (\boldsymbol{y}_n, \boldsymbol{v}_n)\}$; $\boldsymbol{y}_i$, $\boldsymbol{v}_i$, both vectors. Given a new observation, with $\boldsymbol{v}$ known, predict the corresponding value of $\boldsymbol{y}$. Formally, compute $p\{\boldsymbol{y} \,|\, \boldsymbol{v}, (\boldsymbol{y}_1, \boldsymbol{v}_1), \ldots (\boldsymbol{y}_n, \boldsymbol{v}_n)\}$.
  - Need a model $\{p(\boldsymbol{y} \,|\, \boldsymbol{v}, \boldsymbol{\theta}), \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{\theta} \in \Theta\}$ which makes precise the probabilistic relationship between $\boldsymbol{y}$ and $\boldsymbol{v}$. The simplest option assumes a *linear dependency* of the form $p(\boldsymbol{y} \,|\, \boldsymbol{v}, \boldsymbol{\theta}) = \mathrm{N}(\boldsymbol{y} \,|\, \boldsymbol{V}\boldsymbol{\beta}, \Sigma)$, but far more complex structures are common in applications.
  - *Univariate linear regression on k covariates*. $Y \subset \Re$, $\boldsymbol{v} = \{v_1, \ldots, v_k\}$. $p(y \,|\, \boldsymbol{v}, \boldsymbol{\beta}, \sigma) = \mathrm{N}(y \,|\, \boldsymbol{v}\boldsymbol{\beta}, \sigma^2)$, $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_k\}^t$. Data $\boldsymbol{x} = \{\boldsymbol{y}, \boldsymbol{V}\}$, $\boldsymbol{y} = \{y_1, \ldots, y_n\}^t$, and $\boldsymbol{V}$ is the $n \times k$ matrix with the $\boldsymbol{v}_i$'s as rows. $p(\boldsymbol{y} \,|\, \boldsymbol{V}, \boldsymbol{\beta}, \sigma) = \mathrm{N}_n(\boldsymbol{y} \,|\, \boldsymbol{V}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}_n)$; reference prior $\pi^*(\boldsymbol{\beta}, \sigma) = \sigma^{-1}$.

    Predictive posterior is the Student density

    $$p(y \,|\, \boldsymbol{v}, \boldsymbol{y}, \boldsymbol{V}) = \mathrm{St}(y \,|\, \boldsymbol{v}\hat{\boldsymbol{\beta}},\ s\sqrt{f(\boldsymbol{v}, \boldsymbol{V})\,\tfrac{n}{n-k}},\ n-k)$$

    $$\hat{\boldsymbol{\beta}} = (\boldsymbol{V}^t\boldsymbol{V})^{-1}\boldsymbol{V}^t\boldsymbol{y}, \quad ns^2 = (\boldsymbol{y} - \boldsymbol{v}\hat{\boldsymbol{\beta}})^t(\boldsymbol{y} - \boldsymbol{v}\hat{\boldsymbol{\beta}})$$

    $$f(\boldsymbol{v}, \boldsymbol{V}) = 1 + \boldsymbol{v}(\boldsymbol{V}^t\boldsymbol{V})^{-1}\boldsymbol{v}^t$$

- *Example: Simple linear regression*

  - One covariate and a constant term; $p(y \mid v, \boldsymbol{\beta}, \sigma) = \mathrm{N}(y \mid \beta_1 + \beta_2 v, \sigma)$
    Sufficient statistic is $\boldsymbol{t} = \{\overline{v}, \overline{y}, s_{vy}, s_{vv}\}$, with $n\overline{v} = \Sigma v_j$, $n\overline{y} = \Sigma y_j$, $s_{yv} = \Sigma v_j y_j / n - \overline{v}\,\overline{y}$, $s_{vv} = \Sigma v_j^2 / n - \overline{v}^2$.

    $$p(y \mid v, \boldsymbol{t}) = \mathrm{St}(y \mid \hat{\beta}_1 + \hat{\beta}_2\, v,\ s\sqrt{f(v, \boldsymbol{t})\,\tfrac{n}{n-2}},\ n-2)$$

    $$\hat{\beta}_1 = \overline{y} - \hat{\beta}_2 \overline{v}, \quad \hat{\beta}_2 = \frac{s_{vy}}{s_{vv}},$$

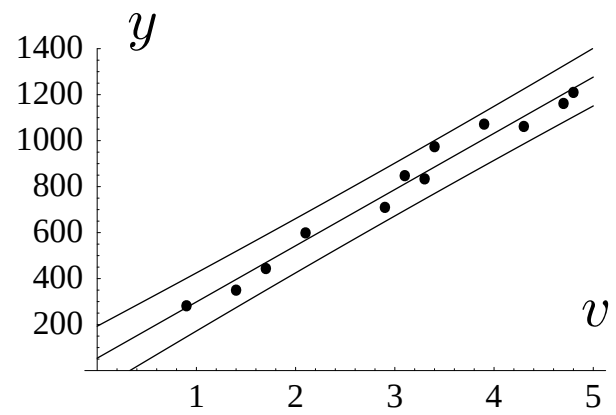
    $$n s^2 = \sum_{j=1}^{n} (y_j - \hat{\beta}_1 - \hat{\beta}_2 x_j)^2$$

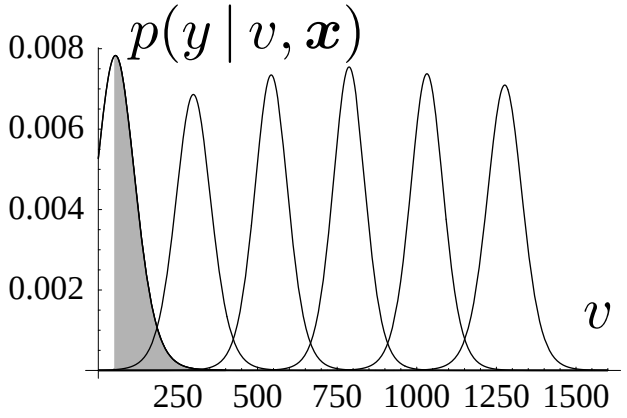
    $$f(v, \boldsymbol{t}) = 1 + \frac{1}{n}\frac{(v-\overline{v})^2 + s_{vv}}{s_{vv}}$$

  - Pollution density ($\mu gr/m^3$), and wind speed from source (m/s).

y_j	1212	836	850	446	1164	601
v_j	4.8	3.3	3.1	1.7	4.7	2.1
y_j	1074	284	352	1064	712	976
v_j	3.9	0.9	1.4	4.3	2.9	3.4

$$\Pr[y > 50 \mid v = 0, \boldsymbol{x}] = 0.66$$





2.4. Hierarchical Models

- *Exchangeability*

  - Random quantities are often "homogeneous" in the precise sense that only their *values* matter, not the *order* in which they appear. Formally, this is captured by the notion of *exchangeability*. The set of random vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is exchangeable if their joint distribution is invariant under permutations. An infinite sequence $\{\boldsymbol{x}_j\}$ of random vectors is exchangeable if all its finite subsequences are exchangeable.

  - *Any random sample from any model is exchangeable*. The *representation theorem* establishes that if observations $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ are exchangeable, they are a *a random sample* from some model $\{p(\boldsymbol{x} \mid \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta\}$, labeled by a *parameter vector* $\boldsymbol{\theta}$, *defined* as the limit (as $n \to \infty$) of some function of the $\boldsymbol{x}_i$'s. Information about $\boldsymbol{\theta}$ in prevailing conditions C is *necessarily* described by *some* probability distribution $\pi(\boldsymbol{\theta} \mid C)$.

  - Formally, the joint density of any finite set of exchangeable observations $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ has an *integral representation* of the form $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \mid C) = \int_\Theta \prod_{i=1}^n p(\boldsymbol{x}_i \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid C)\, d\boldsymbol{\theta}$.

- *Structured Models*

  - Complex data structures may often be usefully described by partial exchangeability assumptions.
  - *Example: Public opinion.* Sample k different regions in the country. Sample n_i citizens in region i and record whether or not ($y_{ij} = 1$ or $y_{ij} = 0$) citizen j would vote A. Assuming exchangeable citizens within each region implies
    $p(y_{i1}, \ldots, y_{in_i}) = \prod_{j=1}^{n_i} p(y_{ij} \mid \theta_i) = \theta_i^{r_i}(1-\theta_i)^{n_i - r_i}$,
    where θ_i is the (unknown) proportion of citizens in region i voting A and $r_i = \Sigma_j y_{ij}$ the number of citizens voting A in region i.

    Assuming regions exchangeable within the country similarly leads to
    $p(\theta_1, \ldots, \theta_k) = \prod_{i=1}^{k} \pi(\theta_i \mid \boldsymbol{\phi})$
    for some probability distribution $\pi(\theta \mid \boldsymbol{\phi})$ describing the political variation within the regions. Often choose $\pi(\theta \mid \boldsymbol{\phi}) = \text{Be}(\theta \mid \alpha, \beta)$.
  - The resulting *two-stages hierarchical Binomial-Beta model*
    $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$, $\boldsymbol{y}_i = \{y_{i1}, \ldots, y_{in_i}\}$, random from $\text{Bi}(y \mid \theta_i)$,
    $\{\theta_1, \ldots, \theta_k\}$, random from $\text{Be}(\theta \mid \alpha, \beta)$
    provides a far richer model than (unrealistic) simple binomial sampling.

- *Example: Biological response.* Sample k different animals of the same species in specific environment. Control n_i times animal i and record his responses $\{\boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}\}$ to prevailing conditions. Assuming exchangeable observations within each animal implies
  $p(\boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}) = \prod_{j=1}^{n_i} p(\boldsymbol{y}_{ij} \,|\, \boldsymbol{\theta}_i)$.
  Often choose $p(\boldsymbol{y}_{ij} \,|\, \boldsymbol{\theta}_i) = \mathrm{N}_r(\boldsymbol{y} \,|\, \boldsymbol{\mu}_i, \Sigma_1)$, where r is the number of biological responses measured.

  Assuming exchangeable animals within the environment leads to
  $p(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k) = \prod_{i=1}^{k} \pi(\boldsymbol{\mu}_i \,|\, \boldsymbol{\phi})$
  for some probability distribution $\pi(\boldsymbol{\mu} \,|\, \boldsymbol{\phi})$ describing the biological variation within the species. Often choose $\pi(\boldsymbol{\mu} \,|\, \boldsymbol{\phi}) = \mathrm{N}_r(\boldsymbol{\mu} \,|\, \boldsymbol{\mu}_0, \Sigma_2)$.
- The *two-stages hierarchical multivariate Normal-Normal model*
  $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$, $\boldsymbol{y}_i = \{\boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}\}$, random from $\mathrm{N}_r(\boldsymbol{y} \,|\, \boldsymbol{\mu}_i, \Sigma_1)$,
  $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k\}$, random from $\mathrm{N}_r(\boldsymbol{\mu} \,|\, \boldsymbol{\mu}_0, \Sigma_2)$
  provides a far richer model than (unrealistic) simple multivariate normal sampling.
- Finer subdivisions, *e.g.*, subspecies within each species, similarly lead to hierarchical models with more stages.

- *Bayesian analysis of hierarchical models*

  - A *two-stages hierarchical model* has the general form
    $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$, $\boldsymbol{y}_i = \{\boldsymbol{z}_{i1}, \ldots, \boldsymbol{z}_{in_i}\}$
    $\boldsymbol{y}_i$ random sample of size n_i from $p(\boldsymbol{z} \mid \boldsymbol{\theta}_i)$, $\boldsymbol{\theta}_i \in \Theta$,
    $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_k\}$, random of size k from $\pi(\boldsymbol{\theta} \mid \boldsymbol{\phi})$, $\boldsymbol{\phi} \in \Phi$.
  - Specify a *prior distribution* (or a reference prior function)
    $\pi(\boldsymbol{\phi})$ for the *hyperparameter vector* $\boldsymbol{\phi}$.
  - Use *standard probability theory* to compute all desired
    posterior distributions:
    $\pi(\boldsymbol{\phi} \mid \boldsymbol{x})$ for inferences about the hyperparameters,
    $\pi(\boldsymbol{\theta}_i \mid \boldsymbol{x})$ for inferences about the parameters,
    $\pi(\psi \mid \boldsymbol{x})$ for inferences about the any function $\psi = \psi(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_k)$
    of the parameters,
    $\pi(\boldsymbol{y} \mid \boldsymbol{x})$ for predictions on future observations,
    $\pi(t \mid \boldsymbol{x})$ for predictions on any function $t = t(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$
    of m future observations
  - *Markov Chain Monte Carlo* based *software* available for the necessary
    computations.

3. Decision Making

3.1 Structure of a Decision Problem

- *Alternatives, consequences, relevant events*

  - A decision problem if two or more possible courses of action; $\mathcal{A}$ is the class of possible *actions*.
  - For each $a \in \mathcal{A}$, Θ_a is the set of *relevant events*, those may affect the result of choosing a.
  - Each pair $\{a, \boldsymbol{\theta}\}$, $\boldsymbol{\theta} \in \Theta_a$, produces a consequence $c(a, \boldsymbol{\theta}) \in \mathcal{C}_a$. In this context, $\boldsymbol{\theta}$ if often referred to as the *parameter of interest*.
  - The class of pairs $\{(\Theta_a, \mathcal{C}_a), a \in \mathcal{A}\}$ describes the *structure* of the decision problem. Without loss of generality, it may be assumed that the possible actions are mutually exclusive, for otherwise the appropriate Cartesian product may be used.
  - In many problems the class of relevant events Θ_a is the same for all $a \in \mathcal{A}$. Even if this is not the case, a comprehensive *parameter space* Θ may be defined as the union of all the Θ_a.

- *Foundations of decision theory*

  - Different sets of principles capture a minimum collection of logical rules required for "rational" decision-making.

    These are axioms with strong intuitive appeal.
    Their basic structure consists of:

    - The *Transitivity* of preferences:
      If $a_1 \succ a_2$ given C, and $a_2 \succ a_3$ given C,
      then $a_1 \succ a_3$ given C.
    - The *Sure-thing principle*:
      If $a_1 \succ a_2$ given C and E, and $a_1 \succ a_2$ given C and not E
      then $a_1 \succ a_2$ given C.
    - The existence of *Standard events*:
      There are events of known plausibility.
      These may be used as a unit of measurement, and
      have the properties of a probability measure

  - These axioms are not a description of human decision-making,
    but a *normative* set of principles defining *coherent* decision-making.

- *Decision making*

  - Many different axiom sets.
    All lead basically to the same set of conclusions, namely:

    • The consequences of wrong actions should be evaluated in terms of a real-valued *loss* function $\ell(a, \boldsymbol{\theta})$ which specifies, on a numerical scale, their undesirability.

    • The uncertainty about the parameter of interest $\boldsymbol{\theta}$ should be measured with a *probability distribution* $\pi(\boldsymbol{\theta} \,|\, C)$

$$\pi(\boldsymbol{\theta} \,|\, C) \geq 0, \quad \boldsymbol{\theta} \in \Theta, \qquad \int_{\Theta} \pi(\boldsymbol{\theta} \,|\, C)\, \mathrm{d}\boldsymbol{\theta} = 1,$$

    describing all available knowledge about its value, given the conditions C under which the decision must be taken.

    • The relative undesirability of available actions $a \in \mathcal{A}$ is measured by their expected loss: *the optimal action minimizes the expected loss*.

$$\ell[a \,|\, C] = \int_{\Theta} \ell(a, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, C)\, d\boldsymbol{\theta}, \quad a \in \mathcal{A}.$$

    (alternatively, one may *maximize expected utility*)

- *Intrinsic loss functions: Intrinsic discrepancy*

  - The loss function is typically *context dependent*.
  - In mathematical statistics, *intrinsic* loss functions are used to measure the distance between between statistical models.

    They measure the *divergence between the models* $\{p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}\}$ and $\{p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}\}$ as some *non-negative* function of the form $\ell\{p_1, p_2\}$ which is zero if (and only if) the two distributions are equal almost everywhere.
  - The *intrinsic discrepancy* between two statistical models is simply the intrinsic discrepancy between their sampling distributions, *i.e.*,

$$\begin{aligned}\delta\{p_1, p_2\} &= \delta\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\} \\ &= \min\left\{\int_{\mathcal{X}_1} p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1) \log \frac{p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1)}{p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2)} \, \mathrm{d}\boldsymbol{x}, \int_{\mathcal{X}_2} p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2) \log \frac{p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2)}{p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1)} \, \mathrm{d}\boldsymbol{x}\right\}\end{aligned}$$

  - The intrinsic discrepancy is an *information-based, symmetric, invariant intrinsic loss*.

3.2 Point and Region Estimation

- *Point estimation as a decision problem*

  - Given statistical model $\{p(\boldsymbol{x} \,|\, \boldsymbol{\omega}), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}, \boldsymbol{\omega} \in \Omega\}$, quantity of interest $\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\omega}) \in \Theta$. A *point estimator* $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}(\boldsymbol{x})$ of $\boldsymbol{\theta}$ is some function of the data to be regarded as a proxy for the unknown value of $\boldsymbol{\theta}$.
  - To choose a point estimate for $\boldsymbol{\theta}$ is a *decision problem*, where the action space is $\mathcal{A} = \Theta$.
  - Given a *loss function* $\ell(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta})$, the posterior expected loss is

$$\ell[\tilde{\boldsymbol{\theta}} \,|\, \boldsymbol{x}] = \int_{\Theta} \ell(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta},$$

    The corresponding *Bayes estimator* is the function of the data,

$$\boldsymbol{\theta}^* = \boldsymbol{\theta}^*(\boldsymbol{x}) = \arg \inf_{\tilde{\boldsymbol{\theta}} \in \Theta} \ell[\tilde{\boldsymbol{\theta}} \,|\, \boldsymbol{x}],$$

    which minimizes that expectation.

- *Conventional estimators*

  - The *posterior mean* and the *posterior mode* are the Bayes estimators which respectively correspond to a *quadratic* an a *zero-one* loss functions.

    • If $\ell(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = (\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})^t(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})$, then, assuming that the mean exists, the Bayes estimator is the *posterior mean* $E[\boldsymbol{\theta} \mid \boldsymbol{x}]$.

  - • If the loss function is a zero-one function, so that $\ell(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = 0$ if $\tilde{\boldsymbol{\theta}}$ belongs to a ball of radius ε centered in $\boldsymbol{\theta}$ and $\ell(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = 1$ otherwise then, assuming that a unique mode exists, the Bayes estimator converges to the *posterior mode* $\text{Mo}[\boldsymbol{\theta} \mid \boldsymbol{x}]$ as the ball radius ε tends to zero.

  - If θ is *univariate and continuous*, and the loss function is *linear*,

$$\ell(\tilde{\theta}, \theta) = \begin{cases} c_1(\tilde{\theta} - \theta) & \text{if} \quad \tilde{\theta} \geq \theta \\ c_2(\theta - \tilde{\theta}) & \text{if} \quad \tilde{\theta} < \theta \end{cases}$$

    then the Bayes estimator is the *posterior quantile* of order $c_2/(c_1 + c_2)$, so that $\Pr[\theta < \theta^*] = c_2/(c_1 + c_2)$.

    In particular, if $c_1 = c_2$, the Bayes estimator is the *posterior median*.

  - Any θ value may be optimal: *it all depends on the loss function*.

- *Intrinsic point estimation*

  - Given the statistical model $\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta}), \boldsymbol{x} \in \boldsymbol{\mathcal{X}}, \boldsymbol{\theta} \in \Theta\}$ the intrinsic discrepancy $\delta(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ between two parameter values $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ is the intrinsic discrepancy $\delta\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1), p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2)\}$ between the corresponding probability models.

    This is symmetric, non-negative (and zero iff $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$), invariant under reparametrization and invariant under bijections of $\boldsymbol{x}$.

  - The intrinsic estimator is the *reference* Bayes estimator which corresponds to the loss defined by the *intrinsic discrepancy*:

    - The expected loss with respect to the reference posterior distribution

$$d(\tilde{\boldsymbol{\theta}} \,|\, \boldsymbol{x}) = \int_{\Theta} \delta\{\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}\}\, \pi^*(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta}$$

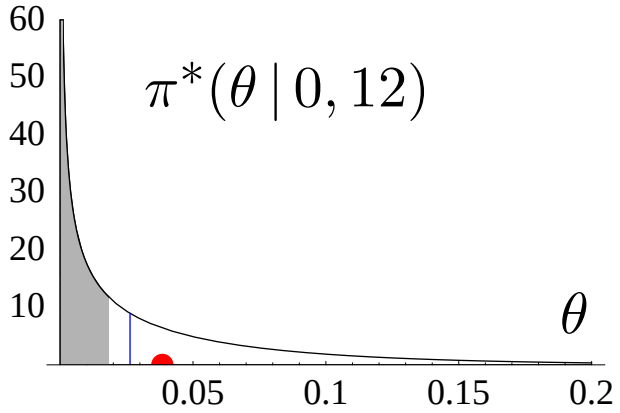
    is an objective measure, in information units, of the *expected* discrepancy between the model $p(\boldsymbol{x} \,|\, \tilde{\boldsymbol{\theta}})$ and the true (unknown) model $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$.

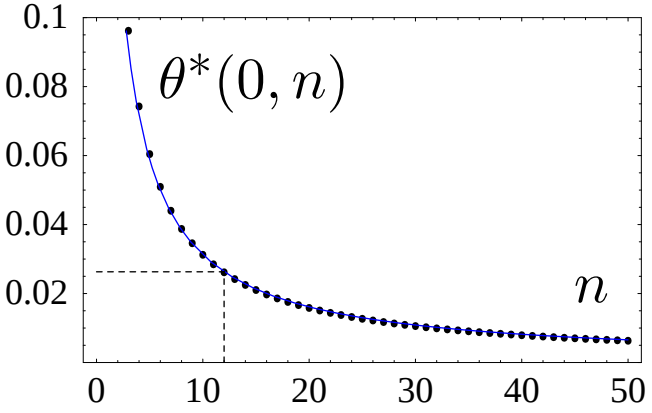
    - The *intrinsic estimator* $\boldsymbol{\theta}^* = \boldsymbol{\theta}^*(\boldsymbol{x})$ is the value which minimizes such expected discrepancy,

$$\boldsymbol{\theta}^* = \arg \inf_{\tilde{\boldsymbol{\theta}} \in \Theta} d(\tilde{\boldsymbol{\theta}} \,|\, \boldsymbol{x}).$$

- *Example: Intrinsic estimation of the Binomial parameter*

  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, random from $p(x \,|\, \theta) = \theta^x (1-\theta)^{1-x}$, $r = \Sigma x_j$. Intrinsic discrepancy $\delta(\tilde{\theta}, \theta) = n \, \min\{k(\tilde{\theta} \,|\, \theta), k(\theta \,|\, \tilde{\theta})\}$,
    $k(\theta_1 \,|\, \theta_2) = \theta_2 \log \frac{\theta_2}{\theta_1} + (1-\theta_2) \log \frac{1-\theta_2}{1-\theta_1}$, $\quad \pi^*(\theta) = \text{Be}(\theta \,|\, \frac{1}{2}, \frac{1}{2})$
    $\pi^*(\theta \,|\, r, n) = \text{Be}(\theta \,|\, r + \frac{1}{2}, n - r + \frac{1}{2})$.
  - Expected reference discrepancy
    $d(\tilde{\theta}, r, n) = \int_0^1 \delta(\tilde{\theta}, \theta) \, \pi^*(\theta \,|\, r, n) \, \text{d}\theta$
  - Intrinsic estimator
    $\theta^*(r, n) = \arg\min_{0<\tilde{\theta}<1} d(\tilde{\theta}, r, n)$
    From invariance, for any bijection $\phi = \phi(\theta)$, $\phi^* = \phi(\theta^*)$.
  - Analytic approximation
    $\theta^*(r, n) \approx \frac{r+1/3}{n+2/3}, \quad n > 2$
  - $n = 12, r = 0, \quad \theta^*(0, 12) = 0.026$
    $\text{Me}[\theta \,|\, \boldsymbol{x}] = 0.018$, $\text{E}[\theta \,|\, \boldsymbol{x}] = 0.038$





- *Intrinsic region (interval) estimation*

  - The *intrinsic q-credible region* $R^*(q) \subset \Theta$ is that q-credible reference region which corresponds to minimum expected intrinsic loss:

    (i) $\int_{R^*(q)} \pi^*(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta} = q$

    (ii) $\forall \boldsymbol{\theta}_i \in R^*(q),\ \forall \boldsymbol{\theta}_j \notin R^*(q), \qquad d(\boldsymbol{\theta}_i \,|\, \boldsymbol{x}) < d(\boldsymbol{\theta}_j \,|\, \boldsymbol{x})$

  - Binomial examples: $d(\boldsymbol{\theta}_i \,|\, \boldsymbol{x}) = d(\theta_i \,|\, r, n)$

$r = 0, n = 12,$

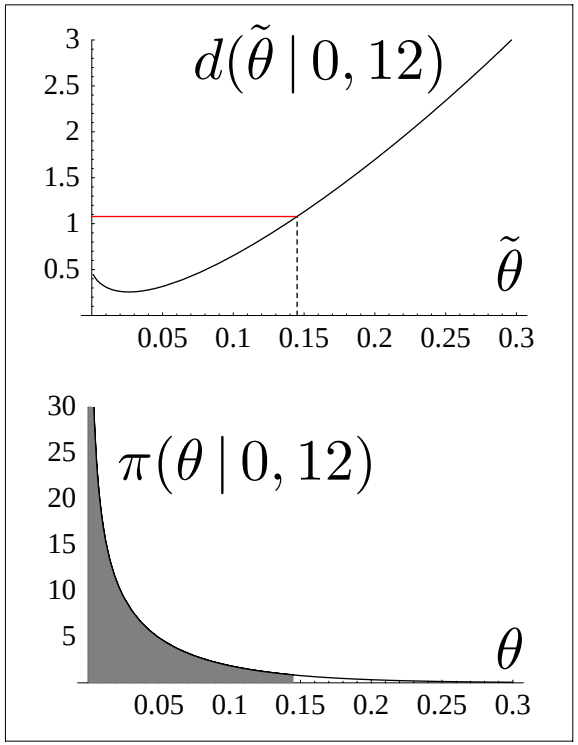
$\theta^* = 0.0263;$

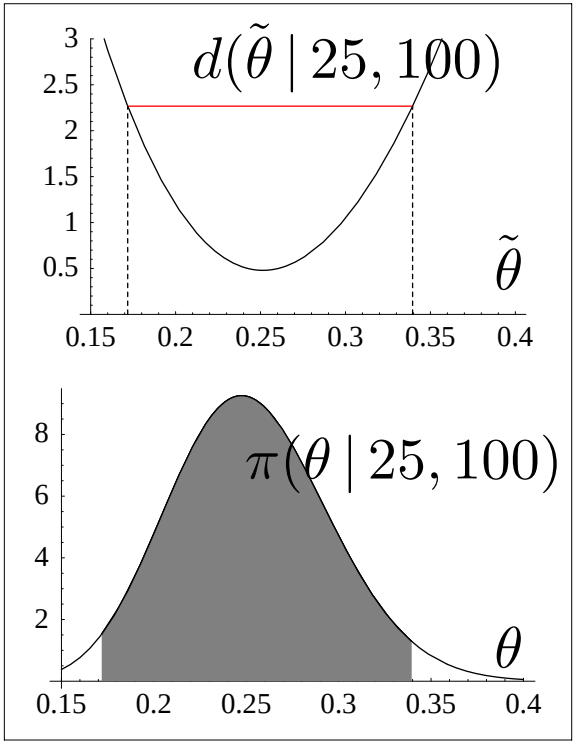
$R^*_{0.95} = [0, 0.145];$

$r = 25, n = 100,$

$\theta^* = 0.2514;$

$R^*_{0.95} = [0.172, 0.340];$





3.3 Hypothesis Testing

- *Precise hypothesis testing as a decision problem*

  - The posterior $\pi(\boldsymbol{\theta} \mid D)$ conveys intuitive information on the values of $\boldsymbol{\theta}$ which are *compatible* with the observed data $\boldsymbol{x}$: those with a *relatively high probability density*.
  - Often a particular value $\boldsymbol{\theta}_0$ is suggested for special consideration:
    - Because $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ would greatly simplify the model
    - Because there are context specific arguments suggesting that $\boldsymbol{\theta} = \boldsymbol{\theta}_0$

    More generally, one may analyze the *restriction* of parameter space Θ to a subset Θ_0 which may contain more than one value.
  - Formally, testing the hypothesis $H_0 \equiv \{\boldsymbol{\theta} = \boldsymbol{\theta}_0\}$ is a *decision problem* with just two possible actions:
    - a_0: to *accept* H_0 and work with $p(\boldsymbol{x} \mid \boldsymbol{\theta}_0)$.
    - a_1: to *reject* H_0 and keep the general model $p(\boldsymbol{x} \mid \boldsymbol{\theta})$.
  - To proceed, a *loss* function $\ell(a_i, \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$, describing the possible consequences of both actions, must be specified.

- *Structure of the loss function*

  - Given data $\boldsymbol{x}$, optimal action is to reject H_0 (action a_1) *iff* the expected posterior loss of accepting, $\int_\Theta \ell(a_0, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, d\boldsymbol{\theta}$, is *larger* than the expected posterior loss of rejecting, $\int_\Theta \ell(a_1, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, d\boldsymbol{\theta}$, *i.e.*, iff

    $$\int_\Theta [\ell(a_0, \boldsymbol{\theta}) - \ell(a_1, \boldsymbol{\theta})]\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, d\boldsymbol{\theta} = \int_\Theta \Delta\ell(\boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, d\boldsymbol{\theta} > 0.$$

    Therefore, only the loss difference $\Delta\ell(\boldsymbol{\theta}) = \ell(a_0, \boldsymbol{\theta}) - \ell(a_1, \boldsymbol{\theta})$, which measures the *advantage* of rejecting H_0 as a function of $\boldsymbol{\theta}$, has to be specified: The hypothesis should be rejected whenever the *expected* advantage of rejecting is positive.

  - The advantage $\Delta\ell(\boldsymbol{\theta})$ of rejecting H_0 as a function of $\boldsymbol{\theta}$ should be of the form $\Delta\ell(\boldsymbol{\theta}) = l(\boldsymbol{\theta}_0, \boldsymbol{\theta}) - l^*$, for some $l^* > 0$, where

    - $l(\boldsymbol{\theta}_0, \boldsymbol{\theta})$ measures the *discrepancy* between $p(\boldsymbol{x} \mid \boldsymbol{\theta}_0)$ and $p(\boldsymbol{x} \mid \boldsymbol{\theta})$,
    - l^* is a positive *utility constant* which measures the advantage working with the simpler model when it is true.

  - The Bayes criterion will then be: *Reject* H_0 if (and only if)

    $\int_\Theta l(\boldsymbol{\theta}_0, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \mid \boldsymbol{x})\, d\boldsymbol{\theta} > l^*$, that is if (and only if)
    the *expected discrepancy* between $p(\boldsymbol{x} \mid \boldsymbol{\theta}_0)$ and $p(\boldsymbol{x} \mid \boldsymbol{\theta})$ is *too large*.

- *Bayesian Reference Criterion*

  - An good choice for the function $l(\boldsymbol{\theta}_0, \boldsymbol{\theta})$ is the *intrinsic discrepancy*,
    $\delta(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \min\{k(\boldsymbol{\theta}_0 \,|\, \boldsymbol{\theta}),\, k(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0)\}$,
    where $k(\boldsymbol{\theta}_0 \,|\, \boldsymbol{\theta}) = \int_{\boldsymbol{\mathcal{X}}} p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) \log\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta})/p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)\} d\boldsymbol{x}$.
    If $\boldsymbol{x} = \{x_1, \ldots, x_n\} \in \mathcal{X}^n$ is a random sample from $p(x \,|\, \boldsymbol{\theta})$, then
    $k(\boldsymbol{\theta}_0 \,|\, \boldsymbol{\theta}) = n \int_{\mathcal{X}} p(x \,|\, \boldsymbol{\theta}) \log \frac{p(x \,|\, \boldsymbol{\theta})}{p(x \,|\, \boldsymbol{\theta}_0)}\, dx$.
  - For objective results, exclusively based on model assumptions and data, the *reference* posterior distribution $\pi^*(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ should be used.
  - Hence, *reject if (and only if) the expected reference posterior intrinsic discrepancy $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ is too large*,
    $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x}) = \int_{\Theta} \delta(\boldsymbol{\theta}_0, \boldsymbol{\theta})\, \pi^*(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, \mathrm{d}\boldsymbol{\theta} > d^*$, for some $d^* > 0$.
    This is the *Bayesian reference criterion (BRC)*.
  - The *reference test statistic* $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ is nonnegative, it is invariant both under reparametrization and under sufficient transformation of the data, and it is a measure, in natural information units (nits) of the expected discrepancy between $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$ and the true model.

- *Calibration of the BRC*

  - The reference test statistic $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ is the posterior expected value of the intrinsic discrepancy between $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$ and $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$.

    • A reference test statistic value $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x}) \approx 1$ suggests that data are clearly compatible with the Hypotheis that $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.

    • A test statistic value $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})\ \log(10) = 2.303$ nits implies that, given data $\boldsymbol{x}$, the *average* value of the likelihood ratio *against* the hypothesis, $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})/p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$, is expected to be about 10: *mild evidence* against $\boldsymbol{\theta}_0$.

    • Similarly, $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x}) \approx \log(100) = 4.605$ (expected likelihood ratio against $\boldsymbol{\theta}_0$ about 100), indicates *strong evidence* against $\boldsymbol{\theta}_0$, and $\log(1000) = 6.908$, *conclusive evidence* against $\boldsymbol{\theta}_0$.

  - Strong connections between BRC and intrinsic estimation:

    • The *intrinsic estimator* is the value of $\boldsymbol{\theta}$ with minimizes the reference test statistic: $\boldsymbol{\theta}^* = \arg\inf_{\boldsymbol{\theta} \in \Theta} d(\boldsymbol{\theta} \,|\, \boldsymbol{x})$.

    • The regions defined by $\{\boldsymbol{\theta};\ d(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \leq d^*\}$ are invariant *reference posterior $q(d^*)$-credible regions* for $\boldsymbol{\theta}$. For regular problems and large samples, $q(\log(10)) \approx 0.95$ and $q(\log(100)) \approx 0.995$.

- *A canonical example: Testing a value for the Normal mean*

  - In the simplest case where the variance σ^2 is known,

    $$\delta(\mu_0, \mu) = n(\mu - \mu_0)^2/(2\sigma^2), \qquad \pi^*(\mu \,|\, \boldsymbol{x}) = \mathrm{N}(\mu \,|\, \overline{x}, \sigma/\sqrt{n}),$$

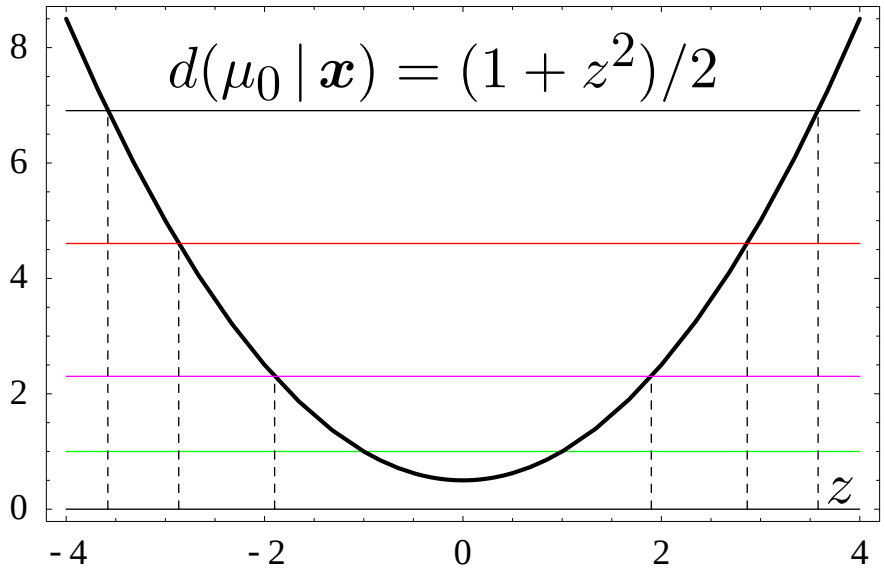
    $$d(\mu_0 \,|\, \boldsymbol{x}) = \tfrac{1}{2}(1 + z^2), \qquad z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

    Thus rejecting $\mu = \mu_0$ if $d(\mu_0 \,|\, \boldsymbol{x}) > d^*$ is equivalent to rejecting if $|z| > \sqrt{2d^* - 1}$ and, hence, to a conventional two-sided frequentist test with significance level $\alpha = 2(1 - \Phi(|z|))$.

d^*	$\lvert z\rvert$	α
$\log(10)$	1.8987	0.0576
$\log(100)$	2.8654	0.0042
$\log(1000)$	3.5799	0.0003

  - The expected value of $d(\mu_0 \,|\, \boldsymbol{x})$ if the hypothesis is true is

    $$\int_{-\infty}^{\infty} \tfrac{1}{2}(1 + z^2)\mathrm{N}(z \,|\, 0, 1)\, \mathrm{d}z = 1$$



- *Fisher's tasting tea lady*

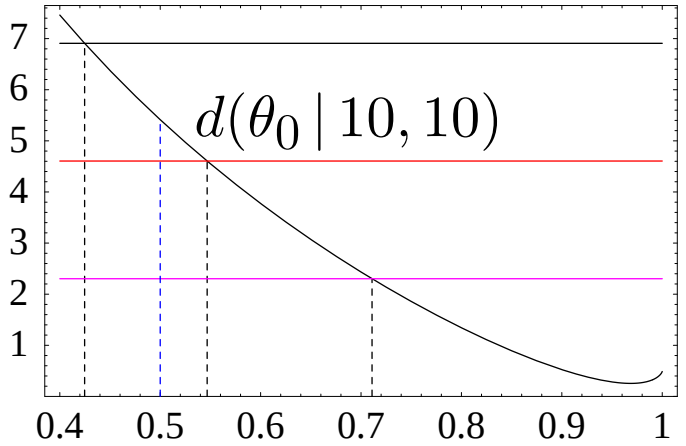
  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, random from $p(x \,|\, \theta) = \theta^x (1-\theta)^{1-x}$, $r = \Sigma x_j$. Intrinsic discrepancy $\delta(\theta_0, \theta) = n \, \min\{k(\theta_0 \,|\, \theta), k(\theta \,|\, \theta_0)\}$,
    $k(\theta_1 \,|\, \theta_2) = \theta_2 \log \frac{\theta_2}{\theta_1} + (1-\theta_2) \log \frac{1-\theta_2}{1-\theta_1}$, $\quad \pi^*(\theta \,|\, r, n) = \text{Be}(\theta \,|\, r + \frac{1}{2}, n - r + \frac{1}{2})$

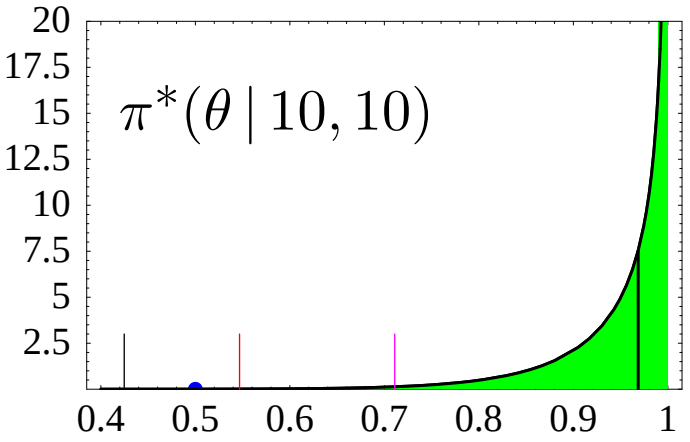
    Intrinsic test statistic
    $d(\theta_0 \,|\, r, n) = \int_0^1 \delta(\tilde{\theta}, \theta) \, \pi^*(\theta \,|\, r, n) \, \mathrm{d}\theta$

  - Fisher's example: $\boldsymbol{x} = \{10, 10\}$,
    Test $\theta_0 = 1/2$, $\theta^*(\boldsymbol{x}) = 0.9686$
    $d(\theta_0 \,|\, 10, 10) = 5.414 = \log[224]$

    Using $d^* = \log[100] = 4.61$,
    the value $\theta_0 = 1/2$ is rejected.
    $\Pr[\theta < 0.5 \,|\, \boldsymbol{x}] = 0.00016$

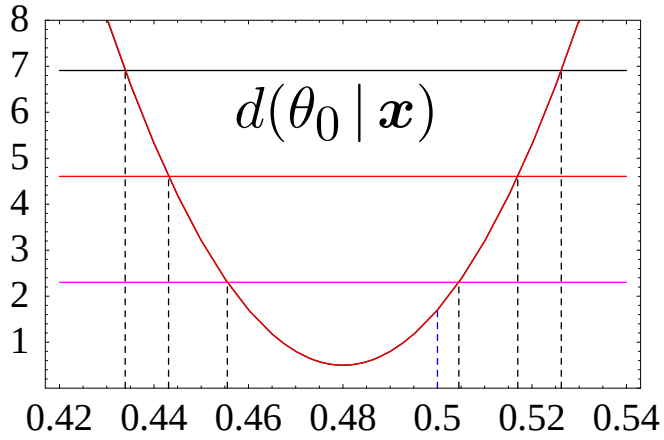
$d(\theta^* \,\|\, \boldsymbol{x})$	θ^*	$\Pr[\theta < \theta^* \,\|\, \boldsymbol{x}]$
$\log[10]$	0.711	0.00815
$\log[100]$	0.547	0.00043
$\log[1000]$	0.425	0.00003





- *Asymptotic approximation*

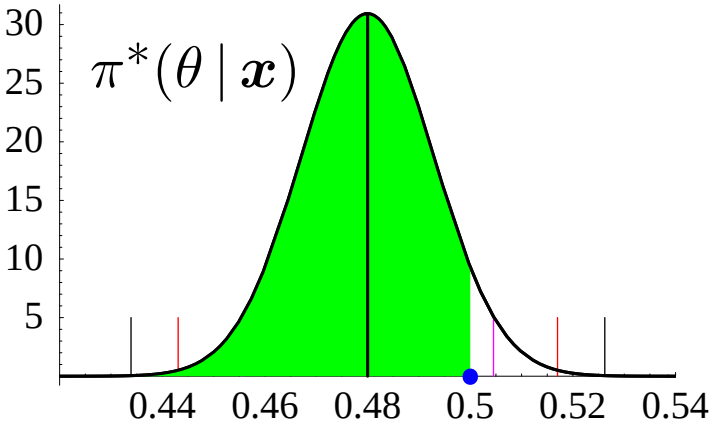
  - For large samples, the posterior approaches $\mathrm{N}(\theta \,|\, \hat{\theta}, (nF(\hat{\theta}))^{-1/2})$, where $F(\theta)$ is Fisher's function. Changing variables, the posterior distribution of $\phi = \phi(\theta) = \int F^{1/2}(\theta)\,\mathrm{d}\theta = 2 \arcsin \sqrt{\theta})$ is approximately normal $\mathrm{N}(\phi \,|\, \hat{\phi}, n^{-1/2})$. Since $d(\theta, \boldsymbol{x})$ is invariant, $d(\theta_0, \boldsymbol{x}) \approx \frac{1}{2}[1 + n\{\phi(\theta_0) - \phi(\hat{\theta})\}^2]$.



- *Testing for a majority ($\theta_0 = 1/2$)*

$$\boldsymbol{x} = \{720, 1500\}, \quad \theta^*(\boldsymbol{x}) = 0.4800$$

$d(\theta^* \,\|\, \boldsymbol{x})$	$R = (\theta_0^*, \theta_1^*)$	$\Pr[\theta \in R \,\|\, \boldsymbol{x}]$
log[10]	(0.456, 0.505)	0.9427
log[100]	(0.443, 0.517)	0.9959
log[1000]	(0.434, 0.526)	0.9997



Very mild evidence against $\theta = 0.5$:
$d(0.5 \,|\, 720, 1500) = 1.67$
$\Pr(\theta < 0.5 \,|\, 720, 1500) = 0.9393$

Basic References

Many available on line at **www.uv.es/bernardo**

- *Introductions*

Bernardo, J. M. and Ramón, J. M. (1998).
An introduction to Bayesian reference analysis. *The Statistician* **47**, 1–35.

Bernardo, J. M. (2003). Bayesian Statistics.
Encyclopedia of Life Support Systems (EOLSS): Probability and Statistics, (R. Viertl, ed). Oxford, UK: UNESCO.

Bernardo, J. M. (2005). Reference Analysis.
Handbook of Statistics **25** (D. K. Dey and C. R. Rao eds.)
Amsterdam: Elsevier, 17–90

- *Textbooks*

Gelman, A., Carlin, J. B., Stern, H. and Rubin, D. B. (2003).
Bayesian Data Analysis (2nd ed.) New York: CRC Press.

Bernardo, J. M. and Smith, A. F. M. (1994).
Bayesian Theory. Chichester: Wiley.
2nd ed. to appear in June 2006

- *Research papers on reference analysis (cronological order)*

Bernardo, J. M. (1979). Reference posterior distributions for Bayesian inference. *J. Roy. Statist. Soc. B* **41**, 113—147, (with discussion).
Reprinted in *Bayesian Inference* (N. G. Polson and G. C. Tiao, eds.), Brookfield, VT: Edward Elgar, (1995), 229—263.

Berger, J. O. and Bernardo, J. M. (1992). On the development of reference priors. *Bayesian Statistics 4* (J. M. Bernardo, J. O. Berger, A. P. Dawid and A. F. M. Smith, eds.) Oxford: University Press, 35–60 (with discussion).

Bernardo, J. M. (1997) . Noninformative priors do not exist. *J. Statist. Planning and Inference* **65**, 159—189, (with discussion).

Bernardo, J. M. and Rueda, R. (2002). Bayesian hypothesis testing: A reference approach. *Internat. Statist. Rev.* **70**, 351–372.

Bernardo, J. M. and Juárez, M. (2003). Intrinsic estimation. *Bayesian Statistics 7* (J. M. Bernardo, M. J. Bayarri, J. O. Berger, A. P. Dawid, D. Heckerman, A. F. M. Smith and M. West, eds.) Oxford: University Press, 465–476.

Bernardo, J. M. (2005). Intrinsic credible regions: An objective Bayesian approach to interval estimation. *Test* **14**, 317-384 (invited paper, with discussion).

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- José-Miguel Bernardo contact data

  <jose.m.bernardo@uv.es>

  www.uv.es/bernardo