

A value for cooperative games with a coalition structure*

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Abstract

A value for games with a coalition structure is introduced, where the rules guiding the cooperation among the members of the same coalition are different from the interaction rules among coalitions. In particular, players inside a coalition exhibit a greater degree of solidarity than they are willing to use with players outside their coalition. The Shapley value [Shapley, 1953] is therefore used to compute the aggregate payoffs of the coalitions, and the Solidarity value [Nowak and Radzik, 1994] to obtain the payoffs of the players inside each coalition.

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1 Introduction

There are many settings in cooperative games where players naturally organize themselves into groups for the purpose of negotiating payoffs. This action can be modeled by including

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a *coalition structure* into the game, which consists of an exogenous partition of players into a set of groups or unions. These unions sometimes arise for natural reasons. One example is the existence of a large number of agents. Players are joined into groups of similar interests and characteristics in the case of trade unions, political parties, cartels, lobbies, etc. Another typical reason is due to geographical location as in the case of cities, states and countries.

When groups are formed, the agents interact at two levels: first, bargaining takes place among the unions, and then, bargaining occurs inside each union in order to share what the union has obtained at the first interaction level among their members. Owen (1977) was the first to follow this approach. In his coalitional value, unions play a quotient game among themselves. Each union receives a payoff that is shared among its players in an internal game. The payoffs at each level are given by the Shapley value (Shapley, 1953b). Thus, the same properties (axioms) that govern the interaction between groups also operate among the players of each group. Nevertheless, it could be questioned whether a coalitional value following the same behavior at both levels is a legitimate point of view. A greater degree of solidarity existing between members of the same group than in the interaction between players of different groups seems natural.

For example, take the principle of paying players according to their productivity. This is one of the principles behind the Shapley value. It can be expressed formally by the *marginality* axiom (see Young, 1985), i.e. if the marginal contributions of a player in two games are the same then his value should be the same. Alternatively, it can be expressed by the *null player* axiom, that is, if all the marginal contributions of a player in a game are zero, then the player should obtain zero. What happens if this productivity principle is imposed when sharing rewards at both levels: between unions and inside the unions as is the case with the Owen value?

Let the game¹ be defined by the set of players $N = \{1, 2, 3, 4\}$, and the coalition structure $B = \{\{1\}, \{2, 3\}, \{4\}\}$, that is, players 1 and 4 are isolated and 2 and 3 form a union, and u_T , with $T = \{1, 2\}$, is a unanimity game defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise} \end{cases}$$

Players 3 and 4 are null players in u_T , but player 3 is in the same union as player 2, that is

¹This is only an informal discussion. All the precise definitions of the following concepts will appear in next Sections.

not a null player. In the quotient game, the unions $\{1\}$ and $\{2, 3\}$ are symmetric players, i.e. both contribute the same, so they obtain $1/2$, and union $\{4\}$ is a null player in the quotient game and then obtains zero. Inside the union $\{2, 3\}$, player 3 is a null player in u_T , hence he must obtain zero. Therefore, the payoffs associated to the Owen value are

$$Ow_1 = \frac{1}{2}, \quad Ow_2 = \frac{1}{2}, \quad Ow_3 = 0, \quad Ow_4 = 0.$$

Here, there is no difference for player 3 between belonging to the union $\{2, 3\}$ or being isolated.

Our purpose is to consider coalitional values where player 3 obtains a *positive* gain from its membership of union $\{2, 3\}$. One proposal was presented in Kamijo (2009) and is known as *the two-step Shapley value*. This value satisfies most of the properties that support the Shapley value in the setting of games without coalition structure. Therefore, it can be considered as an alternative value extension of the Shapley value to the coalition structure setting. In Calvo and Gutiérrez (2010), this value has been characterized by means of a *population solidarity* principle, which can be expressed informally as follows: if we add or draw players out of the union, then all members of the union change their value equally. In our example, the two-step Shapley value yields

$$K_1 = \frac{1}{2}, \quad K_2 = \frac{1}{4}, \quad K_3 = \frac{1}{4}, \quad K_4 = 0.$$

Player 3 now obtains the same as 2. However, this seems rather unfair from the productivity point of view, as 3 is a null player that does not contribute to the rewards of the union.

We have then two different principles for rewarding players: paying according to the productivity of players (marginal contributions), which is a good incentive rule; and redistributing positive rewards to all members of a union, which is a natural cohesion group property. Can we make both principles compatible? In this paper, we propose a new coalitional value inspired in the Solidarity value introduced by Nowak and Radzik (1994). This value takes into account the productivity principle as well, as the players' marginal contributions are used in the calculation. However, it also exhibits a redistribution effect, as it not only takes into account his own marginal contribution to the coalition that player belongs to, but also the marginal contributions of the remaining players, in such a way that the own marginal contribution is replaced by the average of the marginal contributions of all players in the coalition. According to this approach, the value is obtained in

two steps. First, unions play a quotient game among themselves and each union receives a payoff given by the Shapley value; and second, the outcome obtained by the union is shared among its members by paying the solidarity value in the internal game. In this way, the payoffs obtained in our unanimity example are

$$\xi_1 = \frac{1}{2}, \xi_2 = \frac{3}{8}, \xi_3 = \frac{1}{8}, \xi_4 = 0,$$

which are more in keeping with the approach of joining productivity and cohesion principles simultaneously to reward the players inside each union.

A paradigmatic example is the case of simple games, which has been important in applications to political sciences. Our value seems an interesting alternative to the Owen value for the computation of the power that political parties have in parliaments under different coalition configurations. In particular, this coalitional value gives support to empirical evidence that sometimes government coalitions are not minimal winning coalitions (as might be expected according to Riker's (1962) *size principle* or the *minimal winning coalition principle*), but rather include more parties than would be necessary to have a majority of the votes in parliaments of proportional representation (see Strom, 1990; and Schofield, 1993).

The rest of the paper is organized as follows. Section 2 is devoted to definitions and notation. We also show how to compute the solidarity value by the random order approach. Section 3 introduces the new coalitional value. We provide the axiomatic characterization of this value in Section 4. The independence of this axiom system is proved in Section 5 and a new characterization of the Owen value is also put forward as a by-product.

2 Definitions

2.1 Shapley value

Let $U = \{1, 2, \dots\}$ be the (infinite) set of potential players. A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subseteq U$ is a nonempty and finite set and $v : 2^N \rightarrow \mathbb{R}$ is a *characteristic function*, defined on the power set of N , satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and every nonempty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S , and it is interpreted

as the total payoff that the coalition S , if it forms, can obtain for its members. Let \mathcal{G}^N denote the set of all cooperative TU-games with player set N .

For all $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v) . For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$.

A *value* is a function γ which assigns to every TU-game (N, v) and every player $i \in N$, a real number $\gamma_i(N, v)$, which represents an assessment made by i of his gains from participating in the game.

Let (N, v) be a game. For all $S \subseteq N$ and all $i \in S$, define

$$\partial^i(v, S) := v(S) - v(S \setminus i).$$

We call $\partial^i(v, S)$ the *marginal contribution of player i to coalition S in the TU-game (N, v)* . The *Shapley value* (Shapley, 1953b) of the game (N, v) is the payoff vector $Sh(N, v) \in \mathbb{R}^N$ defined by

$$Sh_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{(n-s)!(s-1)!}{n!} \partial^i(v, S), \quad \text{for all } i \in N.$$

where $s = |S|$ and $n = |N|$.

Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *null player* in (N, v) if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$. For any two games (N, v) and (N, w) , the game $(N, v + w)$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

Consider the following properties of a value γ in \mathcal{G}^N :

Efficiency: For all (N, v) , $\sum_{i \in N} \gamma_i(N, v) = v(N)$.

Additivity: For all (N, v) and (N', w) with $N = N'$, $\gamma(N, v + w) = \gamma(N, v) + \gamma(N, w)$.

Symmetry: For all (N, v) and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v) , then $\gamma_i(N, v) = \gamma_j(N, v)$.

Null player axiom: For all (N, v) and all $i \in N$, if i is a null player in (N, v) , then $\gamma_i(N, v) = 0$.

The following theorem is due to Shapley (1953b).

Theorem 1 (Shapley, 1953b) *A value γ on \mathcal{G}^N satisfies efficiency, additivity, symmetry and null player axiom if, and only if, γ is the Shapley value.*

Let $\Omega(N)$ be the set of all orders on N . Each $\omega \in \Omega(N)$ is a bijection from N to $\{1, 2, \dots, n\}$ and $\omega(i)$, $i \in N$, denotes the position of player i in the order ω . We denote by P_i^ω the set of all predecessors of i in ω , that is, $P_i^\omega := \{j \in N : \omega(j) < \omega(i)\}$. Let (N, v) be a game, for all $i \in N$ and all $\omega \in \Omega(N)$, the *marginal contribution* that player i receives in ω is defined by

$$m_i^\omega(N, v) := v(P_i^\omega \cup i) - v(P_i^\omega).$$

The Shapley value of the game (N, v) can also be computed by the formula

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\omega \in \Omega(N)} m_i^\omega(N, v), \quad \text{for all } i \in N. \quad (1)$$

For all $T \subseteq N$, the *unanimity game of the coalition T* , (N, u_T) , is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the family of games $\{(N, u_T)\}_{T \subseteq N}$ is a basis for \mathcal{G}^N .

A *weighted value* γ^w is a function that assigns to all (N, v) and all $w \in \mathbb{R}_{++}^N$ a vector $\gamma^w(N, v)$ in \mathbb{R}^N . For each $i \in N$, w_i is the *weight* of player i . We will say that a weighted value γ^w *extends* a value γ if $\gamma^w(N, v) = \gamma(N, v)$ for any weight vector w with $w_i = w_j$ for all $i, j \in N$. The most important weighted generalization of the Shapley value is the *weighted Shapley value* Sh^w (Shapley (1953a), Kalai and Samet (1987)). For all $w \in \mathbb{R}_{++}^N$, the *weighted Shapley value* Sh^w is the linear map $Sh^w : \mathcal{G}^N \rightarrow \mathbb{R}^N$, which is defined for each unanimity game (N, u_T) as follows

$$Sh_i^w(N, u_T) = \begin{cases} \frac{w_i}{\sum_{j \in T} w_j} & \text{for all } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

The weighted Shapley value Sh^w satisfies efficiency, additivity and the null player axiom, but not symmetry.

2.2 Solidarity value

Let (N, v) be a game. For all $S \subseteq N$, define

$$\partial^{av}(v, S) := \frac{1}{s} \sum_{i \in S} \partial^i(v, S).$$

$\partial^{av}(v, S)$ is the *average of the marginal contributions* of players within coalition S in the game (N, v) . The *solidarity value* of the game (N, v) is the payoff vector $Sl(N, v) \in \mathbb{R}^N$ defined by

$$Sl_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{(n-s)!(s-1)!}{n!} \partial^{av}(v, S), \quad \text{for all } i \in N.$$

This value was introduced by Nowak and Radzik (1994) and it satisfies some solidarity principle, since null players can obtain positive payoffs (See example 1.1 in Nowak and Radzik (1994)). They introduced a variation of the null player axiom in order to characterize Sl on \mathcal{G}^N . Player $i \in N$ is an *A-null player* in (N, v) if $\partial^{av}(v, S) = 0$ for all coalition $S \subseteq N$ containing i . There is clearly no relation between the null player and the A-null player concepts. The solidarity value satisfies the following axiom:

A-Null player axiom: For all (N, v) and all $i \in N$, if i is an A-null player, then $\gamma_i(N, v) = 0$.

The following theorem is due to Nowak and Radzik (1994).

Theorem 2 (Nowak and Radzik, 1994) *A value γ on \mathcal{G}^N satisfies efficiency, additivity, symmetry and A-null player axiom if, and only if, γ is the solidarity value.*

We can define a weighted generalization of the solidarity value in the same way as the weighted Shapley value. Nowak and Radzik (1994) defined a new basis for \mathcal{G}^N , denoted by $\{(N, b_T)\}_{T \subseteq N}$. For all $T \subseteq N$, (N, b_T) is defined by

$$b_T(S) = \begin{cases} \left(\frac{|S|}{|T|}\right)^{-1} & \text{if } S \supseteq T \\ 0 & \text{otherwise} \end{cases}$$

They proved that every player $i \in N \setminus T$ is A-null in the game (N, b_T) .

For all $w \in \mathbb{R}_{++}^N$, the *weighted solidarity value* Sl^w is the linear map $Sl^w : \mathcal{G}^N \longrightarrow \mathbb{R}^N$, defined for all (N, b_T) as follows

$$Sl_i^w(N, b_T) = \begin{cases} \frac{w_i}{\sum_{j \in T} w_j} b_T(N) & \text{for all } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

The weighted solidarity value Sl^w satisfies efficiency, additivity and the A-null player axiom, but not symmetry.

Remark 1 *The solidarity value can be obtained recursively (see Calvo, 2008) by*

$$Sl_i(S, v) = \frac{1}{s} \partial^{av}(v, S) + \sum_{j \in S \setminus i} \frac{1}{s} Sl_i(S \setminus j, v), \quad \text{for all } S \subseteq N \text{ and all } i \in S,$$

starting with

$$Sl_i(\{i\}, v) = v(i), \quad \text{for all } i \in N.$$

The solidarity value was thus introduced by Sprumont (1990; Section 5) in order to show that, in the class of increasing average marginal contributions (IAMC) games, i.e. $\partial^{av}(v, S) \leq \partial^{av}(v, T)$, whenever $S \subseteq T$, it is possible to find a population monotonic allocation scheme (PMAS), i.e. a payoff configuration $(x^S)_{S \subseteq N} \in (\mathbb{R}^S)_{S \subseteq N}$ such that

(i) For all $S \subseteq N$, $\sum_{i \in S} x_i^S = v(S)$,

(ii) For all $S, T \subseteq N$ and all $i \in S$, $S \subseteq T \Rightarrow x_i^S \leq x_i^T$.

Sprumont (1990, Proposition 4) shows that the solidarity value is a PMAS in the class of IAMC games.

2.3 Random order approach

In the same way that the Shapley value can be computed by means of orders, we also provide a formula to calculate the solidarity value by orders. Let (N, v) be a game, for all $\omega \in \Omega(N)$ and all $i \in N$, we define

$$\zeta_i^\omega(N, v) := \sum_{\substack{j \in N \\ \omega(j) \geq \omega(i)}} \frac{1}{|P_j^\omega \cup j|} \partial^j(v, P_j^\omega \cup j), \quad \text{for all } i \in N.$$

For example, if $N = \{a, b, c\}$ and ω is such that $\omega(a) = 1$, $\omega(b) = 2$ and $\omega(c) = 3$, this means that the sequence of coalition formation $S_1 \subseteq S_2 \subseteq S_3$ in ω is: at step $r = 1$, $S_1 = \{a\}$, at step $r = 2$, $S_2 = \{a, b\}$, and at step $r = 3$, $S_3 = \{a, b, c\}$. Then the payoffs $\zeta^\omega(N, v)$ are

$$\begin{aligned} \zeta_a^\omega(N, v) &= v(\{a\}) + \frac{v(\{a, b\}) - v(\{a\})}{2} + \frac{v(\{a, b, c\}) - v(\{a, b\})}{3}, \\ \zeta_b^\omega(N, v) &= \frac{v(\{a, b\}) - v(\{a\})}{2} + \frac{v(\{a, b, c\}) - v(\{a, b\})}{3}, \\ \zeta_c^\omega(N, v) &= \frac{v(\{a, b, c\}) - v(\{a, b\})}{3}. \end{aligned}$$

Each player here shares his own marginal contribution with the players in front of him in the order ω .

We have the following theorem.

Theorem 3 For all $(N, v) \in \mathcal{G}^N$, the solidarity value is obtained by

$$Sl_i(N, v) = \frac{1}{n!} \sum_{\omega \in \Omega(N)} \zeta_i^\omega(N, v), \quad \text{for all } i \in N.$$

Proof. Let (N, v) be a game, for all $j \in N$ and all $S \subseteq N$ such that $j \in S$, define

$$\Omega(S, j) := \{\omega \in \Omega(N) : P_j^\omega \cup j = S\}.$$

That is, $\Omega(S, j)$ is the set of all orders for which player j is the last player which completes coalition S . Note that $|\Omega(S, j)| = (s-1)!(n-s)!$. Then, for all $\omega \in \Omega(N)$ and all $i \in N$, player i obtains the sum of $\frac{1}{s} \partial^j(v, S)$ for all j such that $\omega(j) \geq \omega(i)$, with $S = P_j^\omega \cup j$. Therefore, for all $i \in N$,

$$\begin{aligned} \frac{1}{n!} \sum_{\omega \in \Omega(N)} \zeta_i^\omega(N, v) &= \frac{1}{n!} \sum_{\omega \in \Omega(N)} \sum_{\substack{j \in N \\ \omega(j) \geq \omega(i)}} \frac{1}{|P_j^\omega \cup j|} \partial^j(v, P_j^\omega \cup j) = \\ \frac{1}{n!} \sum_{\substack{S \subseteq N \\ S \ni i}} \sum_{j \in S} \sum_{\omega \in \Omega(S, j)} \frac{1}{s} \partial^j(v, S) &= \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(n-s)!(s-1)!}{n!} \partial^{av}(v, S) = Sl_i(N, v). \end{aligned}$$

■

Example 1 Consider the game with player set $N = \{a, b, c\}$ and characteristic function defined by $v(a) = v(b) = v(c) = 0$, $v(a, b) = v(a, c) = 1$, $v(b, c) = 0$, and $v(a, b, c) = 2$. The following table sets out the marginal contributions ζ_i^ω for every order:

orders	players		
ω	ζ_a^ω	ζ_b^ω	ζ_c^ω
(a, b, c)	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{6}$
(a, c, b)	$\frac{5}{6}$	$\frac{2}{6}$	$\frac{5}{6}$
(b, a, c)	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{6}$
(b, c, a)	$\frac{4}{6}$	$\frac{4}{6}$	$\frac{4}{6}$
(c, a, b)	$\frac{5}{6}$	$\frac{2}{6}$	$\frac{5}{6}$
(c, b, a)	$\frac{4}{6}$	$\frac{4}{6}$	$\frac{4}{6}$

Therefore, the payoffs are

$$Sl_a(N, v) = \frac{14}{18}, \quad Sl_b(N, v) = Sl_c(N, v) = \frac{11}{18}.$$

Note that $\partial^{av}(v, \{a, b\}) = \partial^{av}(v, \{a, c\}) = 1$, $\partial^{av}(v, \{b, c\}) = 0$, and $\partial^{av}(v, \{a, b, c\}) = \frac{4}{3}$.

3 Coalitional solidarity value

3.1 The Owen and the two-step Shapley values

For all finite set $N \subseteq U$, a *coalition structure* over N is a partition of N , i.e., $B = \{B_1, B_2, \dots, B_m\}$ is a coalition structure if it satisfies that $\bigcup_{1 \leq k \leq m} B_k = N$ and $B_k \cap B_l = \emptyset$ when $k \neq l$. We also assume $B_k \neq \emptyset$ for all k . The sets $B_k \in B$ are called “unions” or “blocks”. There are two trivial coalition structures: The first, which we denote by B^N , where only the grand coalition forms, that is, $B^N = \{N\}$; and the second is the coalition structure where each union is a singleton and it is denoted by B^n , that is, $B^n = \{\{1\}, \{2\}, \dots, \{n\}\}$. Denote by $\mathcal{B}(N)$ the set of all coalition structures over N . A game (N, v) with coalition structure $B \in \mathcal{B}(N)$ is denoted by (B, v) . Let \mathcal{CSG}^N denote the family of all TU-games with coalition structure with player set N , and let \mathcal{CSG} denote the set of all TU-games with coalition structure.

For all game $(B, v) \in \mathcal{CSG}^N$, with $B = \{B_1, B_2, \dots, B_m\}$, the *quotient game* is the TU-game $(M, v_B) \in \mathcal{G}^M$ where $M = \{1, 2, \dots, m\}$ and $v_B(T) := v\left(\bigcup_{i \in T} B_i\right)$ for all $T \subseteq M$. That is, (M, v_B) is the game induced by (B, v) by considering the coalitions of B as players. Notice that for the trivial coalition structure B^n we have $(M, v_{B^n}) \equiv (N, v)$. For all $\{k, l\} \subseteq M$, we say that B_k and B_l are *symmetric* in (B, v) if k and l are symmetric in the game (M, v_B) . For all $k \in M$, we say that B_k is a *null coalition* if k is a null player in the quotient game (M, v_B) .

Let $B \in \mathcal{B}(N)$. For all $k \in M$ and all $S \subseteq B_k$, denote by $B|_S$ the new coalition structure defined on $(\cup_{j \neq k} B_j) \cup S$, which appears when the complementary of S in B_k leaves the game. That is,

$$B|_S = \{B_1, \dots, B_{k-1}, S, B_{k+1}, \dots, B_m\}.$$

A *coalitional value* is a function Φ that assigns a vector in \mathbb{R}^N to each game with coalition structure $(B, v) \in \mathcal{CSG}^N$. One of the most important coalitional values is the *Owen value* (Owen, 1977). Let $(B, v) \in \mathcal{CSG}^N$ and $k \in M$, for all $S \subseteq B_k$, denote $S' = B_k \setminus S$. Owen (1977) defined a game $(M, v_{B|_S})$ that describes what would happen in the quotient game if union B_k were replaced by S , i.e.,

$$v_{B|_S}(T) = v(\cup_{j \in T} B_j \setminus S') \quad \text{for all } T \subseteq M.$$

Next, he defined an *internal game* (B_k, v_k) by setting $v_k(S) = Sh_k(M, v_{B|_S})$ for all $S \subseteq B_k$. Thus, $v_k(S)$ is the payoff to S in $v_{B|_S}$. The *Owen value* of the game (B, v) is the payoff vector $Ow(B, v) \in \mathbb{R}^N$ defined by

$$Ow_i(B, v) := Sh_i(B_k, v_k), \quad \text{for all } k \in M \text{ and all } i \in B_k. \quad (2)$$

Thus, first union k plays the quotient game (M, v_B) among the unions, and the payoff obtained is shared among its members by playing the internal game (B_k, v_k) . At both levels of bargaining, the payoffs are obtained using the Shapley value Sh . In that sense, we can denote the Owen value as $Ow \equiv \Pi^{(Sh, Sh)}$.

The Owen value satisfies the *quotient game property*:

$$\sum_{i \in B_k} Ow_i(B, v) = Sh_k(M, v_B), \quad \text{for all } k \in M.$$

Note that for the trivial coalition structures B^n and B^N , $Ow(B^N, v) = Ow(B^n, v) = Sh(N, v)$.

The Owen value can also be defined by orders. Let B be a coalition structure over N and $\omega \in \Omega(N)$. We say that ω is *admissible with respect to* B if for all $\{i, j, k\} \subseteq N$ and $l \in M$ such that $\{i, k\} \subseteq B_l$, if $\omega(i) < \omega(j) < \omega(k)$, then $j \in B_l$. In other words, ω is admissible with respect to B if players in the same component of B appear successively in ω . We denote by $\Omega(B, N)$ the set of all admissible orders (on N) with respect to B . The Owen value is given by the formula

$$Ow_i(B, v) = \frac{1}{|\Omega(B, N)|} \sum_{\omega \in \Omega(B, N)} m_i^\omega(N, v), \quad \text{for all } i \in N.$$

We now present the axioms that characterize Ow in \mathcal{CSG}^N . Let Φ be a coalitional value, define

$$\Phi(B, v)[S] = \sum_{i \in S} \Phi_i(B, v), \quad \text{for all } S \subseteq N.$$

(E) *Efficiency*: For all $(B, v) \in \mathcal{CSG}^N$, $\Phi(B, v)[N] = v(N)$.

(A) *Additivity*: For all $(B, v), (B', w) \in \mathcal{CSG}^N$ with $B = B'$, $\Phi(B, v + w) = \Phi(B, v) + \Phi(B, w)$.

(ISy) *Intracoalitional symmetry*: For all $(B, v) \in \mathcal{CSG}^N$, all $k \in M$ and all $\{i, j\} \subseteq B_k$, if i and j are symmetric players in (N, v) , then $\Phi_i(B, v) = \Phi_j(B, v)$.

(CSy) *Coalitional symmetry*: For all $(B, v) \in \mathcal{CSG}^N$ and all $\{k, l\} \subseteq M$, if B_k and B_l are symmetric in (B, v) , then $\Phi(B, v)[B_k] = \Phi(B, v)[B_l]$.

(NP) *Null player axiom*: For all $(B, v) \in \mathcal{CSG}^N$ and all $i \in N$, if i is a null player in (N, v) , then $\Phi_i(B, v) = 0$.

This theorem is due to Owen (1977).

Theorem 4 (Owen, 1977) *A value Φ on \mathcal{CSG}^N satisfies efficiency, additivity, intracoalitional symmetry, coalitional symmetry and null player axiom if, and only if, Φ is the Owen value.*

Let (B, u_T) be the game defined by the set of players $N = \{1, 2, 3, 4\}$, the coalition structure $B = \{\{1\}, \{2, 3\}, \{4\}\}$, and the unanimity game u_T , with $T = \{1, 2\}$. Players 3 and 4 are null players in (N, u_T) , but player 3 is in the same union as player 2, which is not a null player. In the quotient game, unions $\{1\}$ and $\{2, 3\}$ are symmetric players, i.e. both contribute the same, so they obtain $1/2$, and union $\{4\}$ is a null player in the quotient game and then obtain zero. Inside union $\{2, 3\}$, player 3 is a null player in (N, u_T) , hence he must obtain zero. Therefore the payoffs associated to the Owen value are

$$Ow_1(B, u_T) = \frac{1}{2}, \quad Ow_2(B, u_T) = \frac{1}{2}, \quad Ow_3(B, u_T) = 0, \quad Ow_4(B, u_T) = 0.$$

Here, for player 3, there is no difference between belonging to the union $\{2, 3\}$ or being isolated.

Kamijo (2009) defined a new coalitional value, named the *two-step Shapley value*. For all game $(B, v) \in \mathcal{CSG}^N$, the two-step Shapley value of (B, v) is given by the formula:

$$K_i(B, v) = Sh_i(B_k, v) + \frac{1}{|B_k|} [Sh_k(M, v_B) - v(B_k)], \quad \text{for all } k \in M \text{ and all } i \in B_k \quad (3)$$

In Calvo and Gutiérrez (2010) this value has been characterized by means of a *solidarity* principle, that can be expressed informally as follows: if the data of the game change due to factors external to the union, then all members of the union change their value equally. For all $l \in M$ and all $h \in B_l$, define $B_{-h} := (B_1, \dots, B_l \setminus h, \dots, B_m)$.

(PS) *Population solidarity within unions*: For all $(B, v) \in \mathcal{CSG}^N$, all $\{k, l\} \subseteq M$, with $k \neq l$, and all $\{i, j, h\} \subseteq N$, where $\{i, j\} \subseteq B_k$ and $h \in B_l$,

$$\Phi_i(B, v) - \Phi_i(B_{-h}, v) = \Phi_j(B, v) - \Phi_j(B_{-h}, v).$$

Define two additional axioms:

(NC) *Null coalition axiom*: For all $(B, v) \in \mathcal{CSG}^N$ and all $k \in M$, if B_k is a null coalition in (B, v) , then $\Phi(B, v)[B_k] = 0$.

(Coh) *Coherence*²: For all $(B, v) \in \mathcal{CSG}^N$, $\Phi(B^N, v) = \Phi(B, v)$.

Coherence means that games where all players belong to only one union and where all of them act as singletons are indistinguishable.

This theorem is in Calvo and Gutiérrez (2010).

Theorem 5 *A value Φ on \mathcal{CSG} satisfies efficiency, additivity, coalitional symmetry, null coalition axiom, coherence and population solidarity within unions if, and only if, $\Phi \equiv K$.*

The two-step Shapley value in our previous example (B, u_T) yields

$$K_1(B, u_T) = \frac{1}{2}, \quad K_2(B, u_T) = \frac{1}{4}, \quad K_3(B, u_T) = \frac{1}{4}, \quad K_4(B, u_T) = 0.$$

Player 3 now obtains the same as 2, which appears rather unfair from the productivity point of view, as 3 is a null player that does not contribute the rewards of the union.

3.2 Definition of ξ

A new coalitional value is now defined, which can be considered as an extension of the solidarity value to the coalition structure setting.

Definition 1 *For all game $(B, v) \in \mathcal{CSG}^N$, for all $k \in M$ and all $i \in B_k$ we define:*

$$\xi_i(B, v) := Sl_i(B_k, v_k). \quad (4)$$

First union k now plays the quotient game (M, v_B) among the unions, and the payoff obtained (by the Shapley value) is shared among its members by computing the solidarity value in the internal game (B_k, v_k) . Thus, the value ξ can be denoted as $\Pi^{(Sh, Sl)}$.

As the solidarity value satisfies efficiency in the internal game (B_k, v_k) , it follows that ξ also satisfies the *quotient game property*:

$$\sum_{i \in B_k} \xi_i(B, v) = Sh_k(M, v_B), \quad \text{for all } k \in M.$$

²This axiom was called the *Coalitional structure equivalence* in Albizuri (2008).

Moreover, we have that $\xi(B^N, v) = Sl(N, v)$ and $\xi(B^n, v) = Sh(N, v)$.

The structure of the value ξ as a solution of the form $\Pi^{(Sh, Sl)}$ can also be seen in the next proposition.

Proposition 1 For all $(B, v) \in \mathcal{CSG}^N$,

$$\xi_i(B, v) = \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} (Sl_i(B_k, v^T)), \quad \text{for all } k \in M \text{ and all } i \in B_k,$$

where

$$v^T(S) = v \left(S \cup \bigcup_{\substack{r \in T \\ r \neq k}} B_r \right) - v \left(\bigcup_{\substack{r \in T \\ r \neq k}} B_r \right), \quad \text{for all } S \subseteq B_k \text{ and all } T \subseteq M,$$

$m = |M|$ and $t = |T|$.

Proof. Let $(B, v) \in \mathcal{CSG}^N$ be a game. For all $k \in M$, denote $b_k = |B_k|$. Then, for all $i \in B_k$,

$$\begin{aligned} & \xi_i(B, v) = Sl_i(B_k, v_k) = \\ &= \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \frac{1}{s} \sum_{j \in S} (v_k(S) - v_k(S \setminus j)) = \\ &= \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \frac{1}{s} \sum_{j \in S} (Sh_k(M, v_{B|_S}) - Sh_k(M, v_{B|_{S \setminus j}})) = \\ &= \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \frac{1}{s} \sum_{j \in S} \left(\sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} [(v_{B|_S}(T) - v_{B|_S}(T \setminus k)) - (v_{B|_{S \setminus j}}(T) - v_{B|_{S \setminus j}}(T \setminus k))] \right) = \\ &= \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} \left(\frac{1}{s} \sum_{j \in S} (v_{B|_S}(T) - v_{B|_{S \setminus j}}(T)) \right) = \\ &= \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \left(\frac{1}{s} \sum_{j \in S} (v^T(S) - v^T(S \setminus j)) \right) = \\ &= \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} (Sl_i(B_k, v^T)). \end{aligned}$$

■

Remark 2 The structure of the Owen value as $\Pi^{(Sh, Sh)}$ can be proven in a similar way:

$$Ow_i(B, v) = \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} Sh_i(B_k, v^T), \quad \text{for all } k \in M \text{ and all } i \in B_k.$$

Similarly to the Owen value, we can provide a formula to compute the coalitional value ξ by means of orders. Let $(B, v) \in \mathcal{CSG}^N$. For all $\omega \in \Omega(B, N)$, all $k \in M$ and all $i \in B_k$, define

$$\varsigma_i^\omega(B, v) := \sum_{\substack{j \in B_k \\ \omega(j) \geq \omega(i)}} \frac{1}{|(B_k \cap P_j^\omega) \cup j|} \partial^j (v, P_j^\omega \cup j).$$

For example, if $N = \{a, b, c, d\}$, $B = \{B_1, B_2\}$, where $B_1 = \{a, b\}$ and $B_2 = \{c, d\}$; and $\omega \in \Omega(B, N)$ is such that $\bar{\omega} = (a, b, c, d)$, this means that the sequence of coalition formation $S_1 \subseteq S_2 \subseteq S_3 \subseteq S_4$ in ω is: at step $r = 1$, $S_1 = \{a\}$, at step $r = 2$, $S_2 = \{a, b\}$, at step $r = 3$, $S_3 = \{a, b, c\}$, and at step $r = 4$, $S_4 = \{a, b, c, d\}$. Then the payoffs $\varsigma^\omega(B, v)$ are

$$\begin{aligned} \varsigma_a^\omega(B, v) &= v(\{a\}) + \frac{v(\{a, b\}) - v(\{a\})}{2}, \\ \varsigma_b^\omega(B, v) &= \frac{v(\{a, b\}) - v(\{a\})}{2}, \\ \varsigma_c^\omega(B, v) &= (v(\{a, b, c\}) - v(\{a, b\})) + \frac{v(\{a, b, c, d\}) - v(\{a, b, c\})}{2}, \\ \varsigma_d^\omega(B, v) &= \frac{v(\{a, b, c, d\}) - v(\{a, b, c\})}{2}. \end{aligned}$$

Proposition 2 For all $(B, v) \in \mathcal{CSG}^N$,

$$\xi_i(B, v) = \frac{1}{|\Omega(B, N)|} \sum_{\omega \in \Omega(B, N)} \varsigma_i^\omega(B, v), \quad \text{for all } i \in N.$$

Proof. Let $(B, v) \in \mathcal{CSG}^N$ be a game. For all $k \in M$ and all $i \in B_k$,

$$\begin{aligned} & \frac{1}{|\Omega(B, N)|} \sum_{\omega \in \Omega(B, N)} \varsigma_i^\omega(B, v) = \\ &= \frac{1}{|\Omega(B, N)|} \sum_{\substack{T \subseteq M \\ k \in T}} (m-t)!(t-1)! \prod_{r \in M \setminus k} b_r! \sum_{\substack{S \subseteq B_k \\ i \in S}} (b_k - s)!(s-1)! \sum_{j \in S} \frac{1}{s} \partial^j \left(v, S \cup \bigcup_{r \in T \setminus k} B_r \right) = \\ &= \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \sum_{j \in S} \frac{1}{s} \partial^j \left(v, S \cup \bigcup_{r \in T \setminus k} B_r \right) = \\ &= \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} \sum_{\substack{S \subseteq B_k \\ i \in S}} \frac{(b_k - s)!(s-1)!}{b_k!} \sum_{j \in S} \frac{1}{s} \partial^j (v^T, S) = \\ &= \sum_{\substack{T \subseteq M \\ k \in T}} \frac{(m-t)!(t-1)!}{m!} Sl_i(B_k, v^T) = \xi_i(B, v). \end{aligned}$$

■

In our unanimity example (B, u_T) , the payoffs obtained with our coalitional solidarity value are

$$\xi_1(B, u_T) = \frac{1}{2}, \quad \xi_2(B, u_T) = \frac{3}{8}, \quad \xi_3(B, u_T) = \frac{1}{8}, \quad \xi_4(B, u_T) = 0.$$

These are more in keeping with the approach of joining productivity and cohesion principles simultaneously to reward the players inside the union $\{2, 3\}$.

Remark 3 *The solidarity value can be extended in a different way to games with a coalition structure. For example, we can define the values $\Pi^{(Sl,Sl)}$ and $\Pi^{(Sl,Sh)}$ as follows:*

$$\begin{aligned} \Pi_i^{(Sl,Sl)}(B, v) &: = Sl_i(B_k, \bar{v}_k), \quad \text{for all } k \in M \text{ and all } i \in B_k, \\ \Pi_i^{(Sl,Sh)}(B, v) &: = Sh_i(B_k, \bar{v}_k), \quad \text{for all } k \in M \text{ and all } i \in B_k, \end{aligned}$$

where $\bar{v}_k(S) = Sl_k(M, v_{B|_S})$ for all $S \subseteq B_k$. Note that

$$\begin{aligned} \Pi^{(Sl,Sl)}(B^N, v) &= Sl(N, v) & \Pi^{(Sl,Sl)}(B^n, v) &= Sl(N, v) \\ \Pi^{(Sl,Sh)}(B^N, v) &= Sh(N, v) & \Pi^{(Sl,Sh)}(B^n, v) &= Sl(N, v) \\ (Ow \equiv \Pi^{(Sh,Sh)})(B^N, v) &= Sh(N, v) & (Ow \equiv \Pi^{(Sh,Sh)})(B^n, v) &= Sh(N, v) \\ (\xi \equiv \Pi^{(Sh,Sl)})(B^N, v) &= Sl(N, v) & (\xi \equiv \Pi^{(Sh,Sl)})(B^n, v) &= Sh(N, v) \end{aligned}$$

The next table summarizes the payoffs given to the game (B, u_T) by all the aforementioned solutions:

(B, u_T)	Φ_1	Φ_2	Φ_3	Φ_4
$Ow \equiv \Pi^{(Sh,Sh)}$	1/2	1/2	0	0
K	1/2	1/4	1/4	0
$\Pi^{(Sl,Sl)}$	7/18	7/24	7/72	4/18
$\Pi^{(Sl,Sh)}$	7/18	7/18	0	4/18
$\xi \equiv \Pi^{(Sh,Sl)}$	1/2	3/8	1/8	0

The Owen value yields nothing to player 3 as a null player, hence he has no benefits from belonging to the union $\{2, 3\}$. The opposite case is the two-step Shapley value, as it yields the same payoff 1/4 to both players in the union, without taking into account the differences in their productivity. The value ξ is an intermediate option between these two opposite options. The coalitional values $\Pi^{(Sl,Sl)}$ and $\Pi^{(Sl,Sh)}$ do not seem reasonable as union $\{4\}$ is a null coalition in the quotient game and, yet, it obtains a positive payoff in both values. Moreover, player 4 obtains a greater payoff than player 3.

4 Axiomatic approach

This section provides an axiomatic characterization of the coalitional value ξ .

Two new properties are defined in order to characterize ξ .

The first one is a weaker version of the A-null player axiom:

(A-NP*) *A-Null player axiom in B^N* : For all $(N, v) \in \mathcal{G}^N$ and all $i \in N$, if i is an A-null player in (N, v) , then $\Phi_i(B^N, v) = 0$.

Hence, we only require an A-null player in (N, v) to receive zero payoff for the trivial coalition structure $B^N = \{N\}$.

The second is a consistency property. Consistency states the independence of a value with respect to departure of some players with their assigned payoffs. It says that the recommendation made for any problem should always agree with the recommendation made for any problem obtained by imagining some agents leaving with their payoffs, and "reassessing the situation" from the viewpoint of the remaining agents. It has been introduced in different ways depending upon how the payoffs of the agents that leave the game are defined. As the value ξ satisfies the quotient game property (computing the payoffs by using the Shapley value) we follow the approach established by Hart and Mas-Colell (1989). We briefly recall their definitions of reduced game and consistency in \mathcal{G} . Let γ be a value, (N, v) be a game, and $T \subseteq N$. The *reduced game* (T, v_T^γ) is defined as follows:

$$v_T^\gamma(S) := v(S \cup T^c) - \sum_{i \in T^c} \gamma_i(S \cup T^c, v), \quad \text{for all } S \subseteq T,$$

where $T^c = N \setminus T$. A value γ is *consistent* if, for all game (N, v) and all coalition $T \subseteq N$, it holds

$$\gamma_i(T, v_T^\gamma) = \gamma_i(N, v), \quad \text{for all } i \in T.$$

A value γ is *transferable-utility invariant* if, for all pair of games (N, v) and (N, u) , and all real constants $a > 0$ and $\{b_i\}_{i \in N}$,

$$u(S) = av(S) + \sum_{i \in S} b_i, \quad \text{for all } S \subseteq N,$$

implies

$$\gamma_i(N, u) = a\gamma_i(N, v) + b_i, \quad \text{for all } i \in N.$$

Then the following result holds:

Theorem 6 (Hart and Mas-Colell, 1989) Let γ be a value on \mathcal{G}^N . Then:

- (i) γ is consistent;
 - (ii) for two-person games: (a) γ is efficient, (b) γ is transferable-utility invariant, (c) γ satisfies symmetry;
- if and only if γ is the Shapley value.

In Winter (1992), this property is extended to games with a coalition structure as follows: Let Φ be a coalitional value and $(B, v) \in \mathcal{CSG}^N$ be a game where $B = \{B_1, B_2, \dots, B_m\}$. For all $k \in M$ and all $R \subseteq B_k$, the *reduced game* $(R, v_{R,B}^\Phi)$ is defined by

$$v_{R,B}^\Phi(S) := v(N \setminus R \cup S) - \sum_{i \in N \setminus R} \Phi_i(B \mid_{R^c \cup S}, v), \quad \text{for all } S \subseteq R,$$

where $R^c = B_k \setminus R$.

- (C) *Consistency*: For all $(B, v) \in \mathcal{CSG}^N$, all $k \in M$, and all $R \subseteq B_k$, $\Phi_i(B, v) = \Phi_i(B^R, v_{R,B}^\Phi)$, for all $i \in R$.

The interpretation is as follows: The players in S cooperate with the complement of R . The players of $N \setminus R$ get the anticipated payoff according to the value Φ in the game $(B \mid_{R^c \cup S}, v)$, and the players of S get the remaining payoff from $v(N \setminus R \cup S)$. A value Φ is then consistent if each player in R gets in $v_{R,B}^\Phi$ and the trivial coalition structure B^R the same amount he gets in the original game (B, v) . This axiom makes sense for coalitional values defined on \mathcal{CSG} , given that the value in this axiom must be applied on N and also on R , for all $k \in M$ and all $R \subseteq B_k$.

Winter proved that the Owen value satisfies consistency and this property is used in an axiom system that characterizes the value.

Nevertheless, consistency is a too stringent property in order to characterize the coalitional value ξ .

Proposition 3 *The coalitional value ξ does not satisfy consistency.*

Proof. Consider the game $(B, u_T) \in \mathcal{CSG}^N$ where $N = \{1, 2, 3, 4, 5\}$, $T = \{1, 2\}$ and $B = \{B_1, B_2, B_3\}$ with $B_1 = \{1\}$, $B_2 = \{2, 3, 4\}$ and $B_3 = \{5\}$. The coalitional value ξ is

$$\xi(B, u_T) = \left(\frac{1}{2}, \frac{11}{36}, \frac{7}{72}, \frac{7}{72}, 0 \right).$$

Now consider $R = \{2, 3\} \subseteq B_2$. The reduced game $\left(R, (u_T)_{R,B}^\xi\right)$ is given by

$$\begin{aligned}(u_T)_{R,B}^\xi(2) &= \frac{3}{8}, \\ (u_T)_{R,B}^\xi(3) &= 0, \\ (u_T)_{R,B}^\xi(2, 3) &= \frac{29}{72}.\end{aligned}$$

By definition, $\xi\left(B^R, (u_T)_{R,B}^\xi\right) = Sl\left(R, (u_T)_{R,B}^\xi\right)$. Thus,

$$\begin{aligned}\xi_2\left(B^R, (u_T)_{R,B}^\xi\right) &= \frac{85}{288} \neq \frac{11}{36} = \xi_2(B, u_T), \\ \xi_3\left(B^R, (u_T)_{R,B}^\xi\right) &= \frac{31}{288} \neq \frac{7}{72} = \xi_3(B, u_T).\end{aligned}$$

■

Nevertheless, a weaker version of this property can be used, where it would only hold for the particular case where $R = B_k$, for all $k \in M$.

(CC) *Coalitional consistency*: For all (B, v) and all $k \in M$, $\Phi_i(B, v) = \Phi_i(B^{B_k}, v_{B_k}^\Phi)$, for all $i \in B_k$.

We are now ready to present the axiomatic characterization of the value ξ on \mathcal{CSG} .

Theorem 7 *A coalitional value Φ on \mathcal{CSG} satisfies efficiency, additivity, intracoalitional symmetry, coalitional symmetry, null coalition axiom, A-null player axiom in B^N and coalitional consistency if, and only if, $\Phi \equiv \xi$.*

Proof. *Existence.* Let $(B, v) \in \mathcal{CSG}^N$ be a game. Since the Shapley value and the solidarity value satisfy efficiency, for all $k \in M$ we have that $\sum_{i \in B_k} \xi_i(B, v) = v_k(B_k) = Sh_k(M, v_B)$, and then $\sum_{i \in N} \xi_i(B, v) = \sum_{k \in M} \sum_{i \in B_k} \xi_i(B, v) = \sum_{k \in M} Sh_k(M, v_B) = v(N)$. Thus ξ satisfies efficiency. Moreover, since the Shapley and the solidarity values satisfy additivity, ξ also satisfies additivity.

The coalitional value ξ also satisfies intracoalitional symmetry because, for all $k \in M$, if $\{i, j\} \subseteq B_k$ are symmetric in (N, v) , they are also symmetric in (B_k, v_k) , due to the symmetry of the Shapley value. Thus, $\xi_i(B, v) = \xi_j(B, v)$ since the solidarity value is also symmetric. Moreover, ξ satisfies coalitional symmetry and the null coalition axiom because the Shapley value satisfies symmetry and the null player axiom and $\sum_{i \in B_k} \xi_i(B, v) = Sh_k(M, v_B)$, for all $k \in M$.

It is straightforward that ξ satisfies the A-Null player axiom in B^N , because the solidarity value satisfies the A-Null player axiom and, by definition, $\xi(B^N, v) = Sl(N, v)$.

In order to prove that ξ satisfies coalitional consistency, let $k \in M$. Taking into account the definition of ξ , and that ξ satisfies efficiency, for all $S \subseteq B_k$,

$$v_{B_k, B}^\xi(S) = \sum_{i \in S} \xi_i(B |_S, v) = Sh_k(M, v_{B|_S}) = v_k(S).$$

Therefore,

$$\xi_i(B^{B_k}, v_{B_k, B}^\xi) = Sl_i(B_k, v_{B_k, B}^\xi) = Sl_i(B_k, v_k) = \xi_i(B, v), \quad \text{for all } i \in B_k.$$

Uniqueness. Let Φ be a coalitional value satisfying the above axioms, and let $(B, v) \in \mathcal{CSG}^N$.

Suppose that $|B| = 1$, that is, $B = B^N$. Since Φ satisfies efficiency, additivity, intracoalitional symmetry and the A-null player axiom in B^N , then by Theorem 2, it holds that $\Phi(B^N, v) = Sl(N, v)$. Thus Φ is uniquely determined when $|B| = 1$.

Suppose that $|B| \geq 2$. To prove that $\Phi(B, v)$ is uniquely determined, due to the additivity axiom, it is sufficient to show that Φ is uniquely defined on the set of unanimity games. Let $T \subseteq N$ and $B(T) = \{B_k \in B \mid B_k \cap T \neq \emptyset\}$. For all $h \in M$ such that $B_h \notin B(T)$, B_h is a null coalition, then, by the null coalition axiom and intracoalitional symmetry, $\Phi_i(B, u_T) = 0$, for all $i \in B_h$. Hence, only rests to show that $\Phi_i(B, u_T)$ is uniquely determined for players in the coalitions $B_k \in B(T)$.

Let $B_k \in B(T)$. By coalitional consistency, $\Phi_i(B, u_T) = \Phi_i(B^{B_k}, (u_T)_{B_k, B}^\Phi)$, for all $i \in B_k$. We have just proved that $\Phi(B^{B_k}, (u_T)_{B_k, B}^\Phi) = Sl(B_k, (u_T)_{B_k, B}^\Phi)$, then $\Phi_i(B, u_T) = Sl_i(B_k, (u_T)_{B_k, B}^\Phi)$, for all $i \in B_k$. By efficiency and coalitional symmetry, for all $S \subseteq B_k$,

$$(u_T)_{B_k, B}^\Phi(S) = \sum_{i \in S} \Phi_i(B |_S, u_T) = \begin{cases} 0 & \text{if } S \not\supseteq B_k \cap T, \\ \frac{1}{|B(T)|} & \text{if } S \supseteq B_k \cap T, \end{cases}$$

that is, $(u_T)_{B_k, B}^\Phi = \frac{1}{|B(T)|} u_{B_k \cap T}$. Therefore,

$$\Phi_i(B, u_T) = Sl_i(B_k, (u_T)_{B_k, B}^\Phi) = \frac{1}{|B(T)|} Sl_i(B_k, u_{B_k \cap T}), \quad \text{for all } i \in B_k.$$

Thus, $\Phi(B, u_T)$ is uniquely determined. ■

Remark 4 *In the proof of Theorem 7, the null coalition axiom is used only on the set of unanimity games. Therefore, this axiom could be weakened in the following way:*

(NC*) *Null coalition on unanimity games: For all $B \in \mathcal{B}(N)$, all $T \subseteq N$ and all $h \in M$ such that $B_h \cap T = \emptyset$, $\Phi(B, u_T)[B_h] = 0$.*

5 Complementary results

5.1 Independence of the axiomatic system in Theorem 7

The axiom system in Theorem 7 is independent. Indeed:

1. The Owen value satisfies all axioms, except A-null player axiom *in* B^N .
2. The coalitional value $\Pi^{(Sl,Sl)}$ satisfies all axioms, except null coalition axiom. See the example of Section 3: $\{4\}$ is a null coalition, but player 4 does not obtain zero.
3. Let the coalitional value F be defined as

$$F_i(B, v) = Sl_i(B_k, v) + \frac{1}{|B_k|} [Sh_k(M, v_B) - v(B_k)], \quad \text{for all } k \in M \text{ and all } i \in B_k.$$

This value is defined in the same way as the two-step Shapley value, but the solidarity value is applied within each union. Taking into account the properties of the Shapley and the solidarity values, F satisfies E, A, ISy, CSy, NC and A-NP* ($F(B^N, v) = Sl(N, v)$). Nevertheless, F does not satisfy weak coalitional consistency. Consider the example of Section 3. The value is $F(B, u_T) = (1/2, 1/4, 1/4, 0)$. Let $B_2 = \{2, 3\}$, the reduced game $(B_2, (u_T)_{B_2, B}^{F*})$ is given by $(u_T)_{B_2, B}^{F*} = \frac{1}{2}u_{\{2\}}$. Thus, $F(B^{B_2}, (u_T)_{B_2, B}^{F*}) = Sl(B_2, (u_T)_{B_2, B}^{F*}) = (3/8, 1/8) \neq (1/4, 1/4)$.

4. The coalitional value $\Pi^{(Sh,Sl^w)}$ satisfies all axioms, except intracoalitional symmetry.
5. The coalitional value $\Pi^{(Sh^w,Sl)}$ satisfies all axioms, except coalitional symmetry.
6. Let the coalitional value G be defined as $G_i(B, v) = 0$ for all $(B, v) \in \mathcal{CSG}$ and all $i \in N$. It satisfies all axioms except efficiency.
7. Define the *Bounded egalitarian value* as

$$BE_i(N, v) = \begin{cases} \frac{v(N)}{|Carr(N, v)|}, & \text{if } i \in Carr(N, v), \\ 0, & \text{otherwise,} \end{cases}$$

where $Carr(N, v)$ denotes the *carrier* of (N, v) , which is the set of non-null players in (N, v) . Then, $\Pi^{(BE,Sl)}$ satisfies all axioms, except additivity.

5.2 An alternative axiomatic characterization of the Owen value

Following the same lines as in Theorem 7, we can provide a new axiomatic characterization of the Owen value on CSG . We therefore also need a weaker version of the null player axiom as we have done with the A-null player one.

(NP*) *Null player axiom in B^N* : For all $(N, v) \in \mathcal{G}^N$ and all $i \in N$, if i is a null player in (N, v) , then $\Phi_i(B^N, v) = 0$.

Theorem 8 *A coalitional value Φ on CSG satisfies efficiency, additivity, intracoalitional symmetry, coalitional symmetry, null coalition axiom, null player axiom in B^N and coalitional consistency if, and only if, $\Phi \equiv Ow$.*

The proof follows the same lines as in Theorem 7 and it is left to the reader.

It becomes clear from the comparison of Theorems 7 and 8 that the main difference between the Owen and ξ values is based only on the difference between NP^* and $A-NP^*$: when all the players form the grand coalition, null players obtain zero in the Owen value, whereas the A-null players obtain zero in the coalitional value ξ .

The axiomatic system in Theorem 8 is independence. Indeed:

1. The coalitional value ξ satisfies all axioms, except null player axiom in B^N .
2. The coalitional value $\Pi^{(Sl, Sh)}$ satisfies all axioms, except null coalition axiom. See the example of Section 3: $\{4\}$ is a null coalition, but player 4 does not obtain zero.
3. The two-step Shapley value satisfies all axioms, except weak coalitional consistency. Consider the example of Section 3. The value is $K(B, u_T) = (1/2, 1/4, 1/4, 0)$. Let $B_2 = \{2, 3\}$, the reduced game $(B_2, (u_T)_{B_2, B}^{K*})$ is given by $(u_T)_{B_2, B}^{K*} = \frac{1}{2}u_{\{2\}}$. Thus, $K(B^{B_2}, (u_T)_{B_2, B}^{K*}) = Sh(B_2, (u_T)_{B_2, B}^{K*}) = (1/2, 0) \neq (1/4, 1/4)$.
4. The coalitional value $\Pi^{(Sh, Sh^w)}$ satisfies all axioms, except intracoalitional symmetry.
5. The coalitional value $\Pi^{(Sh^w, Sh)}$ satisfies all axioms, except coalitional symmetry.
6. The coalitional value G , defined as $G_i(B, v) = 0$ for all $(B, v) \in CSG$ and all $i \in N$, satisfies all axioms except efficiency.
7. The coalitional value $\Pi^{(BE, Sh)}$ satisfies all axioms, except additivity.

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