The value in games with restricted cooperation[*]

E. Calvo[†]

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Abstract

Given a cooperative game, a *restriction* in cooperation is given by a *set system*, which specifies the set of feasible coalitions that can be formed. In this setting, the Shapley value is defined following the random order approach: the value is the expected marginal contribution of the player to its predecessors in every order. The main difference from previous approaches, which are based on the restriction over the feasible set of orderings, is that all orderings are considered as feasible, and that the set system will determine only which coalitions are formed when players arrive successively in every order.

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1 Introduction

A cooperative game with side payments is a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function. A *restriction* in cooperation is given by a *set system* $(\boldsymbol{F}, N)$, where $\boldsymbol{F} \subset 2^N$, is the set of feasible coalitions that can be formed. Many different restrictions have been considered in the Literature. Some examples are:

Coalition structures, Aumann and Dreze (1974).

Communication Graphs, Myerson (1977).

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[†] University of Valencia. Departament of Economic Analysis and ERI-CES. Spain.

Conference structures, Myerson (1980).

Union stable cooperation structures, Algaba, Bilbao, Borm and López (2000).

Precedence constraints, Faigle and Kern (1992).

Incompatible players, Bergantiños, Carreras and García Jurado (1993).

Partnerships, Carreras (1996).

Convex geometries, Bilbao (1998), Bilbao and Edelman (2000).

Matroids, Bilbao, Driessen, Jiménez Losada and Lebrón (2001).

Antimatroids, Algaba,Bilbao, van den Brink and Jiménez Losada (2003).

Augmenting systems, Bilbao and Ordóñez (2007).

And this family of structures is still growing, which differ by the corresponding set system F. Some of them are generalizations of previous ones. For example, communications graphs is a particular case of conference structures, that in turn is a particular case of union stable cooperation structures. A recursive question that inmediately emerges after a new structure appears is how to extend the Shapley value (see Shapley, 1953) on it. This a natural question if we want to know the expected payoffs that players obtain when playing a game with such a restriction. Looking at the received literature we observe two main lines of answering this question.

In the first approach, the characteristic function v is extended to all unfeasible sets $S \notin \boldsymbol{F}$ in a "natural" way: $v_{\boldsymbol{F}}(S)$ is computed as the sum of the coalition's worth of a feasible partition of S. Now the extended $v_{\boldsymbol{F}}$ is defined over all 2^N and the Shapley value can be computed. This approach was followed by coalition structures, communication graphs, conference structures, partition systems, incompatible players and partnerships.

In the second approach, the value is computed as the player's expected marginal contributions to the coalitions formed by a sequential process: given any ordering of the players, every player joins his predecessors in the order, being all ordersequally likely. Now, given a set system $\boldsymbol{F}$, if a particular coalition of predecessors is not feasible, then the order in which such coalition is formed will not be feasible either. Then, the set of ordersfor which the marginal contributions can be compuded is restricted. This is the approach followed in games with precedence constraints, convex geometries, matroids, antimatroids and augmenting systems.

As we will see in more detail in the next Section, both approaches have their own drawbacks. Firstly, it is unclear how to define $v_{\boldsymbol{F}}(S)$ when $S \notin \boldsymbol{F}$ and several alternative partitions of S are possible. In García Jurado, Carreras and Bergantiños (1993) the partition that maximizes the sum of the coalition payoffs that form the partition is chosen. Unfortunately, this way to define $v_{\boldsymbol{F}}$ does not satisfy the property of additivity and this fact turns out incompatible with the interpretation of $v_{\boldsymbol{F}}$ as an expectation. Secondly, by restricting to the set of orderswhich are compatible with the set system, leave out of consideration many interesting classes of games for which the set of feasible ordersis empty. Moreover, as will be shown in Example 2 below, the allocation payoffs obtained in this way can be rather counterintuitive.

We do not try to enlarge the family by adding a particular structure of games for which the value can be extended unambiguously. Instead, our goal is to extend the value to any set system $(\boldsymbol{F}, N)$ that can be considered. For that reason we only assume that $\emptyset \in F$. Our purpose is to compute the value following also the random order approach: the value is the expected marginal contribution of the player to its predecessors in every order. The main difference from previous approaches, based on the restriction over the feasible set of orders, is that all orders are considered as feasible, and the set system $\boldsymbol{F}$ will determine only which coalitions are formed when players arrive successively in every order.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary definitions and notation. The approaches based on the $v_{\boldsymbol{F}}$ extension and the orders restriction are considered in detail. Then, Section 3 presents the new value. Finally, in Section 4, the value is characterized axiomatically in $(\boldsymbol{F}, N)$ by using the same set of axioms as the ones used to characterize the Shapley value in $(2^N, N)$.

2 Previous approaches

Let $N = \{1, ..., n\}$ be a finite set of players, $|N| = n$, and v: $2^N \longrightarrow \mathbb{R}$ a function satisfying $v(\emptyset) = 0$, where 2^N is the set of all subsets of N. We denote by (N, v) a *transferable utility game* with player set N and *characteristic function* v, and it will be denoted by . For any coalition $S \subseteq N$, $v(S)$ is called the *worth* of S. Denote by G^N the space of games with finite player set N. Given a game (N, v) and a coalition S, we write (S, v) for the subgame obtained by restricting v to subsets of S only (i.e., to 2^S).

A *value* is a function ψ which assigns a real number $\psi_i(N, v)$ to every game (N, v)

and every player $i \in N$. Because partial cooperation within subcoalitions $S \subset N$ is also possible, a value can be also considered as a *payoff configuration* $\psi(v) = (\psi(S,v))_{S \subset N}$.

Let $\Omega(N)$ be the set of all orders in N. Eny $\omega \in \Omega(N)$ is a bijection from N to $\{1, 2, .., n\}$, $|N| = n$, and $\omega(i) < \omega(j)$ means that player i comes before j in the ordering ω. We denote by P_ω^i the set of predecessors of i in ω, that is $P_\omega^i := \{j \in N : \omega(j) < \omega(i)\}$. The *marginal contribution* that every player i receives in any ordering ω is defined by

$$m_i^\omega(N,v) := v\left(P_\omega^i \cup i\right) - v\left(P_\omega^i\right)$$

The Shapley (1953) value (Sh) for the game (N,v) is defined by

$$Sh_i(N,v) = \frac{1}{n!} \sum_{\omega \in \Omega(N)} m_i^\omega(N,v), \quad (i \in N). \tag{1}$$

The payoff $Sh_i(N,v)$ is what player i expect to obtain in v if, for any possible order ω, where all orders are equally likely, the player is rewarded with her marginal contribution to their predecessors.

A *restriction* in the cooperation among the players in N is given by a *set system* $\boldsymbol{F}$, where $\boldsymbol{F} \subset 2^N$ is the set of *feasible* coalitions that can be formed. We only assume that $\emptyset \in \boldsymbol{F}$. Given $S \subset N$ and $\boldsymbol{F} \subset 2^N$, we denote by $\boldsymbol{F}(S)$ the natural restriction of $\boldsymbol{F}$ into S, that is $\boldsymbol{F}(S) := 2^S \cap \boldsymbol{F}$. We will denote a game (N,v) with a restriction $\boldsymbol{F}$ by $(\boldsymbol{F},N,v)$, where $\boldsymbol{F} \subset 2^N$ and $v \in G^N$.

Associated to any ordering $\omega \in \Omega(N)$ there is a sequence of incremental coalitions, $S(0,\omega) \subset S(1,\omega) \subset ... \subset S(n-1,\omega) \subset S(n,\omega)$, starting from $S(0,\omega) := \emptyset$ up to $S(n,\omega) = N$, where players are added one by one following the ordering ω. A set system $\boldsymbol{F}$ will induce a restriction in the set of orders$\Omega(N)$ because it is possible that, for some order ω and step r, the coalition $S(r,\omega)$ is not feasible, $S(r,\omega) \notin \boldsymbol{F}$. We denote by $\Omega(\boldsymbol{F},N)$ the set of orders in N compatible with the set system $\boldsymbol{F}$, that is, $\omega \in \Omega(\boldsymbol{F},N)$ if and only if $P_\omega^i \cup i \in \boldsymbol{F}$ for all $i \in N$.

When $\boldsymbol{F} \neq 2^N$, two approaches have been followed trying to extend equation (1). In the first one, the characteristic function v is extended over all unfeasible coalitions, $S \notin \boldsymbol{F}$, and then a new characteristic function $v_{\boldsymbol{F}}$ is obtained, defined in all 2^N. The second one computes the marginal contributions, $m_i^\omega(N,v)$, only in the set of orderswhich are feasible, $\omega \in \Omega(\boldsymbol{F},N)$, assuming that all of them are equally likely. Both approaches have their own drawbacks, which discuss in detail now.

2.1 The $v_{\boldsymbol{F}}$ modification

This approach is based on the works of Myerson (1977) in graphs, and Myerson (1980) in conference structures. Roughly speaking, the idea is that when $S \notin \boldsymbol{F}$, $v_{\boldsymbol{F}}(S)$ is computed as the sum of the coalition's worth of a *feasible partition* of S.

For any $S \in 2^N$, let $\Pi(\boldsymbol{F}, S)$ be the set of partitions of S within $\boldsymbol{F}$ into maximal components, which are defined as follows:

Definition 1 $\boldsymbol{T} = \{T_1, ..., T_r\} \in \Pi(\boldsymbol{F}, S)$ *if and only if*

- $\boldsymbol{T} \subset \boldsymbol{F}$,
- $\bigcup\limits_{T_k \in \mathbf{T}} T_k \subset S$,
- $T_k \cap T_s = \emptyset$ *for all* $T_k, T_s \in \boldsymbol{T}$,
- *if* $T_k \in \Pi(\boldsymbol{F}, S)$ *then there does not exist* $P \in \boldsymbol{F}$: $T_k \subsetneq P \subset S$.

Note that if $S \in \boldsymbol{F}$ then $\Pi(\boldsymbol{F}, S) = \{\{S\}\}$. In general, when $S \notin \boldsymbol{F}$ such decomposition into feasible subcoalitions can be not unique, and even cannot recover the original coalition S, $\bigcup\limits_{T_k \in \mathbf{T}} T_k \subsetneq S$.

Let $v_{\boldsymbol{F}}(S)$ be defined by $v_{\boldsymbol{F}}(S) := 0$ if $\Pi(\boldsymbol{F}, S) = \emptyset$, and $v_{\boldsymbol{F}}(S) =: \max_{\boldsymbol{T} \in \Pi(\boldsymbol{F}, S)} \sum\limits_{T_k \in \boldsymbol{T}} v(T_k)$ otherwise. That is, players within S make a feasible partition such that the sum of the total payoffs is the biggest possible.

Definition 2 *The value extension* γ *is defined by* $\gamma(\boldsymbol{F}, N, v) := Sh(N, v_{\boldsymbol{F}})$.

Clearly, $\gamma\left(2^N, N, v\right) = \phi(N, v)$. In García Jurado, Carreras and Bergantiños (1993), this definition was used for the case of incompatible players.

It is worth noting that for the cases of coalition structures, graphs, conference structures and partition systems, it happens that the decomposition into maximal partitions is *always unique*, that is $\Pi(\boldsymbol{F}, S) = \{\boldsymbol{S}^*\}$ and $\cup_{S_k^* \in \boldsymbol{S}^*} S_k^* = S$ for all $S \in 2^N$, and then $v_{\boldsymbol{F}}(S) = \sum_{S_k \in \boldsymbol{S}^*} v(S_k)$. This fact implies that the modification $v_{\boldsymbol{F}}$ is *additive* for all these cases, i.e., given $(\boldsymbol{F}, N, v)$, $(\boldsymbol{F}, N, w)$ and $(\boldsymbol{F}, N, v+w)$, it holds that $v_{\boldsymbol{F}} + w_{\boldsymbol{F}} = (v+w)_{\boldsymbol{F}}$, which follows from the fact that

$$v_{\boldsymbol{F}}(S) + w_{\boldsymbol{F}}(S) = \sum_{S_k \in \boldsymbol{S}^*} v(S_k) + \sum_{S_k \in \boldsymbol{S}^*} w(S_k) = \sum_{S_k \in \boldsymbol{S}^*} (v+w)(S) = (v+w)_{\boldsymbol{F}}(S).$$

Unfortunately, the additivity decomposition is lost for more general set systems, as when there are incompatible players (García Jurado *et al.*, 1993). We illustrate this point with the following example.

Example 1. Let $N = \{i, j, k, l\}$ be the grand coalition and the set system $\boldsymbol{F} = \{\emptyset, \{i\}, \{i, j\}, \{k, l\}, \{j, k, l\}\}$. Consider the games v, w, and $v + w$, as given by the next table:

$\boldsymbol{F}$	$\{i, j\}$	$\{k, l\}$	$\{i\}$	$\{j, k, l\}$
v	1	1	2	2
w	2	2	1	1
$v + w$	3	3	3	3

Now we have that

$$\begin{aligned}&v_{\boldsymbol{F}}(N) = v\left(\{i\}\right) + v\left(\{j, k, l\}\right) = 4,\\ &w_{\boldsymbol{F}}(N) = w\left(\{i, j\}\right) + w\left(\{k, l\}\right) = 4,\\ &(v + w)_{\boldsymbol{F}}(N) = 6.\end{aligned}$$

Consider the value extension γ. Then, by efficiency of Sh, we have that

$$\sum_{i \in N} \gamma_i\left(\boldsymbol{F}, N, v\right) = \sum_{i \in N} Sh_i\left(N, v_{\boldsymbol{F}}\right) = v_{\boldsymbol{F}}(N).$$

But then, it cannot be true that $\gamma_i\left(\boldsymbol{F}, N, v + w\right) = \gamma_i\left(\boldsymbol{F}, N, v\right) + \gamma_i\left(\boldsymbol{F}, N, w\right)$ for all $i \in N$, because $(v + w)_{\boldsymbol{F}}(N) \neq v_{\boldsymbol{F}}(N) + w_{\boldsymbol{F}}(N)$. Hence this extension cannot be a linear operator. This fact is incompatible with the interpretation of γ as an *expectation*, that is, γ_i is the von Neumann-Morgenstern utility that a risk neutral player i gets when playing the game $(\boldsymbol{F}, N, v)$ (see Roth, 1977).

2.2 The $\Omega(\boldsymbol{F}, N)$ restriction

The alternative approach restricts the set of ordersin which the marginal contributions of the players are computed to those which are compatible with the set system $\boldsymbol{F}$.

Definition 3 *The value extension* ξ *on* $(\boldsymbol{F}, N, v)$ *is defined by*

$$\xi_i\left(\boldsymbol{F}, N, v\right) := \frac{1}{\left|\Omega\left(\boldsymbol{F}, N\right)\right|} \sum_{\omega \in \Omega(\boldsymbol{F}, N)} m_i^{\omega}\left(N, v\right), \quad \left(i \in N\right). \tag{2}$$

When $\boldsymbol{F} = 2^N$ it holds that $\Omega\left(2^N, N\right) = \Omega(N)$, hence $\xi_i\left(2^N, N, v\right) = Sh(N, v)$.

ξ is the definition used in Faigle and Kern (1992) for the case of precedence constrains, in Bilbao and Edelman (2000) for convex geometries, and in Bilbao and Ordoñez (2008) for augmenting systems. All these structures fall into the class of set systems for which (2) is well defined, since $\Omega(\boldsymbol{F}, N) \neq \emptyset$. This implies that there exists at least a collection of sets $\{S_0, S_1, ..., S_n\} \subset \boldsymbol{F}$ such that $S_0 = \emptyset$, $S_n = N$, and $S_0 \subset S_1 \subset ... \subset S_n$.

We first note that the condition $\Omega(\boldsymbol{F}, N) \neq \emptyset$, leaves out of consideration some interesting classes of set systems, as for example:

- *The grand coalition is not feasible.* Suppose that we want to apply the value as a power index in a weighted majority game that comes from the seats obtained by political parties in a Parliament. And we know that some ideological incompatibilities between two particular parties, make unfeasible those coalitions which contain them, in particular, the grand coalition N.

- *Pure bargaining problems.* Here only the grand coalition can reach agreements. This situation can be expressed by the set system $\boldsymbol{F} = \{\emptyset, N\}$, where $v(N) > 0$.

- *Matching problems.* We have here two sets of players N_1 and N_2, the grand coalition is $N = N_1 \cup N_2$ and the set system is given by $\boldsymbol{F} = \{\{i, j\} : \ i \in N_1, j \in N_2\}$.

- *Economic teams.* Where there is a minimum number of members of a team for which a job can be done.

- *Sport teams.* Where the number of players in a feasible team is fixed.

In all these cases, formula (2) cannot be applied directly, because for any ordering ω there are sizes for which the coalitions are forbidden, and then it is impossible to complete any feasible chain $\{S(r, \omega)\}_{r=0,1,...,n}$.

An additional question is that, for some examples in which ξ is well defined, the payoffs obtained are counterintuitive, as illustrated by the next example.

Example 2. Consider the following game (N, v), where $N = \{i, j, k\}$, and v is defined in 2^N by

$$v(i) = 1, \ v(j) = v(k) = v(\{j, k\}) = 0,$$
$$v(\{i, j\}) = v(\{i, k\}) = 3, \ v(\{i, j, k\}) = 4.$$

Here we can interpret that player i is an expert worker, that players j and k are apprentices, and that any productive job must be done with the help of the expert player.

If we compute the value under the full cooperation setting, 2^N, we obtain

$$Sh_i(N,v) = \frac{8}{3},\ Sh_j(N,v) = Sh_k(N,v) = \frac{2}{3}.$$

Suppose now that a market regulator wants to avoid swindles, and for that reason only coalitions that contain the expert player i are considered as lawful. With such legal restrictions, the new set system is given by $\boldsymbol{F} = 2^N \setminus \{\{j\},\{k\},\{j,k\}\}$. This implies that now we only have two orders compatible with $\boldsymbol{F}$: $\omega = (\omega(i) = 1, \omega(j) = 2, \omega(k) = 3)$ and $\omega' = (\omega'(i) = 1, \omega'(j) = 3, \omega'(k) = 2)$. If we compute the value extension ξ we obtain

$$\xi_i(\boldsymbol{F},N,v) = 1,\ \xi_j(\boldsymbol{F},N,v) = \xi_k(\boldsymbol{F},N,v) = \frac{3}{2},$$

which yields the worst payoffs to the expert player, just in complete contradiction with the intuitive idea that it is the expert player who is in the best bargaining position, due to the fact that she is decisive for any productive job.

This is a general behavior for the ξ extension: Take any monotonic game[1] (N,v) such that for some $S \subsetneq N$, it holds that $v(T) = 0$ for all $T \subset S$. Here players in S are in a weaker bargaining situation that players in $N \backslash S$. Define a new set system by deleting all the coalitions in S, i.e. $\boldsymbol{F} = 2^N \setminus \{\{T\}_{T \subset S}\}$. Now it holds that with the ξ value extension the *payoffs change in the opposite direction* with this kind of transformation: if $v(R) = 0$ for all $R \in 2^S \cap \boldsymbol{F}$, then it holds that $Sh_i(N,v) \leq \xi_i(\boldsymbol{F},N,v)$, for all $i \in S$, and $Sh_j(N,v) \geq \xi_j(\boldsymbol{F},N,v)$, for all $j \in N \backslash S$. This happens because in $\Omega(\boldsymbol{F},N)$ all orders for which players in S enter before than players of $N \backslash S$ are deleted. And in all of these orders, the marginal contributions of players in S are zero.

3 The value extension

To overcome the difficulties showed with the v_F modification and the $\Omega(\boldsymbol{F},N)$ restriction, we propose to make the random order process of coalition formation $\Omega(N)$ independent of the set system $\boldsymbol{F}$. Therefore, we assume that all orders $\Omega(N)$ are feasible, and the set system $\boldsymbol{F}$ will determine only which coalitions are formed when players arrive successively in every order.

For any order $\omega \in \Omega(N)$, there is a sequence of coalition formation made in $n+1$ steps, where $|N| = n$. Denote by $(\boldsymbol{F}_\omega^r)_{r=0,1,..,n}$ such sequence in $\omega \in \Omega(N)$. As initially there are

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[1] A game (N,v) is monotonic if $v(S) \leq v(T)$ whenever $S \subset T$.

no players, in Step 0, $\boldsymbol{F}_\omega^0 := \{\emptyset\}$. Every player $i \in N$ will arrive in Step $\omega(i)$. Let $\boldsymbol{F}_\omega^{\omega(i)}$ be the set of coalitions formed at Step $\omega(i)$. Because the set system $\boldsymbol{F}$ can have a very general structure, we need to specify in detail the rules that determine which coalitions are formed at every $\boldsymbol{F}_\omega^{\omega(i)}$.

Hence, for all $i \in N$, the rules are:

- *Feasibility:* All coalitions formed in every step must be feasible: $\boldsymbol{F}_\omega^{\omega(i)} \subset \boldsymbol{F}\left(P_\omega^i \cup i\right)$.

- *No Splitting:* If a feasible coalition is formed at some step r, in step $r+1$ either that coalition remains unchanged or it is transformed into a new one by adding players: If $S \in \boldsymbol{F}_\omega^{\omega(i)-1}$ then either $S \in \boldsymbol{F}_\omega^{\omega(i)}$ or $S \subset T \in \boldsymbol{F}_\omega^{\omega(i)}$. Therefore we cannot split coalitions previously formed when we want to build additional feasible coalitions.

- *Maximality:* If $S \in \boldsymbol{F}_\omega^{\omega(i)}$ then there does not exist $T \in \boldsymbol{F}\left(P_\omega^i \cup i\right)$ such that $S \subsetneq T$. Hence, when feasible, cooperation involves the maximun number of players,.

- *Arrival Preference:* When there are several feasible coalitions a player can join, the coalition ranked before under the arrival order of the players will be formed. That is, if $i \in S \in \boldsymbol{F}_\omega^{\omega(i)}$ then it cannot be true that there exists $T \in \boldsymbol{F}\left(P_\omega^i \cup i\right)$ and $j \in T$ such that $\omega(j) < \omega(k)$ for all $k \in S$.

For example, let $N = \{i, j, k, l, m\}$, $\boldsymbol{F} = \{\emptyset, \{i\}, \{i, j\}, \{i, j, k, m\}, \{i, j, l, m\}\}$, and $\bar{\omega} = (i, j, k, l, m)^2$. Then we have that $\boldsymbol{F}_\omega^{\omega(k)} = \boldsymbol{F}_\omega^{\omega(l)} = \{\emptyset, \{i, j\}\}$ and $\boldsymbol{F}_\omega^{\omega(m)} = \{\emptyset, \{i, j, k, m\}\}$ because $\omega(k) < \omega(l)$.

Applying the above rules, the process of coalition formation is well defined.

Proposition 1 *Under feasibility, no splitting, maximality and arrival preference, there exists a unique* $\boldsymbol{F}_\omega^r$ *for all* $\omega \in \Omega(N)$ *and* $r = 0, 1, .., n$.

Proof. Firstly, $\boldsymbol{F}_\omega^0 = \{\emptyset\}$ by definition. Let $\omega(i) = r$ and assume by induction that $\boldsymbol{F}_\omega^{r-1}$ exists and that it is unique. Consider the following possibilities: either (a) $P_\omega^i \cup i \in \boldsymbol{F}\left(P_\omega^i \cup i\right)$, or (b) $P_\omega^i \cup i \notin \boldsymbol{F}\left(P_\omega^i \cup i\right)$.

Assume that (a) is true. Then, by feasibility and maximality $\boldsymbol{F}_\omega^{\omega(i)} = \{\emptyset, P_\omega^i \cup i\}$.

Alternatively supose that (b) is true. Let $\boldsymbol{F}_\omega^{\omega(i)-1} = \left\{\emptyset, T_1, ..., T_{k(i)}\right\}$, where $\boldsymbol{T} = \left\{T_1, ..., T_{k(i)}\right\}$ is a partition of P_ω^i within $\boldsymbol{F}$ into maximal components (see Definition (1)). Here we have also two options:

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[2] $\bar{\omega}$ is the inverse of ω, that is $\bar{\omega}(r) = i \Longleftrightarrow \omega(i) = r$.

- (b.1) $\boldsymbol{F}_{\omega}^{\omega(i)-1}=\boldsymbol{F}_{\omega}^{\omega(i)}$, and then $\boldsymbol{F}_{\omega}^{\omega(i)}$ is well defined.

- (b.2) $\boldsymbol{F}_{\omega}^{\omega(i)-1}\neq\boldsymbol{F}_{\omega}^{\omega(i)}$. In this case there exists $S\in\boldsymbol{F}\left(P_{\omega}^{i}\cup i\right)$ such that $i\in S\in\boldsymbol{F}_{\omega}^{\omega(i)}$. By no breakness this implies either (1) $S\cap T_k=\emptyset$, for all $T_k\in\boldsymbol{T}$, or (2) $S\supset\left(\cup T_k\right)_{k\in I}$, where $I\subsetneq\{1,...,k(i)\}$.

If (1) is true, arrival preference and maximality determine S uniquely on $\boldsymbol{F}\left(P_{\omega}^{i}\cup i\right)\setminus\boldsymbol{T}$, and then $\boldsymbol{F}_{\omega}^{\omega(i)}=\left\{\emptyset,T_1,...,T_{k(i)},S\right\}$.

If (2) is true, arrival preference and maximality determine S uniquely on $\boldsymbol{F}\left(P_{\omega}^{i}\cup i\right)\setminus\boldsymbol{T}'$, where $\boldsymbol{T}'=\{T_k\}_{k\in I}$, and then $\boldsymbol{F}_{\omega}^{\omega(i)}=\{\emptyset,S\}\cup(\boldsymbol{T}\setminus\boldsymbol{T}')$. ■

Moreover, it is also easy to check that, for every $S\in\boldsymbol{F}$ and $i\in S$, there exists at least an order ω such that $S\in\boldsymbol{F}_{\omega}^{\omega(i)}$. Hence, any feasible coalition always has a positive chance to be formed, and all players of that coalition always have a positive chance to be the last player which arrives when such coalition is formed.

The marginal contribution of a player to his predecessors will be the diference between the total worth that can be produced with him minus the worth produced before he arrives.

Definition 4 *The* marginal contribution *that every player i receives in any order ω is defined by*

$$m_i^{\omega}\left(\boldsymbol{F},N,v\right):=\sum_{S\in\boldsymbol{F}_{\omega}^{\omega(i)}}v(S)-\sum_{S\in\boldsymbol{F}_{\omega}^{\omega(i)-1}}v(S),\quad\left(\omega\in\Omega(N),\ i\in N\right).\tag{3}$$

We illustrate these definitions with the next example.

Example 3. Let $N=\{i,j,k,l\}$ and the set system given by:

$$\boldsymbol{F}=\{\emptyset,\{i\},\{j\},\{i,j\},\{i,k\},\{j,k\},\{k,l\},\{i,j,l\}\}.$$

Consider the following evaluation of v at $\boldsymbol{F}$:

$$v(\{i\})=v(\{j\})=1,$$
$$v(\{i,j\})=v(\{i,k\})=v(\{j,k\})=v(\{k,l\})=4,$$
$$v(\{i,j,l\})=5.$$

Let the order ω be such that $\bar{\omega}=(i,j,k,l)$. Then we obtain the following sequence

$(\boldsymbol{F}_\omega^r)_{r=0,\ldots,4}$:

$$\begin{aligned}
&\text{Step } 0 &:& \quad \boldsymbol{F}_\omega^0 = \{\emptyset\}, \\
&\text{Step } 1 &:& \quad \boldsymbol{F}_\omega^1 = \{\emptyset, \{i\}\}, \quad m_i^\omega(\boldsymbol{F}, N, v) = v(\{i\}) = 1, \\
&\text{Step } 2 &:& \quad \boldsymbol{F}_\omega^2 = \{\emptyset, \{i,j\}\}, \quad m_j^\omega(\boldsymbol{F}, N, v) = v(\{i,j\}) - v(\{i\}) = 3, \\
&\text{Step } 3 &:& \quad \boldsymbol{F}_\omega^3 = \{\emptyset, \{i,j\}\}, \quad m_k^\omega(\boldsymbol{F}, N, v) = v(\{i,j\}) - v(\{i,j\}) = 0, \\
&\text{Step } 4 &:& \quad \boldsymbol{F}_\omega^4 = \{\emptyset, \{i,j,l\}\}, \quad m_l^\omega(\boldsymbol{F}, N, v) = v(\{i,j,l\}) - v(\{i,j\}) = 1.
\end{aligned}$$

In Step 2, we have $\boldsymbol{F}_\omega^2 = \{\emptyset, \{i,j\}\}$ instead of $\{\emptyset, \{i\}, \{j\}\}$ by maximality, since $\{i\} \subset \{i,j\}$. In Step 3, we have $\boldsymbol{F}_\omega^3 = \{\emptyset, \{i,j\}\}$ instead of $\{\emptyset, \{i\}, \{j,k\}\}$by no splitting. In Step 4, we have $\boldsymbol{F}_\omega^4 = \{\emptyset, \{i,j,l\}\}$instead of $\{\emptyset, \{i,j\}, \{k,l\}\}$ by arrival preference, since $\omega(j) = 2 < \omega(k) = 3$. Hence, the sequence of coalition formation for ω is

$$\{\emptyset\} \xrightarrow{i} \{\emptyset, \{i\}\} \xrightarrow{j} \{\emptyset, \{i,j\}\} \xrightarrow{k} \{\emptyset, \{i,j\}\} \xrightarrow{l} \{\emptyset, \{i,j,l\}\}.$$

Finally, the value is the expected marginal contribution of the player to its predecessors in every order, when all orders are equally likely.

Definition 5 *The value ϕ for the game $(\boldsymbol{F}, v)$ is defined by*

$$\phi_i(\boldsymbol{F}, N, v) := \frac{1}{n!} \sum_{\omega \in \Omega(N)} m_i^\omega(\boldsymbol{F}, N, v), \quad (i \in N). \tag{4}$$

We illustrate the computation of the value ϕ in the next example.

Example 4 Consider the game with player set $N = \{i,j,k\}$, set system $\boldsymbol{F} = \{\emptyset, \{j\}, \{i,j\}, \{j,k\}\}$, and characteristic function $v(\{j\}) = 1$, $v(\{i,j\}) = 5$, $v(\{j,k\}) = 3$. In the table below the marginal contributions for every order are computed:

orders	players		
$(\overline{\omega}(1), \overline{\omega}(2), \overline{\omega}(3))$	m_i^ω	m_j^ω	m_k^ω
(i,j,k)	0	5	0
(i,k,j)	0	5	0
(j,i,k)	4	1	0
(j,k,i)	0	1	2
(k,i,j)	0	3	0
(k,j,i)	0	3	0

Therefore, the payoffs are

$$\phi_i(\boldsymbol{F},N,v)=\frac{2}{3},\ \phi_j(\boldsymbol{F},N,v)=\frac{9}{3},\ \phi_k(\boldsymbol{F},N,v)=\frac{1}{3}.$$

Remark 1 *Equation (4) can be rewritten in a recursive way as follows*

$$\phi_i(\boldsymbol{F},N,v)=\frac{1}{n!}\sum_{\substack{\omega\in\Omega(N)\\ \omega(i)=n}} m_i^{\omega}\left(\boldsymbol{F},N,v\right)+\frac{1}{n}\sum_{j\in N\backslash i}\phi_i(\boldsymbol{F}(N\backslash j),N\backslash j,v),\quad (i\in N)\,.$$

Remark 2 *For all $S\subset N\backslash i$ denote by $|S|=s$, and let $\Omega\left(S,i\right):=\{\omega\in\Omega(N):P_i^{\omega}=S\}$, that is, $\Omega\left(S,i\right)$ is the set of all orders for which the set of predecessors of i is the coalition S. An alternative expression of (4) is given by*

$$\phi_i(\boldsymbol{F},N,v)=\frac{1}{n!}\sum_{S\subset N\backslash i}\sum_{\omega\in\Omega(S,i)} m_i^{\omega}\left(\boldsymbol{F},N,v\right),\quad (i\in N)\,. \tag{5}$$

When $\boldsymbol{F}$=2^N it holds that $m_i^{\omega}\left(\boldsymbol{F},v\right)=v(S\cup i)-v(S)$, for all $\omega\in\Omega\left(S,i\right)$. Note that $\Omega(N)=\cup_{S\subset N\backslash i}\Omega\left(S,i\right)$, and $|\Omega\left(S,i\right)|=s!\left(n-s-1\right)!$. Therefore (5) turns out to be the familiar expression of the Shapley value given by

$$\phi_i(2^N,N,v)=\sum_{S\subset N\backslash i}\frac{s!\left(n-s-1\right)!}{n!}\left[v(S\cup i)-v(S)\right],\quad (i\in N)\,.$$

Remark 3 *For the type of restrictions considered in Example 2, it always happens that $\phi(\boldsymbol{F},N,v)=Sh(N,v)$. To see this invariance property, let v be a game such that for some $S\subsetneq N$, it holds that $v(T)=0$ for all $T\subset S$. Take the set system $\boldsymbol{F}=2^N\backslash\left\{\{T\}_{T\subset S}\right\}$. Then for all $i\in S$ and $\omega\in\Omega(N)$ such that $P_{\omega}^{i}\cup i\subset S$ it holds that $\boldsymbol{F}_{\omega}^{\omega(i)}=\boldsymbol{F}_{\omega}^{\omega(i)-1}=\{\emptyset\}$, and then we have*

$$m_i^{\omega}\left(\boldsymbol{F},N,v\right)=v\left(\emptyset\right)-v\left(\emptyset\right)=0=v\left(P_{\omega}^{i}\cup i\right)-v\left(P_{\omega}^{i}\right)=m_i^{\omega}\left(N,v\right).$$

Remark 4 *There is a wide family of set systems for which the extension $v_{\boldsymbol{F}}$ is well defined and then $\phi(\boldsymbol{F},N,v)=Sh(N,v_{\boldsymbol{F}})$. This family is defined as follows. Given a set system $(\boldsymbol{F},N)$ and a coalition $S\subset N$, let $\Pi(\boldsymbol{F},S)$ be the set of partitions of S in $\boldsymbol{F}$ into maximal components as in Definition (1). Consider the family of set systems $(\boldsymbol{F},N)$ which satisfies:*

(S.1) $|\Pi(\boldsymbol{F},S)|=1$, for all $S\subset N$.

(S.2) If $S\subset T$ then for all $R\in\Pi(\boldsymbol{F},S)$ there exists $H\in\Pi(\boldsymbol{F},T)$ such that $R\subset H$.

Property (S.1) says that partition $\Pi(\boldsymbol{F},S)$ is always unique, and (S.2) says that when a *coalition T is formed from S by adding players, the coalitions of partition $\Pi(\boldsymbol{F},T)$ cannot be made by breaking coalitions of the partition $\Pi(\boldsymbol{F},S)$. Therefore, if a set system $(\boldsymbol{F},N)$ satisfies (S.1) and (S.2) then it is easy to check that $v_{\boldsymbol{F}}$ is well defined for all $v \in G^N$. Moreover, for any ordering $\omega \in \Omega(N)$ and player $i \in N$, we have that*

$$\begin{aligned} m_i^{\omega}(\boldsymbol{F},N,v) &= \sum_{S\in \boldsymbol{F}_{\omega}^{\omega(i)}} v(S) - \sum_{S\in \boldsymbol{F}_{\omega}^{\omega(i)-1}} v(S) = \sum_{S\in \Pi(\boldsymbol{F},P_{\omega}^{i}\cup i)} v(S) - \sum_{S\in \Pi(\boldsymbol{F},P_{\omega}^{i})} v(S) \\ &= v_{\boldsymbol{F}}(P_{\omega}^{i}\cup i) - v_{\boldsymbol{F}}(P_{\omega}^{i}) = m_i^{\omega}(N,v_{\boldsymbol{F}}), \end{aligned}$$

and then $\phi(\boldsymbol{F},N,v) = Sh(N,v_{\boldsymbol{F}})$.

The reader can check[3] that the set system induced by coalition structures (Aumann and Dreze, 1977), communication graphs (Myerson, 1977), conference structures (Myerson, 1980), union stable cooperation structures (Algaba et al., 2000; Bilbao, Jiménez and López, 2006), and partnerships (Carreras, 1996), satisfies (S.1) and (S.2) and hence ϕ coincides with the extension of the Shapley value given in these works.

For the rest of the cases cited in the Introduction (precedence constraints, incompatible players, convex geometries, matroids, antimatroids, and augmenting systems) ϕ will, in general, yield payoffs different from the values proposed in these works.

4 Axiomatic approach

We see now a set of properties that can be used to characterize our value extension.

4.1 Efficiency

For any order ω, let $\boldsymbol{F}_{\omega}^{n}$ be the set of feasible coalitions formed at final step n. Note that if $N \in \boldsymbol{F}$, then $\boldsymbol{F}_{\omega}^{n} = \{N\}$, for all $\omega \in \Omega$. The *expected final worth* that can be obtained by this order approach is

$$E\left[v,\boldsymbol{F}\right](N) = \frac{1}{n!} \sum_{\omega\in\Omega(N)} \sum_{S\in \boldsymbol{F}_{\omega}^{n}} v(S). \tag{6}$$

Definition 6 *(EF)* ***Expected efficiency:*** *A value ψ satisfies expected efficiency if*

$$\sum_{i\in N} \psi_i(\boldsymbol{F},N,v) = E\left[v,\boldsymbol{F}\right](N), \quad \textit{for all } \boldsymbol{F} \subset 2^N \textit{ and } (N,v) \in G^N.$$

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[3] This is the laborious part of the exercice.

This is a generalization of the classical efficiency property.

Proposition 2 *ϕ satisfies expected efficiency.*

Proof. By definition of (3), it is clear that for all $\omega \in \Omega(N)$ it holds that

$$\begin{aligned}\sum_{i\in N} m_i^{\omega}\left(\boldsymbol{F},N,v\right) &= \sum_{i\in N}\left[\sum_{S\in \boldsymbol{F}_{\omega}^{\omega(i)}} v(S) - \sum_{S\in \boldsymbol{F}_{\omega}^{\omega(i)-1}} v(S)\right] = \\ &= \sum_{r=1}^{n}\left[\sum_{S\in \boldsymbol{F}_{\omega}^{r}} v(S) - \sum_{S\in \boldsymbol{F}_{\omega}^{r-1}} v(S)\right] = \sum_{S\in \boldsymbol{F}_{\omega}^{n}} v(S).\end{aligned}$$

Therefore,

$$\sum_{i\in N}\phi_i\left(\boldsymbol{F},N,v\right) = \frac{1}{n!}\sum_{\omega\in\Omega(N)}\sum_{i\in N} m_i^{\omega}\left(\boldsymbol{F},N,v\right) = \frac{1}{n!}\sum_{\omega\in\Omega(N)}\sum_{S\in \boldsymbol{F}_{\omega}^{n}} v(S).$$

■

Let $\boldsymbol{F}^n = \cup_{\omega\in\Omega}\boldsymbol{F}_{\omega}^{n}$ be the set of feasible coalitions that have a positive chance to be formed at the final step for some order. In general, different orders ω will yield different final coalition configurations $\boldsymbol{F}_{\omega}^{n}$, but many of them will coincide, so we can define the class of equivalence $\boldsymbol{F}^n\left(\Omega(N)\right) := \{\boldsymbol{H}\subset \boldsymbol{F} : \boldsymbol{H} = \boldsymbol{F}_{\omega}^{n},\ \omega\in\Omega(N)\}$. The probability that every final configuration $\boldsymbol{H}$ happens is

$$p_{\boldsymbol{F}}(\boldsymbol{H}) = \frac{\left|\{\omega\in\Omega(N) : \boldsymbol{F}_{\omega}^{n} = \boldsymbol{H}\}\right|}{n!}.$$

Hence, the expected final worth can also be rewritten as

$$E\left[v,\boldsymbol{F}\right](N) = \sum_{\boldsymbol{H}\in \boldsymbol{F}(\Omega)} p_{\boldsymbol{F}}(\boldsymbol{H})\cdot\left[\sum_{S\in \boldsymbol{H}} v(S)\right].$$

In Example 4, we have $\boldsymbol{F}^n = \{\{i,j\},\{j,k\}\}$, $\boldsymbol{F}^n\left(\Omega\right) = \{\boldsymbol{H}_1,\boldsymbol{H}_2\}$, where $\boldsymbol{H}_1 = \{\{i,j\}\}$, and $\boldsymbol{H}_2 = \{\{j,k\}\}$. The probabilities are $p_{\boldsymbol{F}}(\boldsymbol{H}_1) = p_{\boldsymbol{F}}(\boldsymbol{H}_2) = 1/2$, and

$$E\left[v,\boldsymbol{F}\right](N) = \frac{1}{2}v(\{i,j\}) + \frac{1}{2}v\left(\{j,k\}\right) = 4.$$

Hence,

$$\phi_i\left(\boldsymbol{F},N,v\right) + \phi_j\left(\boldsymbol{F},N,v\right) + \phi_k\left(\boldsymbol{F},N,v\right) = \frac{2}{3} + \frac{9}{3} + \frac{1}{3} = 4 = E\left[v,\boldsymbol{F}\right](N).$$

4.2 Additivity

Definition 7 *(A)* ***Additivity:*** *A value* ψ *satisfies additivity if*

$$\psi(\boldsymbol{F},N,v+w) = \psi(\boldsymbol{F},N,v) + \psi(\boldsymbol{F},N,w), \quad \textit{for all } \boldsymbol{F} \subset 2^N \textit{ and } (N,v),(N,w) \in G^N.$$

Proposition 3 ϕ *satisfies Additivity.*

The proof of this proposition is straightforwad given definition (4).

Note that $\{(\boldsymbol{F},N,v) : \boldsymbol{F} \subset \mathbf{2}^N, (N,v) \in G^N\}$ is a vector space of dimension $|\boldsymbol{F}|-1$. A unanimity game u_S is defined by $u_S(T)=1$, if $T \supset S$; $u_S(T)=0$, otherwise. It is easy to check that the set of unanimity games $\{(\boldsymbol{F},N,u_S) : S \in \boldsymbol{F}\}$ forms a basis in this space, i.e.:

$$(\boldsymbol{F},N,v) = \sum_{S \in \boldsymbol{F}} d_S \cdot (\boldsymbol{F},N,u_S),$$

where, for all $S \in \boldsymbol{F}$,

$$d_S = v(S) - \sum_{\substack{T \subsetneq S \\ T \in \boldsymbol{F}}} d_T, \text{ starting with } d_\emptyset = 0.$$

Additivity implies that ϕ is completely determined by fixing the payoffs at unanimity games:

$$\phi(\boldsymbol{F},N,v) = \sum_{S \in \boldsymbol{F}} d_S \cdot \phi(\boldsymbol{F},N,u_S).$$

It is instructive to compute ϕ in Example 4 with this type of decomposition. The game can be split into the sum of unanimity games as

$$(\boldsymbol{F},N,v) = (\boldsymbol{F},N,u_{\{j\}}) + 4 \cdot (\boldsymbol{F},N,u_{\{i,j\}}) + 2 \cdot (\boldsymbol{F},N,u_{\{j,k\}}),$$

and the marginal contributions in these games are:

$u_{\{j\}}$				$u_{\{i,j\}}$			$u_{\{j,k\}}$		
orders	players			players			players		
ϖ	m_i^ω	m_j^ω	m_k^ω	m_i^ω	m_j^ω	m_k^ω	m_i^ω	m_j^ω	m_k^ω
(i,j,k)	0	1	0	0	1	0	0	0	0
(i,k,j)	0	1	0	0	1	0	0	0	0
(j,i,k)	0	1	0	1	0	0	0	0	0
(j,k,i)	0	1	0	0	0	0	0	0	1
(k,i,j)	0	1	0	0	0	0	0	1	0
(k,j,i)	0	1	0	0	0	0	0	1	0

Which yields payoffs given by:

$$\begin{aligned}\phi(\boldsymbol{F},N,v) &= \phi(\boldsymbol{F},N,u_{\{j\}})+4\cdot\phi(\boldsymbol{F},N,u_{\{i,j\}})+2\cdot\phi(\boldsymbol{F},N,u_{\{j,k\}})\\ &= (0,1,0)+4\cdot(1/6,2/6,0)+2\cdot(0,2/6,1/6)=(2/3,9/3,1/3).\end{aligned}$$

4.3 Proportionality in Unanimity Games

When we have full cooperation, $F=2^N$, two players $i,j\in N$, are said to be *substitute players* in a game $v\in G^N$, if

$$v(S\cup i)=v(S\cup j),\quad \text{for all } S\subset N\backslash\{i,j\}.$$

This symmetry property can be used in the characterization of the Shapley value in G^N. In a unanimity game u_S any two players $i,j\in S$ are substitutes, so when $\boldsymbol{F}=\mathbf{2}^N$, from the point of view of the characteristic function, there is no particular reason to distinguish between them, and then their payoffs should be equal. But when $\boldsymbol{F}\neq 2^N$, players of S will have different impact in the completion of S, because, in general, not all supercoalitions of S are feasible.

Let $i(\boldsymbol{F},S)$ be the number of orders for which player i is needed for the completion of S given $\boldsymbol{F}$, i.e.:

$$i(\boldsymbol{F},S):=\left|\left\{\omega\in\Omega(N):\left(P_\omega^i\cup i\right)\in\boldsymbol{F}\text{ and } S\subset P_\omega^i\cup i\right\}\right|,\qquad (i\in S\in\boldsymbol{F}).$$

In these orders we say that $i\in S$ is a *decisive player*, and $i(S)$ is a measure of the impact that player i has in the completion of S given $\boldsymbol{F}$. If $\boldsymbol{F}\neq 2^N$, we have in general that $i(\boldsymbol{F},S)\neq j(\boldsymbol{F},S)$, $i,j\in S$, and then players in S are not necessarily symmetric from the point of view of this coalition formation process. Therefore, in a unanimity game u_S the payoffs of the players in the basis S should be *proportional*[4] to this impact in the formation of coalition S.

Definition 8 *(PUG) A value ψ satisfies* proportionality in unanimity games *if*

$$\psi_i(\boldsymbol{F},N,u_S)\cdot j(\boldsymbol{F},S)=\psi_j(\boldsymbol{F},N,u_S)\cdot i(\boldsymbol{F},S),\qquad \textit{for all } i,j\in S\in\boldsymbol{F}.$$

Proposition 4 *ϕ satisfies proportionality in unanimity games*

---

[4] This is the *hieralchical strength axiom* defined in Faigle and Kern (1992).

Proof. It is easy to check that for all $i \in S \in \boldsymbol{F}$, and u_S, it holds that

$$i(\boldsymbol{F}, S) = \left|\left\{\omega \in \Omega(N) : m_i^{\omega}\left(\boldsymbol{F}, N, u_S\right) = 1\right\}\right|,$$

and then

$$\phi_i(\boldsymbol{F}, N, u_S) = \frac{i(\boldsymbol{F}, S)}{n!},$$

so the value ϕ satisfies PUG. ■

4.4 Null Player

The original definition of a null player in a game $v \in G^N$ for the full set system 2^N is as follows: $i \in N$ is a null player in $\left(2^N, v\right)$, if $v(S \cup i) = v(S)$, for all $S \subset N \backslash i$. We extend this definition for every set system $\boldsymbol{F} \subset 2^N$. We say that $i \in N$ is a *null player* in $(\boldsymbol{F}, v)$ if

$$\sum_{S \in \boldsymbol{F}_{\omega}^{\omega(i)}} v(S) - \sum_{S \in \boldsymbol{F}_{\omega}^{\omega(i)-1}} v(S) = 0, \text{ for all } \omega \in \Omega(N).$$

Definition 9 *A value ψ satisfies* null player *if*

$$\psi_i(\boldsymbol{F}, N, v) = 0, \quad \textit{when } i \textit{ is a null player in } (\boldsymbol{F}, N, v).$$

Proposition 5 *ϕ satisfies null player*

Proof. It is straightforward given the definition of ϕ. ■

4.5 Axiomatization

Now we are ready to establish the main theorem of this paper.

Theorem 1 *A value ψ satisfies the properties of expected efficiency, additivity, null player and proportionality at unanimity games, if and only if $\psi(\boldsymbol{F}, N, v) = \phi(\boldsymbol{F}, N, v)$, for all $\boldsymbol{F} \subset 2^N$, and $(N, v) \in G^N$.*

Proof. *Existence.* By the Propositions (1)-(4) above, we know that ϕ satisfies the axioms.

Uniqueness. Let ψ be a solution on $(\boldsymbol{F}, N, v)$ which satisfies the axioms. By additivity

$$\psi(\boldsymbol{F}, N, v) = \sum_{S \in \boldsymbol{F}} d_S \cdot \psi(\boldsymbol{F}, N, u_S).$$

For each $S \in \boldsymbol{F}$, and $j \notin S$, j is a null player at u_S, so

$$\psi_j(\boldsymbol{F}, N, u_S) = 0 = \phi_j(\boldsymbol{F}, N, u_S).$$

Now we have that the expected worth of N at u_S is

$$E\left[u_S, \boldsymbol{F}\right](N) = \frac{1}{n!}\sum_{j \in S} j(\boldsymbol{F}, S),$$

then, by efficiency $\sum_{j \in S} \psi_j(\boldsymbol{F}, N, u_S) = E\left[u_S, \boldsymbol{F}\right](N)$, and by proportionality at unanimity games

$$\frac{\psi_j(\boldsymbol{F}, N, u_S)}{\psi_i(\boldsymbol{F}, N, u_S)} = \frac{j(\boldsymbol{F}, S)}{i(\boldsymbol{F}, S)}$$

for all $i, j \in S$, which implies that

$$\sum_{j \in S} \frac{\psi_j(\boldsymbol{F}, N, u_S)}{\psi_i(\boldsymbol{F}, N, u_S)} = \frac{E\left[u_S, \boldsymbol{F}\right](N)}{\psi_i(\boldsymbol{F}, N, u_S)} = \sum_{j \in S} \frac{j(\boldsymbol{F}, S)}{i(\boldsymbol{F}, S)} = \frac{\frac{1}{n!}\sum_{j \in S} j(\boldsymbol{F}, S)}{\frac{1}{n!} i(\boldsymbol{F}, S)},$$

and finally, $\psi_i(\boldsymbol{F}, N, u_S) = \frac{1}{n!} i(\boldsymbol{F}, S) = \phi_i(\boldsymbol{F}, N, u_S)$, for all $i \in S$. So

$$\psi(\boldsymbol{F}, N, v) = \sum_{S \in \boldsymbol{F}} d_S \cdot \psi(\boldsymbol{F}, N, u_S) = \sum_{S \in \boldsymbol{F}} d_S \cdot \phi(\boldsymbol{F}, N, u_S) = \phi(\boldsymbol{F}, N, v).$$

■

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