

# The Serial Principle and Restricted Balanced Contributions in Discrete Cost Sharing Problems.\*

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## Abstract

We show that the axioms of Efficiency, Serial Principle and Restricted Balanced Contributions, characterize the Moulin's rule (Moulin, 1995) in discrete cost allocation problems.

JEL Classification: C71, D63.

Key words: Cost allocation. Serial cost rule. Shapley value.

## 1 Introduction

We consider the problem of dividing the cost of producing a set of goods among a group of agents, where each good is demanded by only one agent. Thus we can identify each agent with his/her good. This is the *cost sharing* interpretation of the problem. Alternatively, the *surplus sharing* interpretation is also possible: Each agent offer a particular input in the production function, and we wish to divide the total output among the agents. All of this problems can be modeled by a cooperative game in characteristic form: We have a finite set of players  $N = \{1, 2, \dots, n\}$ , and a real valued function  $C : \mathbb{R}_+^N \rightarrow \mathbb{R}$ , with  $C(\theta) = 0$ , where  $\theta = (0, \dots, 0)$ . The problem consists in dividing the worth  $C(q)$  between the  $n$  players for any level  $q = (q_1, \dots, q_n)$  at which the players can participate in the game. *Discrete* cost allocation problems arise when goods are produced in indivisible units, and the level of production is measured by an integer.

The subject of interest in this work is the study of the *Serial cost sharing rule*. This allocation mechanism was introduced in Moulin and Shenker (1992) for the homogeneous case. Moulin (1995) proposes an extension of this mechanism for the discrete-heterogeneous case. In that paper, he characterized this mechanism by using additivity, dummy, symmetry, upper bound, demand monotonicity and cross monotonicity. In this short note we use the serial principle and a restriction of the balanced contributions axiom (Myerson, 1980) to provide an alternative axiomatic characterization. In this way it can be compared several solutions by comparing the differences in the definition of this axiom.

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## 2 The serial cost sharing rule: a characterization

Let  $U$  be the universe of goods and  $N = \{1, 2, \dots, n\}$  a finite set,  $\emptyset \neq N \subset U$ . Let  $e^S \in \mathbb{R}^N$  be defined by  $e_i^S = 1$  if  $i \in S$  and  $e_i^S = 0$  if  $i \in N \setminus S$ , for all  $S \subseteq N$ ; in the particular case  $S = \{j\}$ , we will write  $e^j$  instead of  $e^{\{j\}}$ . Given two vectors  $x, y \in \mathbb{R}^N$ , we define the new vector  $(x \wedge y)$  by  $(x \wedge y)_j = \min\{x_j, y_j\}$ , for any  $j \in N$ ;  $x \leq y$  means that  $x_j \leq y_j$ , for all  $j \in N$ . The positive integers are denoted as usually by  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ .

A *discrete cost allocation problem* (discrete CAP in the following) is a pair  $(q; C)$ , where  $q \in \mathbb{Z}_+^N$ , and  $C$  is a real-valued function  $C : \mathbb{Z}_+^N \rightarrow \mathbb{R}_+$ , such that  $C(\theta) = 0$ , and  $C(p) \leq C(q)$  if  $p \leq q$ .  $Q^N$  denotes the set of all discrete CAP with set of goods  $N$ , and  $Q = \cup_{N \subset U} Q^N$ . A *cost allocation procedure* on  $Q$  is a function  $\eta : Q \rightarrow \cup_{N \subset U} \mathbb{R}_+^N$  such that if  $(q; C) \in Q^N$  then  $\eta(q; C) \in \mathbb{R}_+^N$ . A survey on the discrete problem and different solutions for it can be found in Moulin (1995).<sup>1</sup> In that paper he proposed a cost allocation procedure, that we call the *Moulin discrete serial cost sharing* method, and he characterized it axiomatically. Roughly speaking<sup>2</sup>, the rule assigns to each player an average of the cost increments due to its demand increments along the truncated discrete diagonal,  $(t \cdot e^N) \wedge q$ , where  $t \in \mathbb{Z}_+$ .

In order to characterize the Moulin's solution on discrete CAP we consider the following axioms:

Let  $(q; C)$  be a discrete CAP,  $q \in \mathbb{Z}_+^N$ , and let  $\eta$  be a cost allocation procedure on discrete CAP.

**Axiom 1 (E)** *Efficiency.*

$$\sum_{i=1}^n \eta_i(q; C) = C(q).$$

**Axiom 2 (SP)** *Serial Property.* If  $q_i < q_j$ , then

$$\eta_i(q; C) - \eta_i(q - e^j; C) = 0.$$

That is, decreasing one unit the demand of  $q_j$  has null effect in the cost share of agent  $i$ , when  $q_i < q_j$ .

But what should happens if  $q_i = q_j$ ? The next axiom is a *fairness* property which specifies that this impact must be the same for both agents:

**Axiom 3 (RBC)** *Restricted Balanced Contributions.* If  $q_i = q_j > 0$ , then

$$\eta_i(q; C) - \eta_i(q - e^j; C) = \eta_j(q; C) - \eta_j(q - e^i; C).$$

Note that when  $q_i \in \{0, 1\}$ , for all  $i \in N$ , we are in the case of a Transferable Utility game. In this case RBC is just the *Balanced Contributions* axiom of Myerson (1980), that joint with Efficiency characterizes the Shapley value (Shapley, 1953) in this particular context.

Next Theorem shows that this set of axioms provides a full characterization of the Moulin discrete serial rule.

**Theorem 4** *A cost allocation procedure  $\eta$  on  $Q$  satisfies E, SP and RBC if and only if  $\eta = M$ .*

<sup>1</sup>The discrete case is also considered in Nouweland et al. (1995), Klijn et al. (1999), Calvo and Santos (2000) and Sprumont (2000).

<sup>2</sup>The reader is referred to Moulin (1995) for its original definition.

To prove the theorem, first we show that E with SP imply the next *Null Level* property (i.e.,  $\eta_i(q; C) = 0$  if  $q_i = 0$ ).

**Lemma 5** *Let  $\eta$  be a cost allocation procedure on  $Q$  satisfying E and SP, then it is satisfied the Null Level property.*

**Proof.** Note that by definition  $C(\theta) = 0$  and  $\eta_i(q; C) \geq 0$ , for all  $q$  and  $i$ . Then Efficiency implies that  $\eta_i(\theta; C) = 0$ . Let  $i$  such that  $q_i = 0$ , applying repeatedly SP, we have that  $\eta_i(q; C) - \eta_i(\theta; C) = 0$ , and the proof is complete. ■

Now, we show an alternative way to compute the Moulin discrete serial rule, that will be use in the proof of the theorem. In Albizuri *et al.* (2002), it was proved that the Moulin discrete serial cost rule is equivalent to compute the Shapley value of an associated TU game with set of players  $N$ . Formally, let  $(q; C) \in Q^N$  be a discrete CAP. The  $N$  person TU game  $w^{(q; C)}$  is defined, for each  $T \subseteq N$ , by:

$$w^{(q; C)}(T) = \sum_{s=0}^{q_{\max}-1} (C[(s \cdot e^N + e^T) \wedge q] - C[s \cdot e^N \wedge q]).$$

where  $q_{\max} := \max_{i \in N} q_i$ . Then it holds that

$$Sh(N; w^{(q; C)}) = M(q; C), \tag{A.1}$$

where  $Sh$  denotes the Shapley value and  $M$  the Moulin discrete serial cost sharing method.

Note that this result can be considered as a summation of Shapley values applied to TU games defined at each stage  $s \in \{0, 1, \dots, q_{\max} - 1\}$  as follows: For all  $s$  and all  $T \subseteq N$  define

$$w_s^{(q; C)}(T) = C[(s \cdot e^N + e^T) \wedge q] - C[s \cdot e^N \wedge q].$$

Then  $Sh(N; w^{(q; C)}) = \sum_{s=0}^{q_{\max}-1} Sh(N; w_s^{(q; C)})$ . Therefore, for each  $i \in N$ , this implies that

$$M_i(q; C) = \sum_{s=0}^{q_i-1} Sh_i(N; w_s^{(q; C)}), \tag{A.2}$$

since  $Sh_i(N; w_s^{(q; C)}) = 0$  for all  $s \geq q_i$ .

*Proof of the theorem.* First we show that  $M$  satisfies the axioms. E is straightforward from (A.1) since  $w^{(q; C)}(N) = C(q)$  and the Shapley value satisfies efficiency.

To see RBC, from (A.2) we have that  $M_i(q; C) = \sum_{s=0}^{q_i-1} Sh_i(N; w_s^{(q; C)})$ , for each  $i \in N$ . Hence, if  $q_i = q_j$ , we have

$$M_i(q; C) - M_i(q - e^j; C) = Sh_i(N; w_{q_i-1}^{(q; C)}) - Sh_i(N; w_{q_i-1}^{(q-e^j; C)})$$

and

$$M_j(q; C) - M_j(q - e^i; C) = Sh_j(N; w_{q_i-1}^{(q; C)}) - Sh_j(N; w_{q_i-1}^{(q-e^i; C)}).$$

Then, the axiom holds because the Shapley value satisfies balanced contributions in the game  $(N; w_{q_i-1}^{(q;C)})$ .

To see that  $M$  satisfies SP note that, if  $q_i < q_j$ , it holds that  $w_s^{(q;C)} = w_s^{(q-e^j;C)}$ , for all  $0 \leq s \leq q_i - 1$ . Then we have

$$M_i(q - e^j; C) = \sum_{s=0}^{q_i-1} Sh_i(N; w_s^{(q-e^j;C)}) = \sum_{s=0}^{q_i-1} Sh_i(N; w_s^{(q;C)}) = M_i(q; C),$$

and the axiom holds.

To prove uniqueness, let  $\eta$  be a cost allocation procedure that satisfies the axioms, and  $(q; C) \in Q^N$  be a discrete CAP. If  $q = \theta$ , by Lemma 5 it holds that  $\eta(\theta) = \theta = M$ . For  $q \neq \theta$  we use an induction hypothesis on  $q(N) := q_1 + \dots + q_n$ .

If  $q(N) = 1 = q_i$ , Lemma 5 implies that  $\eta_j(q; C) = 0$ , for all  $j \in N \setminus \{i\}$ . Then E implies that  $\eta_i(q; C) = C(q)$ , and consequently,  $\eta = M$ .

Now, let  $(q; C)$  be a discrete CAP and let  $M(q) = \{i \in N : q_i = q_{\max}\}$ . By induction hypothesis

$$\eta_i((q - e^j); C) = M_i((q - e^j); C),$$

for all  $i, j \in M(q)$ .

Let  $h \in M(q)$  and  $i \in N$ . If  $i \notin M(q)$ , then  $q_h > q_i$ , and SP with the induction hypothesis (*ih*) implies

$$\eta_i(q; C) \stackrel{(SP)}{=} \eta_i(q - e^h; C) \stackrel{(ih)}{=} M_i(q - e^h; C) \stackrel{(SP)}{=} M_i(q; C). \quad (1)$$

If  $i \in M(q)$ , then  $q_h = q_i$ , and the RBC axiom implies

$$\eta_i(q; C) - \eta_h(q; C) = \eta_i(q - e^h; C) - \eta_h(q - e^i; C),$$

and

$$M_i(q; C) - M_h(q; C) = M_i(q - e^h; C) - M_h(q - e^i; C).$$

Hence, by induction hypothesis,

$$M_i(q - e^h; C) = \eta_i(q - e^h; C),$$

and

$$M_h(q - e^i; C) = \eta_h(q - e^i; C),$$

then

$$\eta_i(q; C) - \eta_h(q; C) = M_i(q; C) - M_h(q; C).$$

Now, adding in  $i \in M(q)$ , we have

$$\sum_{i \in M(q)} (\eta_i(q; C) - M_i(q; C)) = m(q) \cdot (\eta_h(q; C) - M_h(q; C)), \quad (2)$$

where  $m(q) = |M(q)| \neq 0$ .

Moreover, from (1),

$$\sum_{i \in N \setminus M(q)} (\eta_i(q; C) - M_i(q; C)) = 0, \quad (3)$$

and by E,

$$\sum_{i \in N} (\eta_i(q; C) - M_i(q; C)) = 0.$$

Now, by using equality (2) and (3), we have

$$0 = \sum_{i \in N} (\eta_i(q; C) - M_i(q; C)) = m(q) \cdot (\eta_h(q; C) - M_h(q; C)).$$

Then  $\eta_h(q; C) = M_h(q; C)$ , for each  $h \in M(q)$ , and the proof is complete.

### 3 Comparisons

Other well known solutions for discrete CAP has been characterized by using similar “adaptations” of the balanced contributions axiom.

In the discrete Aumann-Shapley rule (Moulin, 1995), it is used (see Calvo and Santos, 2000):

**Axiom 6 (BC)** Balanced Contributions: *If  $q_i, q_j > 0$ , then*

$$\frac{[\eta_i(q; C) - \eta_i(q - e^j; C)]}{q_i} = \frac{[\eta_j(q; C) - \eta_j(q - e^i; C)]}{q_j}.$$

The term  $[\eta_i(q; C) - \eta_i(q - e^j; C)]$  is the cost variation in the production of  $q_i$  units of  $i$  when the level of production of good  $j$  diminishes in one unit. Hence, the left hand of the above inequality is the  $j$ 's *marginal contribution* to  $i$ 's *per unit cost* at level  $q$  of production. The axiom says these marginal per unit cost contributions between pairs of goods must be equal.

In the Shapley-Shubik rule (Shubik, 1962), it is used the following axiom<sup>3</sup>:

**Axiom 7 (TBC)** Totally-balanced Contributions: *If  $q_i, q_j > 0$ , then*

$$\eta_i(q; C) - \eta_i(q - q_j \cdot e^j; C) = \eta_j(q; C) - \eta_j(q - q_i \cdot e^i; C).$$

Here, the left hand of the inequality is the  $j$ 's *contribution* to  $i$ 's *total cost* at level  $q$  of production, when we pass to produce 0 from  $q_j$  units of good  $j$ .

In the pseudo-average cost method (Moulin, 1995), it is used (see Larrea and Santos, 2001):

**Axiom 8 (MBC)** Mean-balanced Contributions: *If  $q_i, q_j > 0$ , then*

$$\sum_{a_j=0}^{q_j-1} (\eta_i(q; C) - \eta_i(q - a_j e^j; C)) = \sum_{a_i=0}^{q_i-1} (\eta_j(q; C) - \eta_j(q - a_i e^i; C)).$$

Now, in the left hand we compute the sum of the  $j$ 's *contribution* to  $i$ 's *total cost* at level  $q$  of production, when we pass to produce *all possible intermediate* levels between 0 to  $q_j - 1$  units of good  $j$ .

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<sup>3</sup>That with efficiency and this axiom it is characterized the Shapley-Shubik rule follows directly of the Myerson (1980) characterization of the Shapley value in TU-games.

The following table resume all these characterizations:

rules\Axioms	<i>E</i>	<i>SP</i>	<i>RBC</i>	<i>BC</i>	<i>TBC</i>	<i>MBC</i>
<b>Moulin Serial</b>	x	x	x	-	-	-
<b>Aumann-Shapley</b>	x	-	-	x	-	-
<b>Shapley-Shubik</b>	x	-	-	-	x	-
<b>Pseudo-Average</b>	x	-	-	-	-	x

## 4 References

Albizuri M. J., Santos J. C., and Zarzuelo J. M. (2002). "On the Serial Cost Sharing Rule". International Journal of Game Theory 31, 437-446.

Calvo E., and Santos J. C. (2000). "A Value for Multichoice Games," Mathematical Social Sciences 40, 341-354.

Klijn, F., Slikker, M., and Zarzuelo, J.M. (1999). "Characterizations of a Multi-Choice Value," International Journal of Game Theory 28, 521-532.

Larrea C., and Santos J. C. (2001). "On the Pseudo-Average Cost Method," Discussion Papers, Department of Applied Economics IV, Basque Country University.

Moulin H. (1995). "On Additive Methods to Share Joint Costs," The Japanese Economic Review 46, 303-332.

Moulin H., and Shenker S. (1992). "Serial Cost Sharing," Econometrica 60, 1009-1037.

Myerson R. (1980). "Conference Structures and Fair Allocation Rules," International Journal of Game Theory 9, 169-182.

Nouweland A. Van den, Potters J., Tijs S., and Zarzuelo J. M. (1995). "Cores and Related Solution Concepts for Multi-choice Games," ZOR-Mathematical Methods of Operations Research 41, 289-311.

Shapley L. S. (1953). "A Value for n-Person Games," in Contributions to Theory of Games II (Annals of Mathematics Studies, 28) ed. by H. W. Kuhn and A. W. Tucker. Princeton University Press 307-317.

Shubik, M. (1962). "Incentives, Decentralized Control, the Assignment of Joint Costs and Internal Pricing," Management Science 8, 325-343.

Sprumont, Y. (2000). "Coherent Cost-Sharing Rules," Games and Economic Behavior 33(1), 126-144.