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Random marginal and random removal values

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Abstract We propose two variations of the non-cooperative bargaining model for games in coalitional form, introduced by Hart and Mas-Colell (Econometrica 64:357–380, 1996a). These strategic games implement, in the limit, two new NTU-values: the random marginal and the random removal values. Their main characteristic is that they always select a unique payoff allocation in NTU-games. The random marginal value coincides with the Consistent NTU-value (Maschler and Owen in Int J Game Theory 18:389–407, 1989) for hyperplane games, and with the Shapley value for TU games (Shapley in In: Contributions to the theory of Games II. Princeton University Press, Princeton, pp 307–317, 1953). The random removal value coincides with the solidarity value (Nowak and Radzik in Int J Game Theory 23:43–48, 1994) in TU-games. In large games we show that, in the special class of market games, the random marginal value coincides with the Shapley NTU-value (Shapley in In: La Décision. Editions du CNRS, Paris, 1969), and that the random removal value coincides with the equal split value.

Keywords Shapley value · Solidarity value · NTU-games · Large market games

JEL Classification C71

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1 Introduction

This paper is concerned with the Shapley value extension program.

When utility is transferable across players (TU-games) the most prominent solution concept is the Shapley (1953) *value*. It yields a unique payoff allocation. Shapley's original support for the value was justified by their axiomatic properties. Other axiomatizations are due to Myerson (1980), and to Hart and Mas-Colell (1989). Bargaining models that yield the value in the TU-case have also been proposed. Among these, we should mention Gul (1989), Hart and Moore (1990), Hart and Mas-Colell (1996a), Winter (1994), and Perez-Castrillo and Wettstein (2001).

There are different ways to extend the value when utility is not transferable (NTU-games). The most well known have been suggested by Harsanyi (1963), Shapley (1969),[1] and Maschler and Owen (1992).[2] These three solutions were constructed in such a way that they coincide with the Nash (1950) solution for pure bargaining games and with the Shapley value for TU-games. Axiomatic support for these solutions has been obtained by Aumann (1985) for the Shapley NTU-solution, by Hart (1985) for the Harsanyi NTU-solution, and by de Clippel et al. (2004) and by Hart (2005) for the Consistent NTU-solution. Bargaining games that support these solutions have been proposed for the Consistent NTU-solution in Hart and Mas-Colell (1996a,b), and for the Shapley NTU-solution in Vidal-Puga (2008) (only for the particular case in which the boundary of the grand coalition is a hyperplane).

In this paper we wish to preserve two properties in the value extension program: *Single-valuedness* and *symmetry*. Recall that both, the Shapley value and the Nash bargaining solution, satisfy symmetry and select single payoffs in TU-games, and pure bargaining games, respectively. Nevertheless, the Harsanyi, Shapley, and Consistent NTU-solutions *do not guarantee uniqueness in the solution set*. In particular, it is easy to build examples of *symmetric games* where the application of these three solution concepts results in *asymmetric equilibrium payoffs*. In these cases, where do these asymmetric payoffs come from?

In our view, there is no convincing justification for these payoffs from an axiomatic point of view (there is a contradiction with the symmetry property). So we take a look on the other side of the *Nash program:* The strategic approach, which tries to offer a noncooperative framework giving rise to the cooperative solution payoffs as a result of the players' equilibrium behavior.

In the setting of NTU-games, the only relevant proposal, up to now, has been the bargaining procedure due to Hart and Mas-Colell (1996a). This model is an elegant and simple variation of the alternating offers method, which supports the Consistent NTU-solution. Moreover, when applied to a TU-game, the Shapley value is obtained, whereas the Nash bargaining solution follows when it is a pure bargaining game.[3]

The Hart and Mas-Colell bargaining procedure goes as follows:

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1 The Shapley NTU-value is also known as λ-transfer value.

2 First introduced for hyperplane games in Maschler and Owen (1989), and also called the Consistent NTU-value.

3 Unfortunately, there are not bargaining models that can be associated in a parallel way to the Shapley and Harsanyi NTU-solutions.

> There is a fixed parameter ρ ($0 \leq \rho < 1$). In each round there is a set of "active" players,[4] and a "proposer" which is chosen randomly (from uniform distribution) among them. In the first round, all players are active. The proposer makes a feasible offer. If the rest of the players accept it, then the process ends with this offer as the final payoff. If it is rejected by even one player, we move to the next round where, with probability ρ, the set of active players is the same and, with probability $(1 - \rho)$, the proposer drops out of the game (receiving a payoff of zero), and the remaining players becomes the new active set.

This bargaining procedure is a sequential, perfect information game, and it has stationary subgame perfect equilibrium. Moreover, when the probability ρ goes to one, every limit of the noncooperative equilibria belongs to the set of *consistent payoffs* of the NTU-game.

Our claim is that there is a crucial step in the design of the bargaining model that is responsible for the origin of asymmetric payoffs, and we can enunciate it by using the same authors' words (see Hart and Mas-Colell 1996a,b, Sect. 1):

> The key modeling issue is the specification of what happens if there is no agreement and, as a consequence, the game moves to the next stage. It is at this point that subgroups are made to matter by allowing for the possibility of *partial* breakdown of negotiations. Clearly, there are many ways to model such a partial breakdown. In the body of this paper we concentrate on a particular and simple class: disagreement puts only the proposer in jeopardy. That is, after his proposal is rejected, the proposer may cease to be an active participant.

Note that the cause of a rejection is due to the fact that the proposer offers less than what the responder expects to obtain. But who is the player responsible for such a breakdown? The proposer, offering too little, or the responder, claiming too much? An anonymous rule should not specify who is to blame for this breakdown, except when the rule itself computes what the *right offer must be*; but in that case the rule determines directly the right outcome without the players' help. The Hart and Mas-Colell model identifies the proposer as being the only player responsible for the lack of agreement, giving him a chance $(1 - \rho)$ to leave the game after a rejected proposal.

In this paper we show that it is possible to yield strategic support for single-valued NTU-solutions, which also satisfy symmetry (i.e., yielding symmetric payoffs in symmetric games). For this purpose we only need to change the breakdown procedure in such a way that, after rejection, *the probability of leaving the game be the same for all players (proposer and responders)*, thus making all of them responsible for the lack of agreement. There are several ways in which this can be done. We show two of them.

The simplest way to make this type of modification was already mentioned in Hart and Mas-Colell (1996a), Sect. 6. There, they propose several modifications of the bargaining procedure, and this is case (d) from their list:

[4] From now on, we interpret players in a game as *agents* with *neutral gender*. They can be interpreted as automata, institutions or so on. Therefore, we will avoid choosing their gender every time.

– *Random removal.* All players (proposer and responders) drop out with equal probability. The player that leaves the game receives a zero payoff, and the rest restart the bargaining process.

The authors mention that in the TU-case

> The resulting solution is different from the previous ones (thus, it is neither the Shapley value nor the equal split solution).[5] However, for a large n,[6] it is easy to see that it is close to the equal split solution.

The interest of this modification resides in the solution obtained. It satisfies both requirements: Single-valuedness and symmetry. Moreover, it yields the *Nash bargaining solution* in pure bargaining games, and the *solidarity value* of Nowak and Radzik (1994) in TU-games.

The second modification proposed here has the advantage that the solution obtained fits into the Shapley value generalization program:

– *Random marginal.* A new proposer is chosen (among all of them) with equal probability. The proposer makes an *ultimatum offer*. If the rest of the players agree, then this offer is the final payoff. If it is rejected by even one player, the proposer drops out of the game (receiving a zero payoff), and the remaining players becomes the new active set.

Now, this bargaining procedure supports a new solution in the NTU case that it is again single-valued and satisfies symmetry. In the TU case the Shapley value is selected, and in Hyperplane games it is the Consistent NTU-value. Moreover, in pure bargaining games the solution obtained differs from the Nash bargaining solution. The point selected maximizes utility gains from a breakdown point, so in this way it is similar to the Nash solution, but the breakdown point is an average of the ideal points, used in the definition of the Kalai and Smorodinsky (1975) solution, whereas in the Nash case the breakdown point is the disagreement point. So it comprises elements of both solutions in its definition.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary definitions and notation. Then, Sect. 3 presents the bargaining model of Hart and Mas-Colell as well as the modifications that yield the random marginal and the random removal bargaining models. The proposals corresponding to an equilibrium, for both bargaining models, are characterized in NTU-games. In Sect. 4, the random marginal and random removal values are characterized in the particular case of TU-games. In Sect. 5 is considered the general case of NTU-games. Finally, Sect. 6 explores, in large games, the connections between the random marginal value and the Shapley NTU-value, and between the random removal value and the equal split value.

[5] That is, the payoffs are shared equally between the players of the coalition.

[6] In this context, n means the number of players.

2 Definitions

Let $N = \{1, \dots, n\}$ be a finite set of players. A *coalition* is a subset of N. Let $P(N)$ be the set of all coalitions of N. The cardinality of a coalition S is denoted by s. If $x, y \in \mathbb{R}^N$, we write $x \geq y$ if $x^i \geq y^i$ for all $i \in N$, and $x > y$ if $x^i > y^i$ for all $i \in N$. Given two vectors $x, y \in \mathbb{R}^N$, we use the notation $x \cdot y := \sum_{i \in N} x^i y^i$, and $x * y := (x^i y^i)_{i \in N}$. If $x \in \mathbb{R}^N$ and $\emptyset \neq S \subset N$, we write x^S as the restriction of x to S, i.e., $x^S = (x^i)_{i \in S} \in \mathbb{R}^S$. Let $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \mid x > 0\}$. A set $A \subset \mathbb{R}^N$ is called *comprehensive* if $A - \mathbb{R}^N_+ \subset A$. The boundary of A is denoted by ∂A. The boundary is *non level* if for all $x \in \partial A$ it holds that $\{x\} - \mathbb{R}^N_+ \cap \partial A = \{x\}$.

A non-transferable utility game (NTU-game for short), is a map V assigning to each coalition S, $\emptyset \neq S \subset N$, a subset $V(S) \subset \mathbb{R}^S$ of *attainable payoff vectors* for players in S. Several regularity conditions are imposed such as:

(A.1) $V(S)$ is non-empty, closed, convex and comprehensive.
(A.2) $\partial V(S)$ is non level.
(A.3) $0 \in V(S)$ and $V_0(S) := V(S) \cap \mathbb{R}^S_+$ is bounded.
(A.4) Monotonicity: $V_0(S) \times \{0^{T \setminus S}\} \subset V_0(T)$ whenever $S \subset T$.
(A.5) Positive smoothness: For each $S \subset N$, at each x of $\partial V(S)$ there exists a unique supporting hyperplane to $V(S)$ (i.e., there exist a unique[7] $\lambda \in \mathbb{R}^S_{++}$ such that $V(S) \subset \{y \in \mathbb{R}^S : \lambda \cdot y \leq \lambda \cdot x\}$).

The Assumption A.4 is just the extension to NTU-games of the classical Monotonicity assumption for TU-games. The class of all games that satisfy A.1, A.2, A.3, and A.4, is denoted by $\mathscr{G}$.

For each $i \in N$, let $r^i := \max\{x : x \in V(i)\}$, and let $r = \left(r^i\right)_{i \in N} \in \mathbb{R}^N$. Some relevant classes of NTU-games are:

1. Transferable utility games (TU-games), when for each coalition S, there is a number $v(S)$ such that $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x^i \leq v(S)\}$ for all $S \subset N$. Here $r^i = v(i)$ for all $i \in N$. Risk neutral players who use a totally divisible good to make the coalitional payoffs is an example of this type of games. If V is a TU-game, then it will be also denoted by v.
2. Hyperplane games (H-games), when $\partial V(S)$ is a hyperplane for all $S \subset N$. That is, for each coalition S, there exists a number $v(S)$ and a vector $\lambda_S \in \mathbb{R}^S_{++}$ such that $V(S) = \{y \in \mathbb{R}^S : \lambda_S \cdot y \leq v(S)\}$. Here $r^i = v(i)/\lambda(i)$ for all $i \in N$. For example, *prize games* can be modeled in this way: each coalition $S \subset N$ has a prize π_S. The prize π_S is indivisible, and only one member of S can receive it. The feasible set of each coalition S consists of all lotteries over which players in S get the prize π_S (for more details see Hart 1994).
3. Pure Bargaining games (PB-games), when $0 \in \partial V(S)$, for all $S \neq N$. Pure Bargaining games are usually described by a pair $(0, V(N))$, where $V(N)$ is the utility feasible set attainable by unanimity agreements of all members of N, and 0 is the utility feasible payoffs vector obtained in case of disagreement. The fact that no other coalition than the grand coalition can make agreements is reflected by $0 \in \partial V(S)$, for all $S \neq N$.

[7] We normalize it so that $\sum_{i \in S} \lambda^i = 1$.

Remark In the normalization assumption A.3, it is worth noting that we have been making the implicit assumption that the utilities are previously normalized in such a way that when any player leaves the game, the payoff that it obtains is zero. Therefore, for any subcoalition S, the players have an incentive to cooperate because every player can attain payoffs better than it will obtain being alone, "out off" the game. Note also that the payoff r^i is what player i obtains if the remaining $N\backslash i$ players have left the game. Hence a two-person NTU-game will be a pure bargaining game only when $r = 0$.

A *payoff configuration* is a family $\mathbf{x} = (x_S)_{S \subset N}$ where $x_S \in \mathbb{R}^S$ for all $S \subset N$. It is *efficient* if $x_S \in \partial V(S)$ for all $S \subset N$. A *value* on $\mathcal{G}$ is a function ϕ that assigns a unique payoff configuration to each game belonging to $\mathcal{G}$. Given $V \in \mathcal{G}$, and $S \subset N$, the vector $\phi_S(V)$ is called the *value of* V *for* S. A *solution* on $\mathcal{G}$ is a mapping Ψ that assigns a set of payoff configurations to each game belonging to $\mathcal{G}$. For notational simplicity, we use $S\backslash i$ and $S \cup i$ instead of $S\backslash\{i\}$ and $S \cup \{i\}$, respectively.

3 The bargaining model

The multilateral bargaining procedures presented here try to reflect a plausible negotiation process, in which agents on their own seek a cooperative agreement. The rules of the bargaining should have, a prior, enough appealing properties (like simplicity, fairness, efficiency, etc.) such that the agents be ready to follow. With this in mind, we take into account three basic aspects in the modeling of the negotiation process:

- All agents involved can make offers and counteroffers, up to the moment when a *unanimous* agreement is reached. Therefore there are neither referees, nor players acting as dictators, that can impose the final payoffs. For this reason, the alternating offer method (Rubinstein 1982; Stahl 1972) is used.
- The agreement cannot be delayed indefinitely. This fact is incorporated by a chance that the process can stop after each offer's rejection.
- The final agreement can be partial. This means that if a breakdown of negotiation happens this does not imply the end of the process: *only one* agent leaves the game and the remaining agents restart the bargaining. It is in this way that the attainable payoffs of subcoalitions will have a significant influence over the final outcome.

The two models presented here, the random removal and random marginal models, are based on the Hart and Mas-Colell model, and hence have a similar structure. For this reason we describe the three models at the same time, differing only in what happens if breakdown occurs.

Let an NTU-game $V \in \mathcal{G}$ and $0 \le \rho < 1$ be a fixed parameter:

In each *round* there is a set $S \subset N$ of *active* players, and a *proposer* $i \in S$. In the first round, the active set is $S = N$. The proposer is chosen at random from S, with all players in S being equally likely to be selected. The proposer makes a feasible offer $a_{S,i} \in V(S)$. If all members of S accept it—they are asked in some prespecified order—then the game ends with these payoffs. If it is rejected by even one member of S, then, with probability ρ, we move to a next round where the set of active players is again S and, with probability $1 - \rho$, *breakdown* occurs.

HM-breakdown (*Hart and Mas-Colell*): Proposer i leaves the game, receiving a payoff of zero, and the set of active players becomes $S\backslash i$.

RR-breakdown (*random removal*): A new player j is chosen at random from S to leave the game receiving a payoff of zero, being equally likely to be selected, and the set of active players becomes $S\backslash j$.

RM-breakdown (*random marginal*): A new player j is chosen at random from S to make an *ultimatum offer*, being equally likely to be selected: Proposer j makes a feasible offer $u_{S,j} \in V(S)$. If all members of S accept it—they are asked in some prespecified order—then the game ends with these payoffs. If it is rejected by even one member of S, then j leaves the game, receiving a payoff of zero, and the set of active players becomes $S\backslash j$.

The three models can interpreted as follows. Consider a bargaining process in which players in a room are chosen randomly to make offers, which can be unanimously accepted for all players, or rejected by even some player. If accepted then the game ends with such proposal as final payoffs. Otherwise another proposer is again chosen randomly to make another offer, and so on.

But during the time for reaching an agreement, there is a chance that the process stops (it can be indicated by a green light that turns red). We then move to a breakdown situation.

With the breakdown of the Hart and Mas-Colell model, the *last proposer* whose offer has been rejected leaves the room and the bargaining restarts with the remaining players. Hence it is only the last proposer who takes responsibility for the lack of agreement at that moment.

With the breakdown of the random remove model, a player is chosen randomly lo leave the room, restarting the bargaining with the remaining players. Now all the players, the proposer and the responders, take this responsibility.

Finally, with the random marginal model, when the light turns red the players know that there is time for only *another last offer*: a new proposer is chosen randomly and if its offer is rejected then the proposer leaves the room and the bargaining restarts with the remaining players. Once again, all the players are considered equally responsible for the lack of agreement when the light turned red, because all of them can be chosen to make an ultimatum offer.

Figure 1 displays the three models. The symbol **Y** means that all responders accept the offer, **N** means that at least one responder rejects the offer. The symbol (open diamond) means that a random move is made in the game. In capital (open diamond), the round continues with probability ρ, with $1 - \rho$ it goes to a breakdown. In small (open diamond), a new player is chosen randomly. The symbol (filled square) means that the previous proposer is chosen deterministically.

Note that the key difference between the HM-model and RR-model lies in *who* is the agent that leaves the game after the breakdown: In the HM-model it is the proposer who leaves whereas in the RR-model any player can leave. On the other hand, the main difference between the HM-model and the RM-model lies in the *time perception*

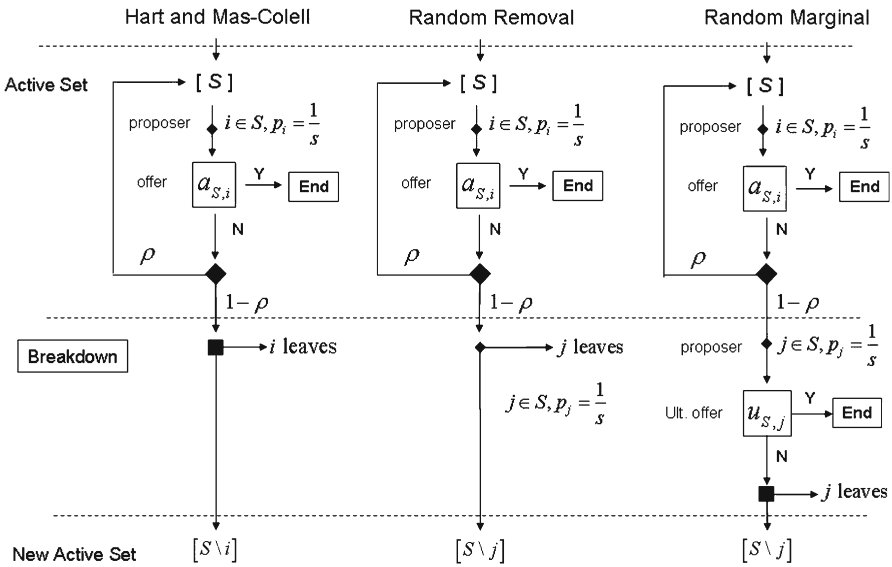

Fig. 1 The three bargaining models

of the breakdown: In the HM-model the proposer does not know in advance whether it leaves or not the game in case that its offer is rejected. By the contrary, this fact is known in advance every time the proposer makes an offer.

The negotiation games have potentially infinitely many periods and, with more than two active players, it is well known that there are many subgame perfect equilibria strategies. Hence, we restrict ourselves to considering only *stationary* strategies. Therefore, the strategies are such the choice at each stage only depends on the set of active players S and on the current proposer i. Given a profile of stationary strategies, we denote by $a_{S,i}(\rho)$ the proposal when the set of active players is S and the proposer is i, for $i \in S \subset N$. The average of these proposals is defined by $a_S(\rho) := (1/s) \sum_{i \in S} a_{S,i}(\rho)$.

We recall first the equations that characterize the stationary subgame perfect equilibrium (SP) of the HM-model (Proposition 1, Hart and Mas-Colell 1996a).

Proposition 1 *The proposals corresponding to an SP equilibrium in the Hart and Mas-Colell model are accepted by all the players in each round, and they are characterized by:*

(HM.1) $a_{S,i}(\rho) \in \partial V(S)$ *for all* $i \in S \subset N$,
(HM.2) $a^j_{S,i}(\rho) = \rho a^j_S(\rho) + (1-\rho) a^j_{S\setminus i}(\rho)$ *for all* $i, j \in S \subset N$ *with* $i \neq j$.

Moreover, these proposals are nonnegative.

The proposition says that proposer i makes an offer such that it will obtain its maximum payoff compatible by giving to the rest of players what they would expect to obtain in the continuation of the game if the offer were rejected, i.e., for every

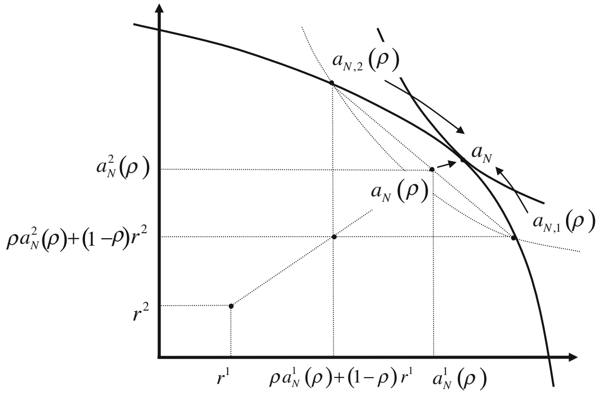

Fig. 2 HM-model

responder $j \neq i$, it will get its expected payoff $a_S^j(\rho)$ when the active player set is S again, with probability ρ, and $a_{S\backslash i}^j(\rho)$ when the active player set is $S\backslash i$, with probability $(1-\rho)$ of i leaving the game. See Fig. 2 for a two-player example.

In the following example we illustrate the multiplicity problem that appears with the HM-model. Consider two players who claim an indivisible good. In addition, assume that this good can be owned either by only one of the players or shared by both. The players have the same preferences and they are risk neutral. The set of *pure* feasible outcomes is $A = \{O_1, O_2, S, E\}$, where O_i $(i = 1, 2)$ means that the good is only for player i, S means that the good is shared by both players, and E means that the good is for nobody. We normalize the utilities as follows:

$$u^1(O_1) = u^2(O_2) = 75,$$
$$u^1(S) = u^2(S) = 50,$$
$$u^1(E) = u^2(E) = 0.$$

Here the set of players is $N = \{1, 2\}$ and the characteristic function V is built as follows: If a player relinquishes its claim, it "leaves" the game, and then the good is for the other player, hence

$$V(\{i\}) = \{x : x \leq r^i\} \text{ where } r^i = 75,\ i \in N.$$

When both players claim the good, they can either agree on any pure outcome in A, or on any lottery in A (for example, tossing a coin to decide if the good is only for player 1 or 2: $\left[O_1, p_1 = \frac{1}{2}; O_2, p_2 = \frac{1}{2}\right]$). Therefore the feasible expected payoffs that both claimants can guarantee by cooperation are established by the convex hull of $u(A) = \{(50, 50), (75, 0), (0, 75), (0, 0)\}$. Then,

$$V(N) = \text{conv}(u(A)) - \mathbb{R}_+^2,$$

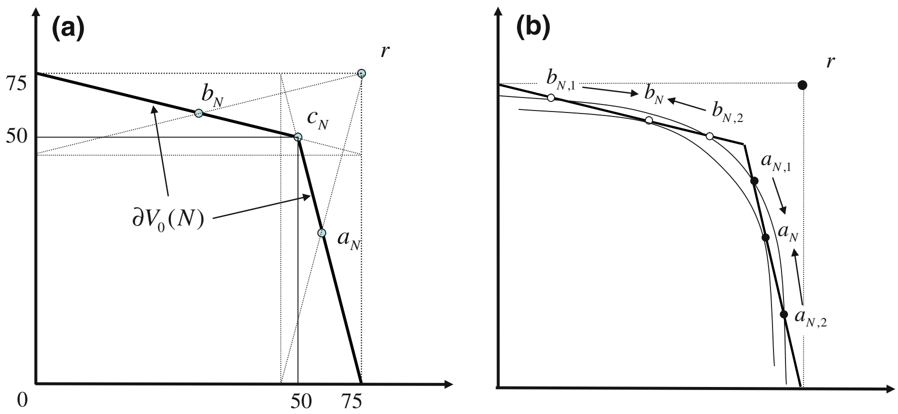

Fig. 3 Multiplicity

("conv" denotes "convex hull"). The sets $V(\cdot)$ are also comprehensive (utility is freely disposable). $V(N)$ is represented in Fig. 3a.

The consistent solution selects three[8] outcomes: $a_N = (56.25, 37.5), b_N = (37.5, 56.25)$, and $c_N = (50, 50)$.

Now, if $x_{N,i}(\rho)$, $i = 1, 2$, are the equilibrium proposals in the HM-model, then it can be checked that they must satisfy:

$$\left(x^1_{N,1}(\rho) - r^1\right)\left(x^2_{N,1}(\rho) - r^2\right) = \left(x^1_{N,2}(\rho) - r^1\right)\left(x^2_{N,2}(\rho) - r^2\right),$$

which, in our example, yield two different solutions: $\{a_{N,1}(\rho), a_{N,2}(\rho)\}$ and $\{b_{N,1}(\rho), b_{N,2}(\rho)\}$ that they converge to a_N and b_N, respectively, when $\rho \to 1$, as can be seen in Fig. 3b.

First, note that this bargaining procedure *does not allow an approximation to all payoff solutions*: In our example, point $c_N = (50, 50)$, which is precisely the symmetric payoff, is excluded. Secondly, we have multiplicity: When $\rho \to 1$ the limits of equilibrium payoffs are a_N and b_N. If we have no previous reasons to discriminate between both players, a way to solve this impasse is to choose with a fair lottery between a_N and b_N by tossing a coin. But then, the expected payoffs are (46.875, 46.875) that are Pareto dominated by (50, 50).

In the next two propositions we characterize the conditions for an SP equilibrium in the random removal and random marginal models. They only differ in what players can expect to get in the continuation of the game in case of breakdown.

For any $S \subset N$, define

$$d^j_S(\rho) := \frac{1}{s} \sum_{k \in S \setminus j} a^j_{S \setminus k}(\rho), \quad j \in S.$$

[8] It can easily be checked that these three points also belong to the set of Shapley and Harsanyi NTU-solutions of this game. For a detailed definition of these solution concepts, the reader can see Hart (2005).

Therefore $d_S(\rho) = (1/s)\sum_{j\in S}(0, a_{S\setminus j}(\rho))$, where $(0, a_{S\setminus j}(\rho)) \in \mathbb{R}^S$ is the payoff vector in which player j leaves the game, receiving 0, and players $k \neq j$ receive $a^k_{S\setminus j}(\rho)$.

Proposition 2 *The proposals corresponding to an SP equilibrium in the random removal model are accepted by all the players in each round, and are characterized by:*

(RR.1) $a_{S,i}(\rho) \in \partial V(S)$ *for all* $i \in S \subset N$,
(RR.2) $a^j_{S,i}(\rho) = \rho a^j_S(\rho) + (1-\rho) d^j_S(\rho)$ *for all* $i, j \in S \subset N$ *with* $i \neq j$.

Moreover, these proposals are nonnegative.

For the random marginal model the proposition is very similar. For any $S \subset N$, we define $u_S(\rho) := (1/s)\sum_{j\in S} u_{S,j}(\rho)$, where $u_{S,j}(\rho) \in \mathbb{R}^S$ is the ultimatum offer when player $j \in S$ is selected as proposer in the breakdown.

Proposition 3 *The proposals corresponding to an SP equilibrium in the random marginal model are accepted by all the players in each round, and are characterized by:*

(RM.1) $a_{S,i}(\rho) \in \partial V(S)$ *for all* $i \in S \subset N$,
(RM.2) $a^j_{S,i}(\rho) = \rho a^j_S(\rho) + (1-\rho) u^j_S(\rho)$ *for all* $i, j \in S \subset N$ *with* $i \neq j$,
(RM.3) $u_{S,i}(\rho) \in \partial V(S)$ *for all* $i \in S \subset N$,
(RM.4) $u^j_{S,i}(\rho) = a^j_{S\setminus i}(\rho)$ *for all* $i, j \in S \subset N$ *with* $i \neq j$.

Moreover, $a_S(\rho) \geq u_S(\rho)$, *and* $a^i_{S,i}(\rho) \geq u^i_S(\rho) \geq 0$, *for all* $i \in S \subset N$.

The proofs of Propositions 2 and 3 are rather similar to the prof for the HM-model [see Proposition 1, in Hart and Mas-Colell (1996a)]. In particular, Proposition 2 is case (d) in Proposition 9 of Hart and Mas-Colell (1996a). So we deal explicitly only with the RM-model.

Proof We proceed by induction. The result is easily checked for the 1-player case. Let $(a_{S,i}(\rho), u_{S,i}(\rho))$, for $i \in S \subset N$, be the proposals of a given SP equilibrium. Assume by induction hypothesis that RM.1–RM.4 are satisfied for $S \neq N$. We will show that RM.1–RM.4 are satisfied.

We see firstly that, in case of breakdown, $u_{S,i}(\rho)$ satisfies RM.3 and RM.4 for any $i \in N$. Since it is assumed that $a_{S,i}(\rho) \in \partial V(S)$ and $a_{S,i}(\rho) \geq 0$ for $i \in S \neq N$, monotonicity and convexity imply that $a_S(\rho) \in V_0(S)$ for all $S \neq N$. Let $i \in N$ be the proposer of an ultimatum offer in case of breakdown. Because each player j in $N\setminus i$ can guarantee $a^j_{N\setminus i}(\rho)$ by rejection, player j only accepts offers such that $u^j_{N,i}(\rho) \geq a^j_{N\setminus i}(\rho)$. Hence the best player i can do is to offer $u^j_{N,i}(\rho) = a^j_{N\setminus i}(\rho)$ for all $j \in N\setminus i$. Let $u^i_{N,i}(\rho)$ be such that $u_{N,i}(\rho) \in \partial V(N)$. By monotonicity, $u^i_{N,i}(\rho) \geq 0$. If $u^i_{N,i}(\rho) > 0$, the best player i can do is to offer $u_{N,i}(\rho)$, which will be accepted by all $j \in N\setminus i$. If $u^i_{N,i}(\rho) = 0$, player i is indifferent between offering $u_{N,i}(\rho)$, which will be accepted, and offering a different proposal $b_{N,i}(\rho) \neq u_{N,i}(\rho)$ such that $b^i_{N,i}(\rho) > 0$. In the latter case, there is $j \neq i$ such that $b^j_{N,i}(\rho) < a^j_{N\setminus i}(\rho)$, because A.1 and A.2 imply that $\partial V(N)$ coincides with the Pareto frontier of $V(N)$. Hence $b_{N,i}(\rho)$

is rejected by player j. In both cases player i obtains 0 and each player $j \in S\backslash i$ obtains $a^j_{N\backslash i}(\rho)$, which again coincides with $u_{N,i}(\rho)$.[9] Let $u_N(\rho) := (1/n)\sum_{i\in N} u_{N,i}(\rho)$. By construction $u_{N,i}(\rho) \in \partial V_0(N)$ for all $i \in N$. Then $u_N(\rho) \in V_0(N)$ by convexity.

Denote by c_N the expected payoff vector for the members of N. By convexity $c_N \in V(N)$, and also $\rho c_N(\rho)+(1-\rho)u_N(\rho) \in V(N)$. Let $d_{N,i}(\rho)$ be the vector such that $d_{N,i}(\rho) \in \partial V(N)$ and $d^j_{N,i}(\rho) = \rho c^j_N(\rho) + (1-\rho)u^j_N(\rho)$ for all $j \in N\backslash i$. Thus, $d^i_{N,i}(\rho) \geq \rho c^i_N(\rho) + (1-\rho)u^i_N(\rho)$. For $j \neq i$, $d^j_{N,i}(\rho)$ is player j's expected payoff following a rejection of i's proposal. Therefore $d_{N,i}(\rho)$ is the proposal which is best for i among the proposals that will be accepted if i is the proposer. Moreover, any proposal of i which is rejected yields to i at most $\rho c^i_N(\rho) + (1-\rho)u^i_N(\rho) \leq d^i_{N,i}(\rho)$. Hence, player i will propose $a_{N,i}(\rho) = d_{N,i}(\rho)$, and the proposal will be accepted. Therefore $c_N(\rho) = a_N(\rho)$, and RM.1 and RM.2 are satisfied. To see that $a_N(\rho) \geq u_N(\rho)$, note that the following strategy will guarantee to any i a payoff of at least $u^i_N(\rho)$: accept only if offered at least $u^i_N(\rho)$, and, when proposing, propose $u_N(\rho)$. This implies that $a^i_N(\rho) \geq u^i_N(\rho)$, and then $a^i_{N,i}(\rho) \geq \rho a^i_N(\rho) + (1-\rho)u^i_N(\rho) \geq u^i_N(\rho)$.

Now here only remains to show that the strategies corresponding to proposals satisfying RM.1–RM.4 do form an SP equilibrium. By the induction hypothesis, this is so in any subgame with player set $S \neq N$. Fix a player i in N. Given the strategies of the other players, as a proposer, i cannot increase its payoff $a^i_{N\backslash i}(\rho)$ from proposals that are accepted, and making proposals that were systematically rejected they could only yield the chance to go to the breakdown stage, which gives $u^i_N(\rho)$ as its expected payoff. Whereas the suggested strategies yields $a^i_{N,i}(\rho)$ which is a better outcome ($a^i_{N,i}(\rho) \geq u^i_N(\rho)$). As a responder, i can only deviate by rejecting the offer $a^i_{N,j}(\rho)$ made by another player j, but its expectation in case of continuation, $\rho a^i_N(\rho) + (1-\rho)u^i_N(\rho)$, is just equal to $a^i_{N,j}(\rho)$. A parallel argument applies in case that players reach to the breakdown stage. □

Consider again our previous example. Now in case of breakdown we have:

Random removal (*RR*)*:* Both players have the same probability to leave the game. If player i leaves the game, then its payoff is zero, and player j receives r^j. Therefore the expected payoff vector d_N is

$$d_N = \frac{1}{2}(0, r^2) + \frac{1}{2}(r^1, 0) = \frac{1}{2}(0, 75) + \frac{1}{2}(75, 0) = (37.5, 37.5)\,.$$

Random marginal (*RM*)*:* Both players have the same probability to be the proposer of an ultimatum offer. If player i proposes $u_{N,i} \in V(N)$, then player j is asked whether it agrees or dissents. If it agrees, $u_{N,i}$ is the final payoff. If it dissents, then player i leaves the game, receiving a payoff of zero, and player j receives r^j. The equilibrium proposals $u_{N,i}$, $i = 1, 2$, are characterized by $u_{N,i} \in \partial V(N)$, for all $i \in N$, and $u^j_{N,i} = r^j$, for $j \neq i$. Therefore, it follows that, the expected payoff vector u_N is $u_N = \frac{1}{2}u_{N,1} + \frac{1}{2}u_{N,2}$. In our example, when a proposer i is compelled to make an ultimatum offer to j, it must offer $r^j = 75$ units because this is what the other player

[9] In this indifferent case, mixed strategies also yield the same outcome.

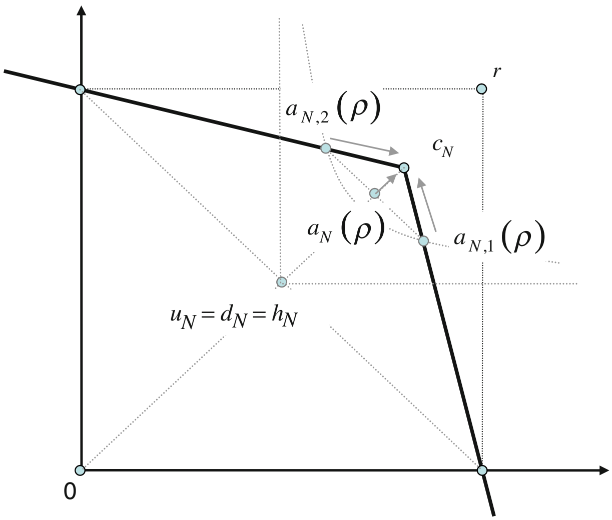

Fig. 4 Limit point in the RM and RR models

would obtain if the proposer is forced to leave the game in case of rejection. Hence the expected payoff vector is

$$u_N = \frac{1}{2}(0, 75) + \frac{1}{2}(75, 0) = (37.5, 37.5)\,.$$

The breakdown expected payoffs of both models, d_N and u_N, coincide, and we denote this vector (37.5, 37.5) by h_N. It follows that in both bargaining models, the equilibrium equations which determine the proposals in N are $x_{N,i}(\rho) \in \partial V(N)$, for all $i \in N$, and $x^j_{N,i}(\rho) = \rho x^j_N(\rho) + (1-\rho)h^j$, for $j \neq i$, where $x_N(\rho) = \frac{1}{2}x_{N,1}(\rho) + \frac{1}{2}x_{N,2}(\rho)$. It can be easily checked that

$$\begin{aligned}\left(x^1_{N,1}(\rho) - h^1\right)\left(x^2_{N,1}(\rho) - h^2\right) &= \left(x^1_{N,2}(\rho) - h^1\right)\left(x^2_{N,2}(\rho) - h^2\right) \\ &= (2-\rho)\rho\left(x^1_N(\rho) - h^1\right)\left(x^2_N(\rho) - h^2\right).\end{aligned}$$

Therefore, when $\rho \to 1$, we have $c_i(\rho) \to c(\rho)$ for $i = 1, 2$, and $c(\rho) \to c = (50, 50)$ (see Fig. 4).

4 TU-games

In this section we analyze the behavior of the RM and RR breakdown models in the case of transferable utility games. The results of this section are particular cases of the results obtained in the next section for NTU-games. Here we find the main difference between the models: The RM-model supports the *Shapley value*, and the RR-model supports the *solidarity value*.

Random marginal

Let v be a TU-game. For each coalition $S \subset N$ and player $i \in S$, define

$$\partial^i(v, S) := v(S) - v(S\backslash i).$$

We call $\partial^i(v, S)$ the *marginal contribution of player i to coalition S in the TU-game v*. The *Shapley value* in the game v is the payoff configuration $\varphi = (\varphi_S(v))_{S\subset N}$ defined by

$$\varphi_S^i(v) = \sum_{\substack{T\subset S \\ T\ni i}} \frac{(s-t)!\,(t-1)!}{s!}\partial^i(v, T), \quad (i \in S \subset N)\,.$$

Alternatively, φ can be obtained recursively[10] by

$$\varphi_S^i(v) = \frac{1}{s}\partial^i(v, S) + \sum_{j\in S\backslash i} \frac{1}{s}\varphi_{S\backslash j}^i(v), \quad (i \in S \subset N)\,, \tag{1}$$

starting with

$$\varphi_{\{i\}}^i(v) = v(i), \text{ for all } i \in N.$$

In Proposition 3 we have seen that the bargaining rules guarantee that the equilibrium payoffs a^S of the negotiation stages will always be greater than, or equal to, the breakdown payoffs u^S, for all $S \subset N$. This fact implies the next Theorem.

Theorem 1 *Let* $(a_S(\rho), u_S(\rho))_{S\subset N}$ *be the equilibrium payoff configuration associated to the random marginal model. Then* $\mathbf{a}(\rho) = (a_S(\rho))_{S\subset N}$ *coincides with the* Shapley *value for TU-games, for any* $0 \le \rho < 1$.

Proof Let v be a TU-game. By RM.3, $\sum_{j\in S} u_{S,i}^j(\rho) = v(S)$, for all $i \in S$, hence $\sum_{j\in S} u_S^j(\rho) = v(S)$, for all $S \subset N$. Moreover, $a_{S,i}(\rho) \ge u_S(\rho)$, and $\sum_{j\in S} a_{S,i}^j(\rho) = v(S)$, for all $i \in S$. Then $a_{S,i}(\rho) = a_S(\rho) = u_S(\rho)$ and $\sum_{j\in S} a_S^j(\rho) = v(S)$. Therefore

$$a_S^i(\rho) = \frac{1}{s}u_{S,i}^i(\rho) + \sum_{j\in S\backslash i} \frac{1}{s}a_{S\backslash j}^i(\rho), \ (i \in S).$$

By RM.3, RM.4, and $\sum_{j\in S\backslash i} a_{S\backslash i}^j(\rho) = v(S\backslash i)$, we have that

$$u_{S,i}^i(\rho) = v(S) - \sum_{j\in S\backslash i} a_{S\backslash i}^j(\rho) = \partial^i(v, S).$$

10 See the Remark of Sect. 4 in Hart and Mas-Colell (1996a).

Therefore

$$a_S^i(\rho) = \frac{1}{s}\partial^i(v, S) + \sum_{j \in S\setminus i} \frac{1}{s} a_{S\setminus j}^i(\rho), \ (i \in S).$$

The payoffs of the single coalitions $\{i\}$, are $a_{\{i\},i}^i(\rho) = v(i)$, for all $i \in N$. Thus they are independent of ρ. This fact implies that the ultimatum payoffs for two-player coalitions are also independent of ρ, and equal to

$$a_S^i = \frac{1}{2}\partial^i(v, S) + \frac{1}{2}v(i), \ \ (i \in S = \{i, j\}).$$

By increasing the size of the coalitions, this recursive argument shows that the equilibrium payoffs are independent of ρ, and equal to

$$a_S^i = \frac{1}{s}\partial^i(v, S) + \sum_{j \in S\setminus i} \frac{1}{s} a_{S\setminus j}^i, \ (i \in S \subset N).$$

Therefore $a_S^i = \varphi_S^i(v)$ for all $i \in S \subset N$. □

Remark It is interesting to note the differences with the results for the HM-model in the TU-case. In the HM-model, we have that $a_S(\rho) = \varphi_S(v)$, and $a_{S,i}(\rho) \neq a_{S,j}(\rho)$ when $\rho < 1$, for all $i, j \in S \subset N$. It means that in the HM-model there is difference between being the proposer or the responder, whereas in the RM-model this is not the case: $a_{S,i}(\rho) = a_{S,j}(\rho) = a_S(\rho)$, for all $\rho < 1$. So, in the TU-case, in the RM-model the bargaining part in which players make proposals and counterproposals is irrelevant, because they are strongly determined by their expectations in the breakdown part. For this reason, if one wishes to support the Shapley value, then the bargaining can be simplified to only two moves, for each round of active players set S: Choose randomly a proposer from S (equally likely). If the proposal is rejected, then randomly choose again another proposer from S (equally likely) to make an ultimatum proposal. If it is rejected, then the last proposer drops out of the game and proceed to a new round with the remaining players as the new active set.

Random removal

Let v be a TU-game. For each coalition $S \subset N$, define

$$\partial^{av}(v, S) := \frac{1}{s}\sum_{i \in S} \partial^i(v, S).$$

We call $\partial^{av}(v, S)$ the *average of marginal contributions of the players in coalition S in the TU-game v*. The *solidarity value* in the game v is the payoff configuration $\psi = (\psi_S(v))_{S\subset N}$ defined by

$$\psi_S^i(v) = \sum_{\substack{T\subset S \\ T\ni i}} \frac{(s-t)!\,(t-1)!}{s!}\partial^{av}(v, T), \ \ (i \in S \subset N).$$

This value was introduced by Nowak and Radzik (1994). Similarly to the Shapley value, it can be easily checked that this value can be obtained recursively by

$$\psi_S^i(v) = \frac{1}{s}\partial^{av}(v, S) + \sum_{j\in S\setminus i} \frac{1}{s}\psi_{S\setminus j}^i(v), \quad (i \in S \subset N),$$

starting with

$$\psi_{\{i\}}^i(v) = v(i), \text{ for all } i \in N.$$

Theorem 2 *Let* $(a_S(\rho))_{S\subset N}$ *be the equilibrium payoff configuration associated to the random removal model. Then* $\mathbf{a}(\rho) = (a_S(\rho))_{S\subset N}$ *coincides with the* solidarity value *for TU-games, for any* $0 \leq \rho < 1$.

Proof Let v be a TU-game. By RR.1, for any $i \in S \subset N$, we have

$$a_{S,i}^i(\rho) = \left(v(S) - \sum_{j\in S\setminus i} a_{S,i}^j(\rho)\right),$$

then

$$a_S^i(\rho) = \frac{1}{s}a_{S,i}^i + \frac{1}{s}\sum_{j\in S\setminus i} a_{S,j}^i(\rho),$$

which yields

$$sa_S^i(\rho) = \left(v(S) - \sum_{j\in S\setminus i} a_{S,i}^j(\rho)\right) + \sum_{j\in S\setminus i} a_{S,j}^i(\rho).$$

Applying RR.2,

$$\begin{aligned} sa_S^i(\rho) &= v(S) - \sum_{j\in S\setminus i}\left(\rho a_S^j(\rho) + (1-\rho)\frac{1}{s}\sum_{k\in S\setminus j} a_{S\setminus k}^j(\rho)\right) \\ &\quad + \sum_{j\in S\setminus i}\left(\rho a_S^i(\rho) + (1-\rho)\frac{1}{s}\sum_{k\in S\setminus i} a_{S\setminus k}^i(\rho)\right) \\ &= v(S) - \sum_{j\in S\setminus i} \rho a_S^j(\rho) - \rho a_S^i(\rho) - \frac{1-\rho}{s}\sum_{j\in S\setminus i}\sum_{k\in S\setminus j} a_{S\setminus k}^j(\rho) \\ &\quad + \sum_{j\in S\setminus i} \rho a_S^i(\rho) + \rho a_S^i(\rho) + \frac{1-\rho}{s}\sum_{j\in S\setminus i}\sum_{k\in S\setminus i} a_{S\setminus k}^i(\rho), \end{aligned}$$

which, applying RR.1 again, yields

$$sa_S^i(\rho) = v(S) - \frac{1}{s}\sum_{j\in S\backslash i}\sum_{k\in S\backslash j} a_{S\backslash k}^j(\rho) + \frac{s-1}{s}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho).$$

Note that

$$\begin{aligned}\sum_{j\in S\backslash i}\sum_{k\in S\backslash j} a_{S\backslash k}^j(\rho) &= \sum_{k\in S\backslash i} a_{S\backslash i}^k(\rho) + \sum_{k\in S\backslash i}\sum_{j\in S\backslash i,} a_{S\backslash k}^j(\rho)\\ &= v(S\backslash i) + \sum_{k\in S\backslash i}\left(v(S\backslash k) - a_{S\backslash k}^i(\rho)\right).\end{aligned}$$

Then, we obtain

$$sa_S^i(\rho) = v(S) - \frac{1}{s}\sum_{k\in S} v(S\backslash k) + \sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho) = \frac{1}{s}\partial^{av}(v,S) + \frac{1}{s}\sum_{k\in S\backslash i} a_{S\backslash k}^i(\rho).$$

The payoff of any single coalition $\{\acute{\imath}\}$, is $a_{\{i\},i}^i(\rho) = v(i)$, for all $i \in N$. So it is independent of ρ. A recursive argument shows that the average equilibrium payoffs a_S^i are independent of ρ, and equal to

$$a_S^i = \frac{1}{s}\partial^{av}(v,S) + \sum_{k\in S\backslash i}\frac{1}{s}a_{S\backslash k}^i,\ \ (i \in S \subset N).$$

Therefore $a_S^i = \psi_S^i(v)$ for all $i \in S \subset N$. □

Remark Note that $a_{s,i}(\rho) \neq a_S$, for all $i \in S \subset N$. But, given assumption A.3, condition RR.2 implies that $a_{s,i}(\rho) \to a_S$ whenever $\rho \to 1$.

5 NTU-games

In this section we show that the RM and the RR-models support each one a single-valued solution in the class of NTU-games.

Random marginal

To characterize the value associated to the RM-model we need to define the concept of *marginal contributions associated to a payoff configuration*. Let V be an NTU-game and let $\mathbf{x} = (x_S)_{S\subset N}$ be an *efficient* payoff configuration. For each coalition S containing player i define

$$\partial_{\mathbf{x}}^i(V,S) := \max\left\{\xi^i \in \mathbb{R} : \left(x_{S\backslash i},\xi^i\right) \in V(S)\right\}.$$

Note that for $V \in \mathcal{G}$ and $x_S \in V_0(S)$, and for all $S \subset N$, $\partial_{\mathbf{x}}^i(V,S)$ is well defined and $\partial_{\mathbf{x}}^i(V,S) \geq 0$. One can interpret this marginal contribution as the maximum that player

i can get in coalition S under the restriction that the others players in S have at their disposal the outside option given by $x_{S\backslash i}$. If V is a TU-game, given by the characteristic function v, for any efficient payoff configuration $\mathbf{x}$ it holds that $\partial^i_{\mathbf{x}}(V,S) = \partial^i(v,S)$ for all $i \in S \subset N$.

Remark The value of the Consistent NTU-solution in Hyperplane games can also be defined recursively[11] by

$$\varphi^i_S(V) = \frac{1}{s}\partial^i_{\mathbf{x}}(V,S) + \sum_{j\in S\backslash i}\frac{1}{s}\varphi^i_{S\backslash j}(V), \quad (i \in S \subset N), \tag{2}$$

where $\mathbf{x} = (\varphi_S(V))_{S\subset N}$, starting with $\varphi^i_{\{i\}}(V) = r^i$, for all $i \in N$.

Since $\partial V(S)$ is a hyperplane, in the RM-model it happens that $u_S(\rho) \in \partial V(S)$, so again we have that $a_{S,i}(\rho) = a_S(\rho) = u_S(\rho)$. By the same arguments as in Theorem 1, we can reproduce a parallel result for the Consistent NTU-solution in Hyperplane games, as Theorem 1 yields for the Shapley value in TU-games.

We now proceed to define the NTU-value supported by the RM-model.

Definition 1 Let $V \in \mathcal{G}$ and $\mathbf{a}= (a_S)_{S\subset N}$ be a payoff configuration. Then $\mathbf{a}$ is the RM-value ζ of V (i.e., $\mathbf{a} = \zeta(V)$) if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$ such that:
(a) $a_S \in \partial V(S)$;
(b) $\lambda_S \cdot a_S = v(S,\lambda_S) := \max\{\lambda_S \cdot c : c \in V(S)\}$; and
(c) $\lambda^i_S\left(a^i_S - u^i_S\right) = \lambda^j_S\left(a^j_S - u^j_S\right)$ for all $i, j \in S$,
where $u^i_S := \frac{1}{s}\left(\partial^i_{\mathbf{a}}(V,S) + \sum_{k\in S\backslash i} a^i_{S\backslash k}\right)$ for all $i \in S$.

Condition (a) states that the payoff vector a_S is efficient for coalition S. Condition (b) ensures that a_S is also λ_S-*utilitarian*, i.e., that it maximizes the sum of the λ_S-rescaled payoffs. Condition (c) is a λ_S- *egalitarian* condition: The gains of the players in a_S with respect to the vector u_S are equal relative to the units given by λ_S. The payoff vector u_S has the following interpretation: The payoff allocation $\left(a_{S\backslash i}, \partial^i_{\mathbf{a}}(V,S)\right)$ specifies the choice of player $i \in S$ when this player has the power to obtain its marginal contribution for S, under the restriction that the other players in S have at their disposal $a_{S\backslash i}$. Vector $u_S = \frac{1}{s}\sum_{i\in S}\left(a_{S\backslash i}, \partial^i_{\mathbf{a}}(V,S)\right)$ gives the expected payoff allocation for players in S if each member has an equal chance of obtaining its marginal contribution.[12]

Proposition 4 *If $V \in \mathcal{G}$, then the RM-value ζ of V exists and it is unique. Moreover, if V is an H-game, then $\zeta(V)$ coincides with the Consistent NTU-value of V, and, if V is a TU-game, then $\zeta(V)$ coincides with the Shapley value of V.*

11 See equation (3) in Hart and Mas-Colell (1996a).

12 See the interpretation of the *Conditional Random Dictatorship* axiom used in de Clippel et al. (2004) for the characterization of the Consistent NTU-solution.

Proof First we prove existence and uniqueness. Let a game $V \in \mathcal{G}$. The payoff configuration $\mathbf{a} = \zeta(V)$ is built recursively as follows: We start with single coalitions $\{i\}$, for all $i \in N$, making $\lambda^i_{\{i\}} > 0$, and $a^i_{\{i\}} = r^i$. Then conditions (a), (b), and (c), trivially hold. Assume by induction hypothesis that the Proposition holds for every $T \subsetneq S$. Then, by the convexity and monotonicity of $V(S)$, it holds that $u_S \in V_0(S)$; and by the unicity assumption of $a_{S\setminus i}$, for all $i \in S$, u_S is determined uniquely. Now there remains to see what happens when either $u_S \in \partial V_0(S)$, or $u_S \in \text{int } V_0(S)$. If $u_S \in \partial V_0(S)$, then making $a_S = u_S$, and taking as λ_S one of the support vectors of $V(S)$ at point a_S, conditions (a), (b), and (c), hold. If $u_S \in \text{int } V_0(S)$, then it is well known that a vector a_S satisfies (a), (b), and (c), if, and only if,

$$a_S := \arg\max \prod_{\substack{i \in S \\ x \geq u_S \\ x \in V_0(S)}} \left(x^i - u^i_S\right),$$

where, in that case, λ_S must be collineal to the vector $(\frac{1}{a^i_S - u^i_S})_{i \in S}$, and point a_S exists and it is unique.

When V is an H-game, or a TU-game, it holds that $u_S \in \partial V_0(S)$, for all $S \subset N$, and therefore $a_S = u_S$. Hence, equations (2) and (1) apply to yield the Consistent and the Shapley values, respectively. □

We now establish the main result of this section.

Theorem 3 *Let $V \in \mathcal{G}$ be an NTU-game. Then for each $0 \leq \rho < 1$ there is an SP equilibrium of the RM-model. Moreover, if V satisfies the additional assumption* A.5*, when $\rho \to 1$, every SP equilibrium payoff configuration $\mathbf{a}(\rho)$ converges to $\mathbf{a} = \zeta(V)$.*

Proof Existence.

Let $V \in \mathcal{G}$. We prove the existence following a recursive argument. Given $0 \leq \rho < 1$, let $a^i_{\{i\},i}(\rho) = u^i_{\{i\},i}(\rho) := r^i \in V_0(\{i\})$, for all $i \in N$. Therefore, for single coalitions, RM.1–RM.4 are satisfied.

Assume by induction that we have already determined $(a_{S,i}, u_{S,i})$ for all $i \in S \subsetneq N$. Let $u_{N,i}(\rho)$ be defined by $u_{N,i}(\rho) \in \partial V(N)$ and $u^j_{N,i}(\rho) := a^j_{N\setminus i}(\rho)$, for all $j \neq i$, where $a_{N\setminus i}(\rho) := \frac{1}{n-1} \sum_{k \in N\setminus i} a_{N\setminus i,k}(\rho)$. Since $a_{N\setminus i}(\rho) \in V_0(N\setminus i)$, by monotonicity it holds that $u_{N,i}(\rho) \in V_0(N)$, and then (RM.3) and (RM:4) are satisfied. Define $u_N(\rho) := \frac{1}{n} \sum_{i \in N} u_{N,i}(\rho)$, then, by convexity, $u_N(\rho) \in V_0(N)$. We now have two possibilities:

If $u_N(\rho) \in \partial V_0(N)$, then making $a_{N,i}(\rho) := u_N(\rho)$, for all $i \in N$, conditions RM.1 and RM.2 are trivially satisfied.

If $u_N(\rho) \notin \partial V_0(N)$, define $V(u_N(\rho), N) := V_0(N) \cap \left(u_N(\rho) + \mathbb{R}^N_+\right)$. Therefore, A.1 and A.3 imply that $V(u_N(\rho), N)$ is a non-empty, compact and convex set. Define n functions $\alpha_i(b)$, $i \in N$, from $V(u_N(\rho), N)$ into itself by letting $\alpha_i(b)$ be defined by: $\alpha_i(b) \in \partial V(u_N(\rho), N)$ and $\alpha^j_i(b) := \rho b^j + (1-\rho) u^j_N(\rho)$, for all $j \neq i$. By non-levelness and A.1, α_i is well-defined and continuous. By convexity of $V(u_N(\rho), N)$, the function $\alpha(\rho) := \frac{1}{n} \sum_{i \in N} \alpha_i(b)$ is a continuous function that maps $V(u_N(\rho), N)$ into itself. Therefore, by Brower's fixed point theorem, there is a vector $a(\rho) \in V(u_N(\rho), N)$ satisfying $a(\rho) = \alpha(a(\rho))$. By construction, letting $a_{N,i}(\rho) := \alpha_i(a(\rho))$, $i \in N$, RM.1 and RM.2 are satisfied.

In this recursive way we prove the existence of payoff configuration proposals $\left(a_{S,i}(\rho), u_{S,i}(\rho)\right)_{i \in S \subset N}$ which satisfy RM.1–RM.4 and, by Proposition 3, they correspond to an SP equilibrium.

Convergence.

Consider a convergence sequence $\{\rho_r\} \to 1$ when $r \to \infty$. Let $\{(a_{S,i}(\rho_r), (u_{S,i}(\rho_r)_{i \in S \subset N}\} \subset \left(\prod_{S \subset N} V_0(S)\right) \times \left(\prod_{S \subset N} V_0(S)\right)$ be their associated SP equilibrium proposals. Since $\left(\prod_{S \subset N} V_0(S)\right) \times \left(\prod_{S \subset N} V_0(S)\right)$ is a compact set, there exists a subsequence $\{\rho'_r\}$ such that their associated proposals converge, i.e. $\{(a_{S,i}(\rho'_r), (u_{S,i}(\rho'_r)_{i \in S \subset N}\} \to \{(a_{S,i}), (u_{S,i})_{i \in S \subset N}\}$. By the compactness assumption, let $(M, \ldots, M) \in \mathbb{R}^S_+$ be an upper bound of $V_0(S)$, then $|a^j_{S,i}(\rho) - a^j_S(\rho)| \leq M(1-\rho)$ for all $i, j \in S$, and all $S \subset N$. Therefore, in the limit, it holds that $a_{S,i} = a_S$, for all $i \in S \subset N$.

Suppose now that V satisfies also the smoothness assumption A.5. Let $\{(a_S), (u_S)_{S \subset N}\}$ be the limit payoff configuration as before, where $u_S = \frac{1}{s} \sum_{i \in S} u_{S,i}$, for all $S \subset N$.

First, note that $a^i_{\{i\}} = r^i$, then it trivially holds that $a_{\{i\}} = \zeta_{\{i\}}(V)$, for all $i \in N$.

Let a coalition $S \subset N$, with $s \geq 2$. Let $\pi_S(\rho) \in \mathbb{R}^S_{++}$ be defined either by any outward normal to $\partial V(S)$ at $u_S(\rho)$ if $u_S(\rho) \in \partial V(S)$, or by the vector $\left(\frac{1}{a^i_S(\rho) - u^i_S(\rho)}\right)_{i \in S}$ if $u_S(\rho) \in \operatorname{int} V(S)$ (note that in this case $a^i_{S,i}(\rho) > u^i_S(\rho)$ for all $i \in S$, and then $a^i_S(\rho) > u^i_S(\rho)$). Therefore, all $a^i_{S,i}(\rho)$, $i \in S$, belong to the hyperplane $H_S(\rho)$ defined by

$$H_S(\rho) := \left\{c \in \mathbb{R}^S : \pi_S(\rho) \cdot c = \pi_S(\rho) \cdot a_S(\rho)\right\}.$$

To check this, if $u_S(\rho) \in \partial V(S)$, then $u_S(\rho) = a_S(\rho) = a_{S,i}(\rho)$, for all $i \in S$. Hence $a_{S,i}(\rho) \in H_S(\rho)$ by definition of $\pi_S(\rho)$. If $u_S(\rho)$ int $V(S)$, we have that

$$\begin{aligned} a^i_{S,i}(\rho) &= s a^i_S(\rho) - \sum_{j \in S \backslash i} a^i_{S,j}(\rho) = s a^i_S(\rho) - \sum_{j \in S \backslash i} \left(\rho a^i_S(\rho) - (1-\rho) u^i_S(\rho)\right) \\ &= (s - \rho(s-1)) \left(a^i_S(\rho) - u^i_S(\rho)\right) + u^i_S(\rho). \end{aligned}$$

Hence,

$$\begin{aligned} \pi_S(\rho) \cdot a_{S,i}(\rho) &= \frac{a^i_{S,i}(\rho)}{a^i_S(\rho) - u^i_S(\rho)} + \sum_{j \in S \backslash i} \frac{a^j_{S,i}}{a^j_S(\rho) - u^j_S(\rho)} \\ &= (s - \rho(s-1)) + \frac{u^i_S(\rho)}{a^i_S(\rho) - u^i_S(\rho)} \end{aligned}$$

$$+\sum_{j\in S\backslash i}\frac{\rho\left(a_S^j(\rho)-u_S^j(\rho)\right)+u_S^j(\rho)}{a_S^j(\rho)-u_S^j(\rho)}$$

$$=s+\sum_{j\in S}\frac{u_S^j(\rho)}{a_S^j(\rho)-u_S^j(\rho)}=\pi_S(\rho)\cdot a_{S,j}(\rho)\,,\ \text{for all } i,j\in S.$$

Denote by $\pi_S^*(\rho)=\pi_S(\rho)\cdot\frac{1}{\sum_{i\in S}\pi_S^i(\rho)}$. Since $a_{S,i}(\rho)\to a_S$, and $a_{S,i}(\rho)\in\partial V(S)$, for all $i\in S$, it holds that $a_S\in\partial V(S)$. By smoothness of $\partial V(S)$, we have that $\pi_S^*(\rho)\to\lambda_S$, where λ_S is the outward unit length normal to $\partial V(S)$ at a_S;[13] λ_S being collineal to the vector $\left(\frac{1}{a_S^i-u_S^i}\right)_{i\in S}$ when $u_S\notin\partial V(S)$. Therefore, for λ_S, a_S, and u_S, conditions (a), (b), and (c), of Definition 1 are satisfied.

The unicity of the limit payoff configuration $\{(a_S), (u_S)_{S\subset N}\}$ follows by a straightforward induction argument. Since $a_{\{i\}}^i=r^i$, for all $i\in N$, the breakdown payoffs u_S are uniquely determined for all $S\subset N$, such that $s=2$. This implies the uniqueness of limit points a_S, for coalitions of size $s=2$. This fact implies again the uniqueness of limit points u_S, and hence the uniqueness of limit points a_S, for coalitions of size $s=3$, and so on and so forth, up to the grand coalition N. □

Remark Asymmetric solutions can be easily defined. We only need to assume that the proposers are chosen with different probabilities. Let $w\in\mathbb{R}_{++}^N$ be a vector of weights, and assume that the proposers are chosen in proportion to these weights. In particular, it holds that the asymmetric solution coincides with the *weighted* Shapley (1953) *value* for TU-games (see Kalai and Samet 1987; Hart and Mas-Colell 1989), and with the *weighted Consistent NTU-value* for H-games (see Maschler and Owen 1989; Calvo et al. 2001).

Random removal

The solidarity value in NTU-games is defined as follows.

Definition 2 Let $V\in\mathscr{G}$ and $\mathbf{a}=(a_S)_{S\subset N}$ be a payoff configuration. Then $\mathbf{a}$ is the Solidarity value ψ of V (i.e., $\mathbf{a}=\psi(V)$) if and only if for each $S\subset N$ there exists a vector $\lambda_S\in\mathbb{R}_{++}^S$ such that:
(a) $a_S\in\partial V(S)$;
(b) $\lambda_S\cdot a_S=v(S,\lambda_S):=\max\{\lambda_S\cdot c : c\in V(S)\}$; and
(d) $\lambda_S^i\left(a_S^i-d_S^i\right)=\lambda_S^j\left(a_S^j-d_S^j\right)$ for all $i,j\in S$,
where $d_S^i:=\frac{1}{s}\sum_{k\in S\backslash i}a_{S\backslash k}^i$ for all $i\in S$.

Condition (d) is also a λ_S-*egalitarian* type condition. The difference with condition (c) of the random marginal value lies in the definition of the breakdown point d_S. The

[13] Note here that for $|S\geq 3|$ the set of equilibrium offers $a_{S,i}(\rho)$ is not necessarily a singleton; and without smoothness on $\partial V(S)$, the convergence to $\zeta_S(V)$ may fail too. See Proposition 8.1, and Remark 3, in Sect. 8, of Thomson and Lensberg (1989).

vector $d_S = \frac{1}{s}\sum_{i\in S}\left(a_{S\setminus i}, 0\right)$ gives the expected payoff allocation for players in S when each member has an equal chance of leaving the game, obtaining a zero payoff.

Parallel results to the random marginal value can be obtained for the solidarity value.

Proposition 5 *If $V \in \mathcal{G}$, then the solidarity value ψ of V exists and it is unique. Moreover, if V is a TU-game, then $\psi(V)$ coincides with the solidarity TU-value of V.*

Theorem 4 *Let $V \in \mathcal{G}$ be an NTU-game. Then for each $0 \leq \rho < 1$ there is an SP equilibrium of the RR-model. Moreover, if V satisfies the additional assumption A.5, when $\rho \to 1$, then every SP equilibrium payoff configuration $\mathbf{a}(\rho)$ converges to $\mathbf{a} = \psi(V)$.*

The proofs are fully identical to the RM-model, exchanging the roles of u_S by d_S, hence they are left to the reader.

Remark The solidarity value has an interesting link with the *equal split* value, relative to $(0, \ldots, 0)$. In Hart and Mas-Colell (1996a) this value is obtained as one of the possible variations of the breakdown technology. In particular, as they point out in Case (b):

Only the responders (but not the proposer) drop out, all with equal probability.

This value is defined in NTU-games as follows:

Definition 3 Let $V \in \mathcal{G}$ and $\mathbf{a}= (a_S)_{S\subset N}$ be a payoff configuration. Then $\mathbf{a}$ is the *equal split value* ϕ of V (i.e., $\mathbf{a} = \phi(V)$) if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$ such that:
(a) $a_S \in \partial V(S)$;
(b) $\lambda_S \cdot a_S = v(S, \lambda_S) := \max\{\, \lambda_S \cdot c : c \in V(S)\}$; and
(e) $\lambda^i_S a^i_S = \lambda^j_S a^j_S$ for all $i, j \in S$.

Note that if V is a TU-game, then $a^i_S = \frac{v(S)}{s}$, for all $i \in S \subset N$. Hart and Mas-Collel mention that, in TU-games, the solidarity value approaches, for a large number of players, the equal split value. We will confirm this assertion for large NTU-games in the next section.

Remark The properties of *Uniqueness* and *Symmetry* are the main motivation of the paper. By *Uniqueness* we mean that the payoff configuration of the solution is single-valued. This property is important as long as we consider a solution as a method to find an agreement when an alternative must be chosen over a set that produces a different ranking of preferences. This lack of unanimity is *not solved* when a solution selects a *subset* of alternatives, because we again have the same ranking problem among players, but now over the subset selected. The example in Sect. 3 shows that the Shapley, Harsanyi and Consistent NTU-values do not satisfy this requirement.[14]

[14] For the Consistent NTU-value a three-person example of non uniqueness can be found in Owen (1994).

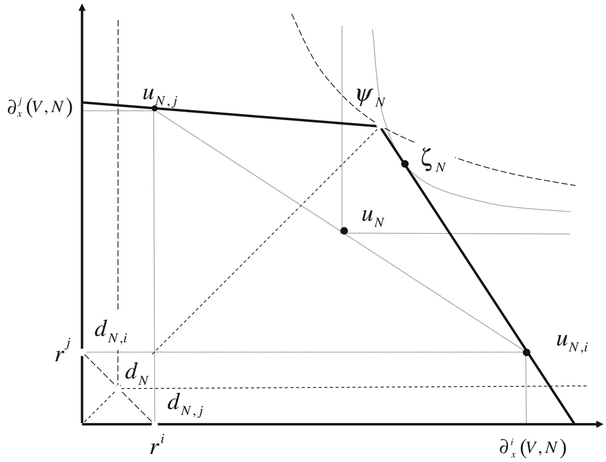

Fig. 5 Differences between the random marginal ζ_N and the random removal ψ_N payoffs

In a TU-game v we say that two players i and j are *substitutes* if $v(S \cup i) = v(S \cup j)$ for all $S \subset N \backslash \{i, j\}$. We extend this property for NTU-games as follows: We say that two players i and j are *substitutes* in a game V if for all coalition $S \subset N \backslash \{i, j\}$, and $x^S \in \mathbb{R}^S$, it holds that $\left(x^S, a\right) \in V(S \cup i)$ if and only if $\left(x^S, a\right) \in V(S \cup j)$. We say that a solution satisfies *Symmetry* if it yields the same payoffs for substitute players, i.e. for every $\mathbf{x} \in \Psi(V)$, $x_S^i = x_S^j$ for all $i, j \in S \subset N$, whenever i and j are substitutes in V. Given the way in which the random marginal and the solidarity NTU-values are built, it can be checked that they also satisfy Symmetry in $\mathscr{G}$.

Remark Although in the example of Sect. 3, our two bargaining procedures support, in the limit, the same point $c = (50, 50)$; this fact is not true in general, as can be seen in the two person problem illustrated in Fig. 5.

Remark As already mentioned, the extensions given by Harsanyi (1963), Shapley (1969), and Maschler and Owen (1992), of the Shapley value to the general NTU-setting are based on the Nash (1950) bargaining solution, because all of them coincide with it when they are applied to pure bargaining problems. Nevertheless, a critical question when comparing pure bargaining solutions is the following: On which feasible points, besides the disagreement point, should a bargaining solution depend? The Nash solution depends only on a small neighborhood of the solution point, more precisely, on the slope of the boundary of the feasible set at the solution point. Kalai and Smorodinsky (1975) criticized the Nash solution exactly for that reason. They axiomatized a solution proposed by Raiffa (1953), which depends also on the utopia points (called here by $u_{N,i}$, $i \in N$).[15]

[15] Other proposals are: the Super-Additive solution, proposed by Perles and Maschler (1981), which depends on the whole Pareto optimal boundary; and the Equal Area solution, proposed by Anbarci and Bigelow (1994), which depends on the whole feasible set.

Note that the RM-value, in its construction, partially incorporate the spirit of the Raiffa proposal, because it changes the original Nash disagreement point by the new one u_N, which is an average of the utopia points $u_{N,i}$, $i \in N$. Hence, the RM-value is sensitive to changes in the utopia points.

Remark It is possible to define a family of values $\zeta(\lambda)$, where $0 \leq \lambda \leq 1$, by changing the conditions (c) and (d) by $\lambda_S^i \left(a_S^i - h(\lambda)_S^i\right) = \lambda_S^j \left(a_S^j - h(\lambda)_S^j\right)$ for all $i, j \in S$, where $h(\lambda)_S = \lambda u_S + (1-\lambda) d_S$.

6 Large games

We now study how the random marginal and the solidarity values should behave in large games with non-transferable utility. More specifically, we will consider only *differentiable market games* for which the *Value equivalence* theorem holds (Aumann 1975); that is, the equivalence between the *value allocations* and the *core allocations*. The basic references and results on this topic can be found in Hart (1994, 2001).

We restrict our analysis to the case of continuum games with finitely many types of players, where each *coalition* is characterized by its composition. Let $N = \{1, \ldots, n\}$ be the set of types. The *profile* of a coalition is a vector $\mu \in \mathbb{R}_+^N$, where μ^i is the mass of players of type i in the coalition. The game form will specify the sets of feasible payoff vectors for all coalitions, i.e., for all $\mu \in \mathbb{R}_+^N$. We consider only *type-symmetric* payoffs, where all players of the same type get the same payoff.[16] For every $\mu \in \mathbb{R}_+^N$, let $V(\mu) \subset \mathbb{R}^N$ be the set of *feasible per-capita payoff vectors* for a coalition with profile μ. This point-to-set map V is called the *NTU-game form*. Given V we define also the set of *feasible total per-type payoffs* by $\hat{V}(\mu) := \{\mu * x : x \in V(\mu)\}$, for all $\mu \in \mathbb{R}_+^N$. Define $\hat{v}(\mu, \lambda) := \sup\{\lambda \cdot (x * \mu) : x \in V(\mu)\} \equiv \sup\{\lambda \cdot y : y \in \hat{V}(\mu)\}$; $\hat{v}(\mu, \lambda)$ is the continuum TU-game associated to V with utility comparison weights λ. We make the following assumptions:

(C.1) For every $\mu \in \mathbb{R}_+^N$, $V(\mu)$ is a non-empty and strict subset of $\mathbb{R}^N$, it is closed, convex, comprehensive and non-level. $V_0(\mu) := V(\mu) \cap \mathbb{R}_+^N$ is non-empty and bounded.

(C.2) *Super-additivity:* $\hat{V}(\mu) + \hat{V}(\gamma) \subset \hat{V}(\mu + \gamma)$ for every $\mu, \gamma \in \mathbb{R}_+^N$.

(C.3) *Differentiability:* For any $\lambda \in \mathbb{R}_{++}^N$, the gradient $\nabla_\mu \hat{v}(\bar{\mu}, \lambda)$ exists for all $\bar{\mu} \in \mathbb{R}_{++}^N$. $\nabla_\mu \hat{v}(\bar{\mu}, \lambda)$ is uniformly bounded and uniformly positive for every bounded subset of $\mathbb{R}_{++}^N$. Moreover $\hat{v}(\mu, \lambda)$ is C^2 on its domain.

(C.4) *Homogeneity:* $V(t\mu) = V(\mu)$ for every $\mu \in \mathbb{R}_+^N$ and $t > 0$ (in terms of total payoffs, $\hat{V}(t\mu) = t\hat{V}(\mu)$, and $\hat{v}(t\mu, \lambda) = t\hat{v}(\mu, \lambda)$).

Assumptions C.1 and C.2 are standard. Differentiability simplifies the exposition. It allows us to establish the condition that characterizes the values in terms of a first order partial differential equations system. Note that the differentiability of $\hat{v}(\mu, \lambda)$

16 Since the random marginal and random removal values yield the same payoffs to substitute players, this is not a real restriction.

implies both the smoothness of $\partial V(\mu)$, and the strict convexity of $V(\mu)$. Therefore, there is a unique $x \in V(\mu)$ such that it holds $\lambda \cdot (x * \mu) = \hat{v}(\mu, \lambda)$. Conversely, for any $x \in \partial V(\mu)$ there is a unique λ, such that it holds $\lambda \cdot (x * \mu) \geq \lambda \cdot (y * \mu)$, for all $y \in V(\mu)$. That is, there is a unique supporting hyperplane to $\hat{V}(\mu)$ at $x * \mu$, being λ its corresponding outward normal vector (support vector). The homogeneity assumption is stronger, usually associated with economic considerations. For example, in *market games* that correspond to *pure exchange economies* , where each type i possesses a utility function u_i concave and non-decreasing, with the slope everywhere bounded away from 0 and infinity, and $u_i(\omega_i) = 0$ (where ω_i is the initial endowment). This economy is represented by a game which satisfies C.1–C.4.

The first extension of the value in this setting was the Shapley NTU-value (see Shapley 1969; Shapley and Shubik 1969). We recall its characterization for finite games:

Definition 4 Let $V \in \mathcal{G}$ and $a = (a_S)_{S \subset N}$ be a payoff configuration. Then a_N is the Shapley NTU-value φ of V if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$ such that:

(a') $a_N \in \partial V(N)$;
(b) $\lambda_S \cdot a_S = v(S, \lambda_S)$; and
(f) $\lambda^i_S \left(a^i_S - a^i_{S \setminus j}\right) = \lambda^j_S \left(a^j_S - a^j_{S \setminus i}\right)$ for all $i, j \in S$.

Note that, in the payoff configuration a, condition (a') of efficiency (and hence, feasibility) is only required for the grand coalition N. Condition (f) is the λ-balanced Contributions property. The main drawback of this definition is given by the possible non-feasibility of the payoffs a_S, for $S \neq N$ (except in the TU-case). The construction of the continuum Shapley NTU-value is as follows: Given a continuum game V, and a vector of weights $\lambda \in \mathbb{R}^N_{++}$, build the continuum TU-game $\hat{v}(\mu, \lambda)$. Under differentiability, the Aumann and Shapley (1974) value of $\hat{v}(\mu, \lambda)$ is defined by

$$\varphi^i\left(\hat{v}(\mu, \lambda)\right) = \int_0^1 \frac{\partial \hat{v}(t\mu, \lambda)}{\partial \mu^i} dt, \quad \text{for all } i \in N.$$

Therefore, we have:

Definition 5 Let V be a continuum game with a finite type of players N, satisfying (C.1), (C.2), and (C.3). Let $x(\mu)$ be a per capita payoff configuration, $\mu \in \mathbb{R}^N_{++}$. Then $x(\mu)$ is a *continuum Shapley NTU-value* of V at μ, if there exists $\lambda(\mu) \in \mathbb{R}^N_{++}$ such that
(1) $x(\mu) \in \partial V(\mu)$; and
(2) $\lambda(\mu) * x(\mu) = \varphi(\hat{v}(\mu, \lambda(\mu)))$.

Note that φ satisfies efficiency, so $\mu \cdot \varphi(\hat{v}(\mu, \lambda(\mu))) = \hat{v}(\mu, \lambda)$. Replacing the finite differences condition (f) by derivatives, we have the equivalent definition:

Definition 6 Let V be a continuum game with a finite type of players N, satisfying (C.1), (C.2), and (C.3). Let $x(\mu)$ be a per capita payoff configuration, $\mu \in \mathbb{R}^N_{++}$. Then

$x(\mu)$ is a *continuum Shapley NTU-value* of V at μ, if there exists $\lambda(\mu) \in \mathbb{R}^N_{++}$ such that

(i) $x(\mu) \in \partial V(\mu)$;
(ii) $\lambda(\mu) \cdot (\mu * x(\mu)) = \hat{v}(\mu, \lambda(\mu))$; and
(iii) $\lambda^i(\mu)\frac{\partial x^i(\mu)}{\partial \mu^j} = \lambda^j(\mu)\frac{\partial x^j(\mu)}{\partial \mu^i}$, for all $i, j \in N$.

Conditions (i) and (ii) are just the efficient and λ-utilitarian conditions. Condition (ii) is the λ-Balanced Contributions property.[17] A very remarkable result is that, under homogeneity, value allocations and core allocations coincide (Aumann 1975; see also Hart and Mas-Colell (1996b), Corollary VII.2).

For the Consistent value, its characterization for finite NTU-games is given by (Hart and Mas-Colell 1996a):

Proposition 6 *Let $V \in \mathcal{G}$ and* $\mathbf{a}= (a_S)_{S\subset N}$ *be a payoff configuration. Then* $\mathbf{a}$ *is the* consistent NTU-value *of V if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$ such that:*

(a) $a_S \in \partial V(S)$*;*
(b) $\lambda_S \cdot a_S = v(S, \lambda_S)$*; and*
(g) $\sum_{j\in N\setminus i} \lambda^i_S \left(a^i_S - a^i_{S\setminus j}\right) = \sum_{j\in N\setminus i} \lambda^j_S \left(a^j_S - a^j_{S\setminus i}\right)$ *for all $i \in S$.*

Where condition (g) is a short of averaged λ-balanced contributions property. In Owen (1996) and Leviatan (2002), this definition is extended to the continuum setting as follows:

Definition 7 Let V be a continuum game with a finite type of players N, satisfying (C.1), (C.2), and (C.3). Let $x(\mu)$ be a C^1 per capita payoff configuration, $\mu \in \mathbb{R}^N_{++}$. Then $x(\mu)$ is a *continuum consistent NTU-value* of V at μ, if there exists $\lambda(\mu) \in \mathbb{R}^N_{++}$ such that

(i) $x(\mu) \in \partial V(\mu)$;
(ii) $\lambda(\mu) \cdot (\mu * x(\mu)) = \hat{v}(\mu, \lambda(\mu))$; and
(iv) $\sum_{k\in N} \lambda^i(\mu)\mu^k \frac{\partial x^i(\mu)}{\partial \mu^k} = \sum_{k\in N} \lambda^k(\mu)\mu^k \frac{\partial x^k(\mu)}{\partial \mu^i}$, for all $i \in N$.

It is proven in both papers that, assuming homogeneity, the Consistent and the Shapley NTU-values coincide in the continuum setting.[18]

We will see now the definitions of the random marginal and random removal values in the continuum.

Random marginal

Firstly, note that condition (c) in Definition 1:

[17] Note that this is the Perron-Frobenius integrability condition for the existence of a Potential. See Hart and Mas-Colell (1989) for the characterization of the Shapley value in terms of a potential function (see also Calvo and Santos 1997).

[18] This is not the case for the continuum extension of the Harsanyi NTU-value, as can be seen in Hart and Mas-Colell (1996b).

(c) $\lambda_S^i \left(a_S^i - u_S^i\right) = \lambda_S^j \left(a_S^j - u_S^j\right)$ for all $i, j \in S$, where $u_S^i := \frac{1}{s}\Big(\partial_{\mathbf{a}}^i (V, S) + \sum_{k \in S \setminus i} a_{S \setminus k}^i\Big)$ for all $i \in S$,
can be rewritten as

(c')

$$\lambda_S^i \left[\left(a_S^i - \partial_{\mathbf{a}}^i (V, S)\right) + \sum_{k \in S \setminus i} \left(a_S^i - a_{S \setminus k}^i\right) \right] = \lambda_S^j \left[\left(a_S^j - \partial_{\mathbf{a}}^j (V, S)\right) + \sum_{k \in S \setminus i} \left(a_S^j - a_{S \setminus k}^j\right) \right],$$

for all $i, j \in S$.

This leads to the following definition:

Definition 8 Let V be a continuum game with a finite type of players N, satisfying (C.1), (C.2), and (C.3). Let $x(\mu)$ be a C^1 per capita payoff configuration, $\mu \in \mathbb{R}_{++}^N$. Then $x(\mu)$ is a *continuum RM-value* of V at μ, if there exists $\lambda(\mu) \in \mathbb{R}_{++}^N$ such that
(i) $x(\mu) \in \partial V(\mu)$;
(ii) $\lambda(\mu) \cdot (\mu * x(\mu)) = \hat{v}(\mu, \lambda(\mu))$; and
(v) $\lambda^i(\mu) \left(x^i(\mu) - \frac{1}{\lambda^i(\mu)} \frac{\partial \hat{v}(\mu, \lambda(\mu))}{\partial \mu^i} + \sum_{k \in N} \mu^k \frac{\partial x^i(\mu)}{\partial \mu^k}\right)$
$= \lambda^j(\mu) \left(x^j(\mu) - \frac{1}{\lambda^j(\mu)} \frac{\partial \hat{v}(\mu, \lambda(\mu))}{\partial \mu^j} + \sum_{k \in N} \mu^k \frac{\partial x^j(\mu)}{\partial \mu^k}\right)$,
for all $i, j \in N$.

We now see the relationship between the RM-value and the Shapley NTU-value in continuum games, assuming homogeneity:

Theorem 5 *Let V be a continuum game with a finite type of players N, satisfying (C.1), (C.2), (C.3), and (C.4). Let $x(\mu)$ be a homogeneous C^1 per-capita payoff configuration, $\mu \in \mathbb{R}_{++}^N$. Then $x(\mu)$ is a continuum RM-value if and only if it is a Shapley NTU-value of V at μ.*

Proof Let $x(\mu)$ be a continuum MR-value of V at μ, and assume that it is homogenous of degree zero. Then, by Euler's formula, $\sum_{k \in N} \mu^k \frac{\partial x^i(\mu)}{\partial \mu^k} = 0$, and hence condition (v) is equivalent to

$$\lambda^i(\mu) x^i(\mu) - \frac{\partial \hat{v}(\mu, \lambda(\mu))}{\partial \mu^i} = k; \quad \text{for all } i \in N,$$

where $k \in \mathbb{R}$. By multiplying this expression by μ^i, and adding it over all i, we have

$$\sum_{i \in N} \lambda^i(\mu) \mu^i x^i(\mu) - \sum_{i \in N} \mu^i \frac{\partial \hat{v}(\mu, \lambda(\mu))}{\partial \mu^i} = \left(\sum_{i \in N} \mu^i\right) k.$$

By condition (ii), $\sum_{i\in N}\lambda^i(\mu)\mu^i x^i(\mu) = \hat{v}(\mu,\lambda(\mu))$. And since $\hat{v}(\mu,\lambda(\mu))$ is homogeneous of degree 1, by Euler's formula again, $\sum_{i\in N}\mu^i\frac{\partial\hat{v}(\mu,\lambda(\mu))}{\partial\mu^i}=\hat{v}(\mu,\lambda(\mu))$. As $\sum_{i\in N}\mu^i > 0$, it follows that $k = 0$, which implies that

$$\lambda^i(\mu)x^i(\mu) = \frac{\partial\hat{v}(\mu,\lambda(\mu))}{\partial\mu^i}, \quad \text{for all } i \in N.$$

On the other hand, under homogeneity,[19]

$$\varphi^i\left(\hat{v}(\mu,\lambda(\mu))\right) = \int_0^1 \frac{\partial\hat{v}(t\mu,\lambda(\mu))}{\partial\mu^i}dt = \frac{\partial\hat{v}(\mu,\lambda(\mu))}{\partial\mu^i}, \quad \text{for all } i \in N, \tag{3}$$

which finally yields

$$\lambda(\mu) * x(\mu) = \varphi\left(\hat{v}(\mu,\lambda(\mu))\right).$$

Since in homogenous and differentiable games the Shapley NTU-value is a homogenous and differentiable mapping, it turns out that it satisfies conditions (i), (ii), and (v), hence both values coincide. □

This result has an easy interpretation. By using the vector of weights $\lambda(\mu)$ for comparing the utilities among players, the term $\lambda^i(\mu)x^i(\mu)$ measures the utility obtained by player i. the term $\hat{v}(\mu,\lambda(\mu))$ is total welfare of society, and $\frac{\partial\hat{v}(\mu,\lambda(\mu))}{\partial\mu^i}$ is the contribution of player i to total welfare, at the margin. Under homogeneity, it turns out that both terms coincide and then the value is characterized by the condition that players get *exactly* their contribution to total welfare.

Remark Theorem 5 is easily extended for any degree r of homogeneity, i.e., $V(t\mu) = t^r V(\mu)$ for every $\mu \in \mathbb{R}^N_+$ and $t > 0$ (in terms of total payoffs, $\hat{V}(t\mu) = t^{r+1}\hat{V}(\mu)$, and $\hat{v}(t\mu,\lambda) = t^{r+1}\hat{v}(\mu,\lambda)$). In that case the vector of weights $\lambda(\mu)$ is again of degree 0, the continuum RM-values $x(\mu)$ are of degree r, and take into account, in condition (v), that $\sum_{k\in N}\mu^k\frac{\partial x^i(\mu)}{\partial\mu^k} = rx^i(\mu)$, and, in (3), that $\frac{\partial\hat{v}(t\mu,\lambda(\mu))}{\partial\mu^i} = t^r\frac{\partial\hat{v}(\mu,\lambda(\mu))}{\partial\mu^i}$, they are characterized by

$$\lambda^i(\mu)x^i(\mu) = \frac{1}{r+1}\frac{\partial\hat{v}(\mu,\lambda(\mu))}{\partial\mu^i} = \varphi^i\left(\hat{v}(\mu,\lambda(\mu))\right), \quad \text{for all } i \in N.$$

So, in general, players are paid *in proportion* to what they contribute to total welfare of society.

If the continuum game V is non-homogeneous, then the RM-value, the Shapley NTU-value, and the consistent NTU-value, will generally yield different payoffs. What

[19] Not that the vector of weights $\lambda(\mu)$ associated to $x(\mu)$ is also homogeneous of degree zero.

the right definition of the value should be in the general non homogeneous setting, remains a challenging open question.

Random removal

Condition (d) for the solidarity value is equivalent to

(d') $\lambda_S^i \left(x_S^i + \sum_{k \in S \setminus i} \left(x_S^i - x_{S \setminus k}^i \right) \right) = \lambda_S^j \left(x_S^j + \sum_{k \in S \setminus i} \left(x_S^j - x_{S \setminus k}^j \right) \right)$, for all $i, j \in S$.

Therefore, by using similar arguments as in the RM-value, we can define the solidarity value in continuum games as follows:

Definition 9 Let V be a continuum game with a finite type of players N, satisfying (C.1), (C.2), and (C.3). Let $x(\mu)$ be a C^1 per capita payoff configuration, $\mu \in \mathbb{R}_{++}^N$. Then $x(\mu)$ is a *continuum solidarity value* of V at μ, if there exists $\lambda(\mu) \in \mathbb{R}_{++}^N$ such that

(i) $x(\mu) \in \partial V(\mu)$;

(ii) $\lambda(\mu) \cdot (\mu * x(\mu)) = \hat{v}(\mu, \lambda(\mu))$; and

(vi) $\lambda^i(\mu) \left(x^i(\mu) + \sum_{k \in N} \mu^k \frac{\partial x^i(\mu)}{\partial \mu^k} \right) = \lambda^j(\mu) \left(x^j(\mu) + \sum_{k \in N} \mu^k \frac{\partial x^j(\mu)}{\partial \mu^k} \right)$, for all $i, j \in N$.

If we also assume homogeneity as in Theorem (5), by Euler's formula, $\sum_{k \in N} \mu^k \frac{\partial x^j(\mu)}{\partial \mu^k} = 0$. Hence, condition (vi) turns into condition

(vii) $\lambda^i(\mu) x^i(\mu) = \lambda^j(\mu) x^j(\mu)$, for all $i, j \in N$,

which, jointly with conditions (i) and (ii), characterize the *continuum equal split NTU-value*. Thus, in homogeneous games, the random removal and the equal split values coincide. Moreover, from (vi) it holds that $\lambda^i(\mu) x^i(\mu) = k$; for all $i \in N$, and then

$$\sum_{i \in N} \lambda^i(\mu) \mu^i x^i(\mu) = \left(\sum_{i \in N} \mu^i \right) k = \hat{v}(\mu, \lambda(\mu)).$$

Therefore

$$\lambda^i(\mu) x^i(\mu) = \frac{\hat{v}(\mu, \lambda(\mu))}{\sum_{i \in N} \mu^i}. \tag{4}$$

Hence, all players get exactly the same utility which is equal to the per capita welfare of society.

Note again that this result can be extended to any degree of homogeneity, as in the previous remark for the RM-values.

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