Research in Mathematics Education
Vol. 11, No. 2, September 2009, 199–211

BOOK REVIEWS

Visual thinking in mathematics: an epistemological study, by Marcus Giaquinto, Oxford, Oxford University Press, 2007, ISBN 978-0-19-928594-5, 304 pp, Hardback, £40.

When approaching the uses and roles of visualisation in doing, teaching or learning mathematics, several positions can be adopted. Mathematics educators know that visual thinking and visualisation are key elements in learning mathematics at any educational level, so they pay special attention to the places of visual (i.e., mental) images in the processes of learning mathematics, the relevance of students' visualisation processes and abilities, or ways to improve teaching by designing visual learning environments. On the other hand, while mathematicians usually only show their outcomes in the abstract style typical of logico-deductive formal proofs, several studies show that visualisation plays a central role in mathematicians' processes of creation, discovery or proof of new results. Additionally, some positions from outside mathematics and mathematics education about the role played by visualisation in mathematics also contribute to producing a detailed picture of the field.

Giaquinto's book adds an interesting opinion about this question. From his speciality as a philosopher interested in the philosophy of science and, in this book, the epistemology of mathematics, Giaquinto makes a detailed analysis of the place of visualisation in the processes of learning and doing mathematics. His core point is that visual images and visualisation are not just accessory elements for mathematicians, teachers and students, but they play a relevant role, since images may help us understand a new concept or suggest a way to prove a new conjecture. Then Giaquinto adopts an epistemological position to analyse the influence of visual images in creating new beliefs and knowledge.

This is not a book on mathematics education, but it should be of interest to mathematics educators since it adopts a point of view that is complementary to the mathematics educator's focus on the processes of doing, teaching and learning mathematics. When reading the book, the text quite often brought to my mind different theories or results established in mathematics education related to, and coherent with, Giaquinto's points.

Chapters 2 to 5 reflect on several aspects of geometry. Chapters 2 and 3 are devoted to analysing the role of visualisation in the acquisition of simple 2-dimensional geometric concepts, using the square as a paradigmatic example. To answer the question "how can we acquire *basic* geometrical knowledge?" several aspects of shape perception and concept formation are considered. Visual perception of geometric shapes induces the acquisition of geometrical beliefs that, in turn, are narrowly related to the formation of geometrical concepts. Variables like position, orientation, reference systems, or intrinsic properties like symmetry influence the formation of the geometrical concepts. Giaquinto differentiates between *perceptual*

ISSN 1479-4802 print/ISSN 1754-0178 online
DOI: 10.1080/14794800903063802
http://www.informaworld.com

and *geometrical* concepts. Perceptual concepts are vague, based on visual experience, sensitive to the above mentioned variables and dependent on the perfection of specific examples observed (e.g., to what extent can a shape with non-straight sides be accepted as an example of a square?). Conversely, geometrical concepts are less based on pure perception and require some characteristics like reliability and rationality of beliefs.

From a mathematics education point of view, this description of the acquisition of geometric concepts and the two types of concepts are related to Van Hiele levels 1 and 2 of mathematical reasoning (Burger and Shaughnessy, 1986; Fuys, Geddes, and Tischler, 1988). As for perceptual concepts, reasoning at level 1 is very dependent on physical (mainly visual) characteristics of the examples shown. Level 2 reasoning implies, as for geometrical concepts, the recognition of geometrical properties (i.e. ideal properties) as the main source of knowledge. In parallel with Van Hiele levels, Vinner's model of the acquisition of mathematical concepts (concept image and concept definition; Vinner, 1991) is also related to the distinction made by Giaquinto between the formation of perceptual concepts (related to acquisition of concept images) and geometrical concepts (related to a more integrated acquisition of concept image and definition).

The two main activities in mathematics are to discover new mathematical conjectures and to prove them. Chapters 4 and 5 are devoted to analysing the role of visualisation and diagrams in those tasks. The main question analysed in Chapter 4 is whether visualisation and diagrams are valid means of geometrical discovery. According to Giaquinto, a mathematical discovery requires three conditions: first, independence, that is that the discovery is made by oneself and one believes in its truth by one's own reasoning; secondly, reliability of the procedures and reasoning used to make the discovery; and thirdly, rationality, that is that the new discovery, doesn't contradict one's epistemic rationality. Quite often, geometric experiments leading to a discovery are based on visualising structures, like a 3-dimensional object, or processes, like folding a part of the object. In these cases, the belief is sustained by inductive inferences made after the visualisation. There is the danger that such induction is based on wrong visual transformations or on specific cases that may provide false information and, therefore, it may produce a wrong discovery. After alerting the reader to the potential pitfalls in the use of visualisation as source of discoveries, and analysing different common types of errors produced when visualising, the author's conclusion is that visual elements are an important part of the processes of mathematical discovery.

If we look at this disquisition from an educational point of view (which is not the case of the author), his analysis and conclusions should be a little different. The author's position is close to that of the mathematician's, but far from that of students of mathematics. He expects a mathematician to be careful to control the mentioned possible sources of errors besides independence, reliability and epistemic rationality, but such an expectation is unrealistic when discoveries are made by students, particularly by primary or secondary students. An educational implication of this chapter is that mathematics teachers should promote the use of visualisation and diagrams by their pupils to discover new mathematical content and, at the same time, they should teach the student to be careful in controlling the sources of errors or mistakes when using such tools.

Chapter 5 is devoted to discussing objections to the use of diagrams and visualisation as part of mathematical proofs. Again, Giaquinto analyses possible objections and shows that they are not consistent. However, in this case, his points to support the role of diagrams and visualisation are weak. While in previous chapters Giaquinto has put himself in the position of a learner or a producer of new geometrical knowledge, in this chapter his position is of a reader of proofs, that is a receiver of information. The chapter first of all identifies and analyses the possible roles of a diagram as part of a geometric proof presented to be followed and understood by others different from its author. He identifies three main roles of a diagram in a proof, and presents examples of each: first, superfluous diagrams; secondly, non-superfluous but replaceable diagrams (some information is included only in the diagram but it might be added to the text and then the diagram becomes superfluous); and thirdly, neither superfluous nor replaceable diagrams.

In his refutation of objections to diagrams in proofs, Giaquinto succeeds in showing that, in some cases, diagrams need to be part of proofs as they provide the ground for generalisation and formalisation. However, there is a significant gap when he argues against the claim that in some contexts any argument containing a generalisation from a diagram will be insufficiently transparent to count as a proof in a formal system. To show that this claim is false, Giaquinto presents a "formal diagrammatic system of Euclidean geometry", aimed at providing a formal axiomatic system, based on graphic representations of points, circles, lines, etc., allowing graphical proofs of the theorems of Euclidean geometry. The example shown in the book is a graphic proof of Euclid's proposition I.I (given a segment, an equilateral triangle can be obtained). Unfortunately, the gap in Euclid's proof (it cannot be proved that the two circles intersect) is also present in this graphical proof. Nelsen (2000) provides many real visual proofs, for instance the well-known visual proof that $\Sigma\ 1/4^n = 1/4 + 1/16 + 1/64 + \ \ldots\ = 1/3$.

As a complement to Giaquinto's point in this chapter, mathematics educators know that diagrams are critical elements of mathematical proofs, particularly when they have to be followed or made by secondary school students. Balacheff's categories of empirical and deductive proofs show how the role of examples (in particular diagrams) in proofs changes with students' understanding of the concept of proof and how to make proofs (Balacheff, 1988). A part of such a learning process is to become aware of the objections to, and conditions of, the use of diagrams in mathematical proofs.

Chapters 6 to 8 are devoted to reflections on aspects of numbers and arithmetic. Chapter 6 approaches the use of number lines as a tool to learn numbers and early arithmetic. Number lines are present in every mathematics textbook, and they have an even more relevant position in coordinate systems. In this chapter, Giaquinto focuses on young children's learning of basic number facts, counting and arithmetic. A first question analysed is whether there is an innate sense of number and cardinality. Several experiments trying to answer this question are presented, concluding that some variables, neuropsychological, educational and cultural, seem to influence subjects' ability to deal with numbers and their visual images. For instance, depending on their culture, students find it easier to move on the number line from left to right, or vice versa; size of numbers also influences tasks like ordering or comparison.

An aspect especially relevant for visualisation is the infiniteness of the number line, since both drawings and visual images generated by most people show finite parts of the line. An author's conjecture is that number lines visualised are usually calibrated, like lines drawn on textbooks, and subjects produce different kinds of images, where calibration may or not be regular, to try to visualise an endless line.

Chapter 7 advances into the analysis of relationships between visualisation and arithmetic by focusing on calculations. The fact that geometry is much more "visual" than arithmetic influences the quite different styles of the chapters devoted to geometry and arithmetic. The first case presented is the learning of numbers and addition of one-digit numbers by young children. Giaquinto summarises some elements of Fuson's theories (well known by mathematics educators and elementary school teachers) to show how children use a combination of visual images, motor actions with fingers (the objects to be visualised) and verbal recitations to count and calculate additions or subtractions. An important issue raised is the epistemic value of evidence and conviction of such manipulations: children believe that, if it is true that 4 fingers plus 3 fingers add up to 7 fingers, then the same result is true for every $4 + 3$ objects added. An objection that can be made to the author's point is that he analyses young children's actions and beliefs from an adult's perspective, as shown when he suggests that most people (children in particular) have a tacit appreciation of the set theoretic principle relating set union to cardinal addition, and principles related to the other arithmetic operations. Mathematics teachers and educators know that those principles were applied in the 1970s and 80s to design the 'Modern Mathematics' curricula, but subsequent research showed that the hypothesis that primary school children would relate set theory to whole number arithmetic was completely wrong.

Multi-digit addition and multiplication are also analysed to show the relevance of visualisation and visual images in the learning of the usual paper and pencil algorithms. Giaquinto identifies several visualisation abilities, namely shape orientation (to differentiate 6 from 9), perception of horizontal and vertical relative positions (to differentiate 18 from 81, and to align the digits correctly). Other visualisation abilities should be added to these, like eye-motor coordination (to focus accurately on different digits or places) and figure-ground perception (to isolate two digits from the complete numbers) (Del Grande, 1990).

The last chapter devoted to arithmetic, Chapter 8, focuses on visual representations of numbers as used to represent, for instance, divisibility properties, equations or algebraic relationships. Giaquinto raises the difficulty that visual thinking strategies used to generalise geometric relationships and properties are not useful in arithmetic, because there is an infinite number of figures sharing all their mathematical properties but no two numbers share all of them. This chapter is devoted to showing that, in any case, visualisation is very useful in aiding the discovery, understanding and proof of many arithmetic and algebraic theorems. The examples used by the author are the well-known visual proof that $1+2+ \dots +n = n(n+1)/2$, and a representation of square numbers by dot arrays. The author analyses two main objections raised against the use of visualisation in this context: the particularity of representations (a graphical representation of a relationship between numbers is necessarily based on specific, and unique, numbers) and the unintended exclusions (when making a visual representation, we may inadvertently visualise an example that does not represent part of the cases). The author revisits those objections, and argues against them to show that, certainly, visual representations may be too partial or biased, but good representations of number patterns or properties help to generalise or to prove the properties. Images are vehicles of information, and transformations, operations, or properties in an image are ways to reliably access general transformations, operations, or properties of numbers.

In a progression to analysing more complex mathematical contexts, Chapter 9 is devoted to the role of visual thinking in basic functional analysis. The argument in this chapter changes: while previous chapters showed that visual thinking is necessary or, at least, helpful in different areas of mathematics, now Giaquinto shows examples where visualisation is untrustworthy and arguments based on visual images lead to wrong conclusions. He differentiates between the roles of diagrams and figures as illustrations of a concept or property and as grounds for a mathematical proof. While the first use is valid in most cases, the second one is clearly inadequate in most situations related to infinite processes like limits or local properties like continuity or derivation. This chapter is devoted to presenting several examples of concepts (continuity, limit and derivative) and theorems (Rolle's and Bolzano's) where figures and mental images are unreliable representations of the processes involved in those definitions or proofs.

Earlier in this review, I mentioned the relevance of Vinner's theory in explaining the relationship between the formation of students' conceptions and the quality of their visual thinking. Any mathematics educator would be reminded again of Vinner's theory when reading this chapter, since it provides a complementary viewpoint to analysing the uncertain role of visualisation in the area of analysis (Vinner and Dreyfus, 1989; Vinner, 1991): visual images are necessary to begin to create adequate concept images of analytical concepts, but they are not sufficient to create correct conceptions in the above-mentioned topics.

Giaquinto presents in chapter 10, as the question to be explored, the suggestion that many manipulations of mathematical expressions may be based on visual relationships among the symbols in the expression. In highly formalised mathematics areas, like algebra or calculus, the activity of writing proofs and, in general, manipulating expressions, is purely textual and symbolic. Such symbolic manipulations are supposed to be based on the application of axioms, definitions, theorems, etc, that allow the manipulations, but Giaquinto's conjecture is that visual thinking may support such manipulations of symbols instead of elements of the axiomatic system. In this case, visualisation is related to symbolic expressions more than to geometrical shapes.

The author suggests that operations in a symbolic expression like commutativity, distributivity or simplification of a fraction are made quite often by applying visual transformations (e.g., rotation or matching) on the symbols in the expression. He even introduces a formal system to present examples of symbolic transformations of algebraic expressions and formal productions of proofs based on visual manipulation of symbols in a semantically meaningless context. This is an issue studied in mathematics education, since there are important students' errors related to it, like the wrong transformations $\frac{a+b}{a+c} = \frac{b}{c}$ or $(a+b)^2 = a^2 + b^2$.

Mathematics educators most often reject Giaquinto's proposal since they use geometric representations of the abstract descontextualised expressions as a way to make students understand and learn them. An example, which can be found in any textbook, is the representation of the identity $(a+b)^2 = a^2 + 2ab + b^2$ by means of a square with side $a+b$ divided into four rectangles whose areas are $a \times a$, $b \times a$, $a \times b$, and $b \times b$. Conversely, some mathematics educators adopt a position near to Giaquinto's. For instance, Kirshner and Awtry (2004) suggest that students use visual pattern matching even when algebraic expressions are presented within graphical contexts. These authors classify algebraic expressions found in secondary school textbooks as 'visually salient' or 'non-visually salient', the identity $(a \times b)^2 = a^2 \times b^2$ being an example of the former, and $(a + b)^2 = a^2 + 2ab + b^2$ of the latter. From this point of view, an educational problem is the natural tendency of students to apply transformations that look correct visually, like those shown in the previous paragraph. However, this is an open question, since both Giaquinto and Kirshner and Awtry coincide in admitting that it is not clear that visual thinking really plays a significant role in learning to carry out such symbolic manipulations.

After having discussed the position of visual thinking in symbolic manipulation, Chapter 11 pays attention to the function of visual thinking and other cognitive functions to dealing with mathematical structures. Starting with structures that can easily be graphically represented, like those based on small sets, the chapter analyses the possibilities of showing the fundamentals of a structure by a diagram. The author presents several examples of finite structures of increasing complexity as grounds for his analysis. On the one hand, he points to the fact that different structures may have equivalent diagrammatic representations while, at the same time, a single structure may admit several representations. This section of the chapter doesn't make a significant addition to knowledge about this issue.

Structures based on infinite sets make a big difference. Giaquinto considers first the set N of whole numbers, whose structure may be represented by the number line (already discussed in a previous chapter), and then considers the set **R** of real numbers. **R** may also be represented by the number line but, as the author mentions, there is a major difference between the two structures represented by the mathematical property of completeness of **R**. Therefore, while each natural number may be represented by a dot in the number line, it is not possible to do the same for real numbers, this fact making the real line an inadequate representation for **R**.

The last part of this chapter is devoted to the structures of transfinite numbers ω, ω^2, ..., reflecting on whether any visual representation of these structures could help us to attain some kind of cognitive comprehension of their structures and properties: something that, for the moment, seems impossible. As a synthesis of this chapter, the author concludes that we can attain cognitive understanding of some simple structures from their visual representations, but, for most structures, this is not possible because they are too big or too complex.

Chapter 12 is presented by the author as a global perspective on types of mathematical thinking. He identifies two categories, algebraic thinking and geometric thinking, and he considers each type of reasoning to identify its strengths and weaknesses. The chapter presents several examples of theorems or relationships worked symbolically or diagrammatically where, therefore, algebraic or geometric reasoning is used. If we compare the different examples, we find that some clearly induce algebraic thinking while others clearly require geometric thinking. However, for some examples, the classification is not so easy, because they require both kinds of thinking.

Mathematics educators who specialise in visualisation have studied this question for many years and, after Krutetskii's (1976) research, they have identified three types of mathematical thinking, namely analytic thinking, visual thinking, and harmonic thinking. This construct agrees with Giaquinto's conclusions, although he doesn't state the existence of harmonic thinking so clearly.

In conclusion, we are presented with an interesting book that raises and analyses, from an epistemic point of view, many questions of interest to mathematics educators and mathematicians. The fact that the author is not a mathematics educator can be seen some times as a flaw of the book, since he elaborates on issues that are well researched and settled in the mathematics education literature. On the whole, however, his distance from our field is a virtue of the book, since Giaquinto presents us with fresh points of view and different approaches on many questions still being discussed by mathematics educators, related to the use of visual reasoning and visualisation in learning mathematics.

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Opening the research text: critical insights and in(ter)ventions into mathematics education, edited by Elizabeth de Freitas and Kathleen Nolan, New York, Springer, 2008, ISBN: 978-0-387-75463-5, 256 pp, Hardback, £55.99.

This book is structured around nine substantive chapters that the editors term 'post-structural', and that share a concern with social justice and mathematics education. These chapters are interspersed by collections of responses from the editors, other mathematics education researchers, mathematicians, teachers, students training to be teachers, masters and doctoral students, and even, in one case, the authors talking back to themselves. While all the chapters take a 'traditional' approach to form, the responses are more varied, including conversations and 'arts-based' approaches drawing on fiction, poetry, cartoons and paintings.