The mercury-in-glass thermometer of non-uniform capillary tube

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The physical model of the *ideal* mercury-in-glass thermometer is usually considered in the first-year undergraduate courses. It is clear that the number of *non-ideal* effects affecting the readings of a mercury-in-glass thermometer is high, and this thermometer is not usually employed when an accuracy better than $\pm 0.5\,^{\circ}\mathrm{C}$ is required. However, effects like the volume change of the glass envelope and the non-uniformity of the capillary tube seem to be of 'academic' interest, and are usually proposed in the 'problems' section of undergraduate textbooks. Though both effects have a very small impact in practical situations, the task of evaluating, inserting and comparing these corrections may be of instructional relevance. Indeed, the process of learning physics calls not only for proposing very simple, ideal models but also for giving the student some procedures to incorporate refinements in the model and judge the need for the corrections in each practical case. Although the influence of the volume change of the glass envelope is certainly well known, this seems not to be the case for the non-uniformity of the capillary tube (Brochard (1963) and Soo (1962) contain two proposed problems concerning this question, but a complete study like the one carried out here appears to be lacking). Therefore, we will consider here the effects that the above-mentioned non-uniformity exerts on the temperature measurement. To this end, we will assume that the cross section of the capillary tube varies linearly with the length of the tube.

For a mercury-in-glass thermometer with regularly spaced marks from 0 to 100 °C, the relationship between the temperature and the length of the mercury column can be written as

$$t(x) = 100\,\frac{x - x_0}{x_{100} - x_0} \qquad (1)$$

where x is the distance between a reference mark and the position of the end of the mercury column, x_0 and x_{100} being the values of x that correspond to the normal melting point of ice and normal boiling point of water respectively. In equation (1), the length x is the thermometric property, and the linear thermometric function $t(x)$ determines the temperature scale (Zemansky and Dittman 1981).

For a thermometer of uniform capillary tube (hereafter called a uniform thermometer) the temperature t defined through equation (1) coincides with the actual temperature. However, if the cross section, S, of the capillary tube varies with x, for example, linearly as in equation (2)

$$S(x) = S_0[1 + a(x - x_0)] \qquad a > 0 \qquad (2)$$

the temperature measured

$$t' = 100\,\frac{x' - x_0}{x'_{100} - x_0} \qquad (3)$$

differs from the actual one. (The magnitudes related to this non-uniform capillary tube thermometer, or *non-uniform thermometer* for short, will be marked with a prime sign). Note that we have taken $x'_0 = x_0$, S_0 being the cross section of the uniform thermometer. In order to find the relationship between x and x', we state that the volume of the mercury at a given temperature is the same in both thermometers:

$$S_0(x - x_0) = S_0 \int_{x_0}^{x'} [1 + a(x - x_0)]\, dx. \qquad (4)$$

In writing equation (4) we assume that the volume of the mercury below the mark x_0 is the same in both thermometers. This can be accomplished in practice if the mark x_0 is located just on the top of the bulb, and the volume of the bulb is the same for both thermometers. Equation (4) leads to

$$x = x' + (a/2)(x' - x_0)^2 > x'. \qquad (5)$$

Obviously, the difference between the readings t and t' is zero for $x = x_0 = x'$ and $x = x_{100}$, $x' = x'_{100}$. The difference between t and t' is

$$\varepsilon \equiv t' - t = 100\left(\frac{x' - x_0}{x'_{100} - x_0} - \frac{x - x_0}{x_{100} - x_0}\right)$$

$$= 50a \frac{x' - x_0}{x'_{100} - x_0} \frac{x'_{100} - x'}{1 + (a/2)(x'_{100} - x_0)} > 0 \qquad (6)$$

and the position x'_M corresponding to the maximum difference between the reading is

$$\left(\frac{d\varepsilon(x')}{dx'}\right)_{x' = x'_M} = 0 \Rightarrow x'_M = (x'_{100} + x_0)/2. \qquad (7)$$

If we make use of equation (5),

$$x_M = (x_{100} + x_0)/2 - (a/8)(x'_{100} - x_0)^2$$
$$< (x_{100} + x_0)/2. \qquad (8)$$

Equation (7) shows that ε reaches its maximum value just at the central point on the x' scale. Note that equation (6) gives a parabolic dependence on x', with $\varepsilon(x' = x_0) = 0 = \varepsilon(x' = x_{100})$. The value $x' = x'_M$ corresponds to the maximum of this parabola. Thus we see that the two thermometers agree at x_0 and x'_{100}, but they do not agree at any intermediate point. Figure 1 shows the length of the mercury column of both thermometers and the difference between their readings as a function of t.

From equations (1), (6), (7) and (8), the temperature t_M at which the difference between the readings t and t' takes the maximum value is

$$t_M = 50\left(1 - \frac{a}{4} \frac{(x'_{100} - x_0)^2}{x_{100} - x_0}\right) < 50. \qquad (9)$$

The numerical value of this maximum difference can be written as $\varepsilon_M = 50 - t_M$, where $\varepsilon_M = \varepsilon(x' = x'_M)$.

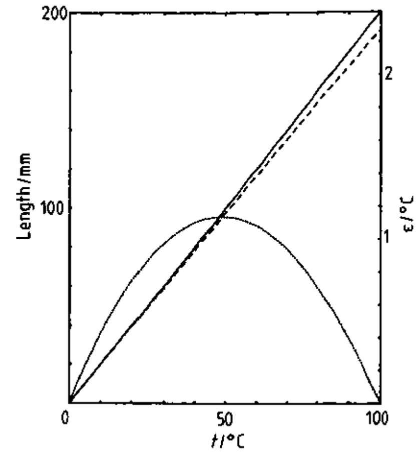

Figure 1. The length of the mercury column in the uniform (——) and non-uniform (– – –) thermometers and the difference $\varepsilon \equiv (t' - t)$ (· · ·) against temperature t; $(x_{100} - x_0) = 200$ mm, $a = 5 \times 10^{-4}\ \mathrm{mm}^{-1}$.

Now, let us take $S_0 = 0.05\ \mathrm{mm}^2$ and $x_0 = 0$, $x_{100} = 200$ mm as typical values. The key parameter of our problem is a (see equation (2)). The *maximum* increase in S as function of parameter a will be:

$$\Delta S/S_0 = S(x'_{100})/S_0 - 1 \cong 200a. \qquad (10)$$

If we consider $a_1 = 5 \times 10^{-5}\ \mathrm{mm}^{-1}$ and $a_2 = 5 \times 10^{-4}\ \mathrm{mm}^{-1}$, then $(\Delta S/S_0)_1 = 0.01$ and $(\Delta S/S_0)_2 = 0.1$. The latter value is probably unrealistically high and has been considered only as a limiting case. It should be noted, however, that if we take $S = \pi R^2$, R being the radius of the capillary tube, then the relative uncertainties, ε_r, of S and R satisfy $\varepsilon_r(R) = \varepsilon_r(S)/2$. Therefore, for $R = (S/\pi)^{1/2} \simeq 0.1$ mm and $\varepsilon_r(S) \simeq 0.01$, R must be determined within a maximum uncertainty of about 1 μm.

Table 1 shows the approximated values for x'_{100}, t_M, ε_M and $\varepsilon_r(t_M)$ resulting from the two values of parameter a considered here. The maximum difference between the readings of the two thermometers is certainly noticeable for $a_2 = 5 \times 10^{-4}\ \mathrm{mm}^{-1}$. In the case of $a_1 = 5 \times 10^{-5}\ \mathrm{mm}^{-1}$, the differences become very small. However, we see that a *maximum* change of only 1% in S will produce an ε_M whose order of magnitude is similar to that of the smaller difference between the marks on the 0–100 °C scale of the mercury-in-glass thermometers currently used in undergraduate labs (0.5 °C).

Note also that we can define a new scale for which the difference between the readings of the two thermometers is zero. This can be done by introducing new marks x_i'' such that each pair of consecutive marks, x''_{i-1} and x''_i, on the scale of the non-uniform thermometer contain *exactly* the same volume

$$\int_{x''_{i-1}}^{x''_i} S(x)\,\mathrm{d}x = \frac{1}{100} \int_{x_0}^{x'_{100}} S(x)\,\mathrm{d}x \qquad (11)$$

$$i = 1, 2, \ldots 100.$$

Equation (11) yields

$$x''_i = \left[\frac{1}{a^2} + i\,\frac{x'_{100} - x_0}{50a} \left(1 + a\,\frac{x'_{100} - x_0}{2} \right) \right]^{1/2} - \frac{1}{a} + x_0 \qquad i = 1, 2, \ldots 100. \qquad (12)$$

As can be expected, the resulting scale is *not* linear any longer. If we take $(x'_{100} - x_0) = 190.9$ mm and $a = 5 \times 10^{-4}$ mm^{-1}, then $(x'_1 - x_0) = 2.00$ mm and $(x''_{100} - x''_{99}) = 1.83$ mm. The distance between the marks on the scale decreases as we move from the bottom to the top of the non-uniform capillary tube because the cross section is increasing along the capillary according to equation (2).

Table 1. Maximum differences between the readings of the two thermometers.

a (mm^{-1})	x'_{100} (mm)	t_M (°C)	ε_M (°C)	$\varepsilon_r(t_M)$
5×10^{-5}	199.01	49.88	0.12	0.24%
5×10^{-4}	190.89	48.86	1.14	2.3%

References

Brochard J 1963 *Thermodynamique* (Paris: Masson) ch 2

Soo S L 1962 *Analytical Thermodynamics* (Englewood Cliffs, NJ: Prentice Hall) ch 2

Zemansky M W and Dittman R H 1981 *Heat and Thermodynamics* (New York: McGraw-Hill) ch 1