Comment on "The electron g factor and factorization of the Pauli equation" by R. J. Adler and R. A. Martin [Am. J. Phys. 60, 837–839 (1992)]

W. E. Baylis
Department of Physics, University of Windsor, Windsor, Ontario, N9B 3P4 Canada

(Received 27 April 1993; accepted 14 June 1993)

In their paper,[1] Adler and Martin have solved the Pauli–Schrödinger equation after factoring it into linear operators. However, the factorization step

$$(\boldsymbol{\pi}\cdot\boldsymbol{\sigma})^2-2m(E-V)=[\boldsymbol{\pi}\cdot\boldsymbol{\sigma}+\sqrt{2m(E-V)}]\times[\boldsymbol{\pi}\cdot\boldsymbol{\sigma}-\sqrt{2m(E-V)}] \quad (1)$$

into a product of linear factors, where $\boldsymbol{\pi}=-i\hbar\nabla-e\mathbf{A}$ is the kinematic (or mechanical) momentum, as in Eq. (10) of Adler and Martin,[1] is not generally valid. Indeed, it is wrong whenever $\nabla V\neq 0$. As a familiar counterexample, take $\mathbf{A}=0$ and $V=kr^2/2$ which describes a three-dimensional harmonic oscillator and whose familiar solutions do not obey the factored equation. This error does not affect the example presented by Adler and Martin since there $V=0$.

[1] R. J. Adler and R. A. Martin, "The electron g factor and factorization of the Pauli equation," Am. J. Phys. 60, 837–839 (1992).

Streak line and path of a particle in introductory fluid mechanics: An example

José A. Manzanares and Salvador Mafé
Department of Thermodynamics, Faculty of Physics, University of Valencia, E-46100 Burjasot, Spain

(Received 8 January 1993; accepted 1 April 1993)

There are two common methods of describing fluid motion.[1] In the Lagrangian method the individual fluid particles are followed during their motion, and the curve of most fundamental importance is the *path of the particle.* In the *Eulerian* method, the basic concept is the state of velocity throughout the whole fluid at some instant, and the *streamline* is the curve of major significance.

The streamlines are very useful to describe fluid motion because they usually are smooth wavy lines. However, the experimental observation of a flow is often performed by dye-injection techniques, i.e., by allowing a fluid of distinctive color to exude slowly into the moving fluid from a small fixed orifice. At any time the exuded fluid lies on a curve that is called *filament line* or *streak line.* It is defined as the locus at a given instant of all the fluid particles which have passed or will pass through a fixed point within the fluid.[1] When the motion is steady, the streamlines and the streak lines are identical with the paths of the particles. However, this does not happen in unsteady flow. The differences among these three concepts in unsteady flow are very important, and make difficult the description of time dependent flow fields from the data of experimental visualization techniques.[2] In fact, visualization techniques like the dye injection described above were initially employed to have a *rough* idea about what was happening in flow fields.[3] Fortunately, powerful image processing techniques now available make it possible for the complete description of flow fields from the visualized streak lines.[4]

The fluid mechanics sections of modern introductory physics textbooks do not consider the above concepts in detail. Some of these books pay little or no attention to them.[5] Others[6] introduce only the concept of streamline using a definition whose validity is restricted to the case of steady flow. These facts can be easily understood, since the space devoted to the fluid mechanics section in introductory physics courses is limited, and the conceptual difficulty associated to ideas like "streamline," "streak line," or "path of a fluid particle" is considerable at this introductory level.

We present here a simple problem involving unsteady fluid motion that introduces the concepts of streak line and path of a particle in a very intuitive way. The problem is very familiar and, after introducing some simplifying assumptions, it involves only elementary concepts from kinematics. Therefore, it could be employed in the introductory physics course where only the concept of streamline for steady flow is usually considered.

Let us consider a cylindrical tank of cross section $S=10$

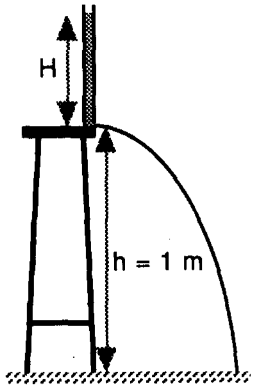

Fig. 1. Schematic diagram of the problem.

cm^2 initially filled with water up to a height $H_0=0.5$ m (see Fig. 1). The tank is placed at a height $h=1$ m above the ground level. Suppose we have water flowing out of a hole of section $\sigma=1$ cm^2 located at the bottom part of the side wall of the tank (see Fig. 1). The pressure at the top of the tank and the sides of the jet is assumed to be the atmospheric one. At a given instant t, the *streak line* (that is, the water jet we see) is formed by water particles that left the tank through the small hole in consecutive instants. If we neglect surface tension effects as well as the frictional force between the water particles and the air, these particles follow the well-known parabolic path. However, each particle flows out of the hole with a different horizontal speed, since this speed depends on the height H of the water level in the tank, and H is decreasing with time.

We will now obtain *both* the equation of the streak line at a given instant t, and the equation of the path of the fluid particle that left the tank at a time t_{out}. In particular, we will consider the paths of the water particles corresponding to the initial, intermediate, and final points of the streak line at time t. In order to arrive at these results we will employ Bernoulli's theorem (specifically, Torricelli's law) which is strictly valid only for steady flow. However, we are within the range of the well-known quasi-steady state approximation[7] and the steady flow equations can be applied with reasonable accuracy. Note that this question is very important, since the introduction of the concepts here involved usually requires the study of unsteady flow situations, and this often leads to mathematical difficulties that make the study inappropriate for introductory physics courses. However, the sort of problems dealt with here (continuous emptying of reservoirs and tanks) permits introducing the relevant concepts while taking the mathematical complexity at a minimum.

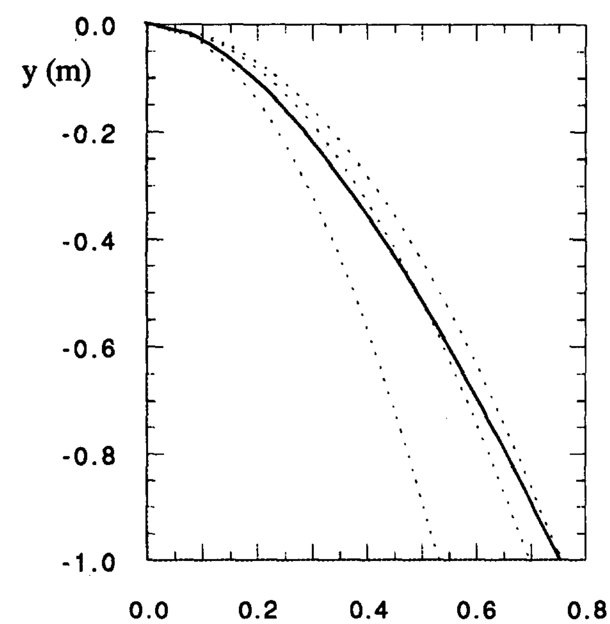

Fig. 2. Streak line (continuous line) and path followed by three fluid particles (dotted lines) of this streak line.

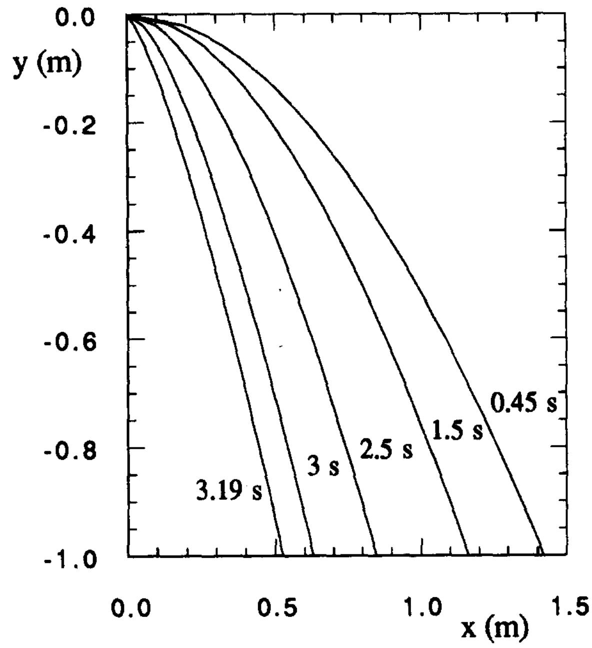

Fig. 3. Streak lines corresponding to different times.

Let us take the origin of coordinates at the position of the hole. The horizontal coordinate x is considered to be positive in the direction of the water jet, and the vertical coordinate y is assumed positive upwards, so that the ground is located at $y=-h$. The origin of time, $t=0$, will correspond to the instant when water begins to flow out of the hole. According to Torricelli's law, the first water particle leaves the tank with a horizontal speed

$$v_0=\sqrt{2gH_0/[1-(\sigma/S)^2]}, \tag{1}$$

where g stands for the gravity acceleration. This particle will follow the path given by equations

$$x=v_0t; \quad y=-(1/2)gt^2, \tag{2}$$

or simply

$$x^2=-y\frac{2v_0^2}{g}. \tag{3}$$

It takes a time

$$T=(2h/g)^{1/2} \tag{4}$$

for the particle to reach the ground, and then the streak line is completely formed. However, the particle that constitutes the initial point of this line (that is, the particle which is leaving the tank at the instant $t=T$) has a horizontal speed

$$v(T)=\sqrt{2gH(T)/[1-(\sigma/S)^2]} \quad (5)$$

and will describe the path given by

$$x^2=-y\frac{2v^2(T)}{g}. \quad (6)$$

At any later instant, the streak line is formed by fluid particles that flowed out of the hole in times ranging from $t_{out}=t-T$ to $t_{out}=t$. The path followed by a particle whose exit time was t_{out} would be

$$x=v(t_{out})(t-t_{out}); \quad y=-(1/2)g(t-t_{out})^2 \quad (7)$$

which can also be written as

$$x^2=-y\frac{2v^2(t_{out})}{g}. \quad (8)$$

From Eq. (7), we know that the particle located at (x,y) at time t flowed out of the hole at the time instant

$$t_{out}=t-\sqrt{2(-y)/g}. \quad (9)$$

Therefore, Eq. (8) gives the path of a particle that left the tank at time $t=t_{out}$. However, if we were able to substitute the *time dependent* exit speed $v(t)$ for $v(t_{out})$ in Eq. (8), we would obtain the streak line equation [note that we can relate t_{out} and t from Eq. (9)].

In order to calculate $v(t)$, we should find previously *how* the height H of the water level in the tank changes with time. To do this, let us write the continuity equation in the form

$$\sigma v=-S(dH/dt), \quad (10)$$

where the minus sign simply states that H is decreasing with time. If we substitute Eq. (5) into Eq. (10) and separate variables, we get

$$-\frac{dH}{\sqrt{H}}=\sqrt{\frac{2g}{(S/\sigma)^2-1}}\,dt. \quad (11)$$

Integration of Eq. (11) yields

$$\sqrt{H}=\sqrt{H_0}-\sqrt{\frac{g}{2[(S/\sigma)^2-1]}}\,t. \quad (12)$$

Now, the exit speed at time $t=t_{out}$ is

$$v(t_{out})=\frac{S}{\sigma}\sqrt{\frac{2g}{(S/\sigma)^2-1}}\left\{\sqrt{H_0}-\sqrt{\frac{g}{2[(S/\sigma)^2-1]}}\,t_{out}\right\}. \quad (13)$$

Finally, substitution of Eq. (13) into Eq. (8), and further substitution for t_{out} from Eq. (9) leads to the equation for the *streak line*

$$x^2=-y\frac{4H_0}{1-(\sigma/S)^2}\left\{1-\frac{\sqrt{(1/2)gt^2}-\sqrt{(-y)}}{\sqrt{H_0[(S/\sigma)^2-1]}}\right\}^2. \quad (14)$$

Equation (14) is valid from $t=T$ to the time required for the tank to become empty. According to Eq. (12), this time is

$$T_{empty}=\sqrt{2H_0[(S/\sigma)^2-1]/g}. \quad (15)$$

On the other hand, the *path of the water particles* corresponding to the initial, intermediate, and final points of the streak line at any instant t are, respectively,

$$x^2=-y\frac{2v(t)^2}{g}=-y\frac{4gH_0}{1-(\sigma/S)^2}\left\{1-\frac{\sqrt{(1/2)gt^2}}{\sqrt{H_0[(S/\sigma)^2-1]}}\right\}^2, \quad (16)$$

$$x^2=-y\frac{2v(t-T/\sqrt{2})^2}{g}=-y\frac{4gH_0}{1-(\sigma/S)^2}\left\{1-\frac{\sqrt{(1/2)gt^2}-\sqrt{h/2}}{\sqrt{H_0[(S/\sigma)^2-1]}}\right\}^2, \quad (17)$$

and

$$x^2=-y\frac{2v(t-T)^2}{g}=-y\frac{4gH_0}{1-(\sigma/S)^2}\left\{1-\frac{\sqrt{(1/2)gt^2}-\sqrt{h}}{\sqrt{H_0[(S/\sigma)^2-1]}}\right\}^2. \quad (18)$$

Figure 2 shows the plots of Eq. (14) (continuous line) and Eqs. (16)–(18) (dotted lines) when $H=1$ cm. This corresponds to the instant

$$t=\sqrt{2[(S/\sigma)^2-1]/g}(\sqrt{H_0}-\sqrt{H})=2.73\ \mathrm{s} \quad (19)$$

for the numerical values considered here. Note that the streak line *does not* follow a parabolic curve though the path of the water particles *do* follow parabolic curves. Likewise, the dotted lines in Fig. 2 are *not* to be confused with the streak lines which are obtained for different times. Figure 3 shows such streak lines. The first curve ($t=T=0.45$ s) corresponds to the *first* complete streak line, and the last curve ($t\approx T_{empty}=3.19$ s) gives the streak line at a time very close to that required for the tank to be empty.

[1]D. J. Triton, *Physical Fluid Dynamics* (Clarendon, Oxford, 1988), Chap. 6; James Lighthill, *An Informal Introduction to Theoretical Fluid Mechanics* (Clarendon, Oxford, 1986), Chap. 2; D. J. Acheson, *Elementary Fluid Dynamics* (Clarendon, Oxford, 1990), Chap. 1.

[2]*Advanced Mechanics of Fluids*, edited by Hunter Rouse (Wiley, London, 1959), Chap. 2; John F. Ross and David R. Williams, "Description of flow fields," Appl. Mech. Rev. **43**, 155–156 (1990), and references therein.

[3]Francis R. Hama, "Streaklines in a perturbed shear flow," Phys. Fluids **5**, 644–650 (1962).

[4]Hiroyuki Hory, Tetsuhiro Tsukiji, and Koji Takahashi, "Numerical processing of a visualized streak line image using a streamline coordinate system," JSME International Journal, Ser. II **34**, 411–416 (1991); *Flow Visualization V*, edited by R. Reznícek (Hemisphere, New York, 1990), Section 2.

[5]Paul A. Tipler, *Physics for Scientists and Engineers*, 3rd ed. (Worth, New York, 1991), Chap. 11; Marcelo Alonso and Edward J. Finn, *Physics*, 2nd ed. (Addison-Wesley, Reading, Massachusetts, 1992), Chap. 14; Jerry B. Marion, *General Physics with Bioscience Essays* (Wiley, New York, 1979), Chap. 7.

[6]David Halliday, Robert Resnick, and J. Walker, *Fundamentals of Physics*, 4th ed. (Wiley, New York, 1993), Chap. 16; Hans C. Ohanian, *Physics*, 2nd ed. (Norton, New York, 1989), Chap. 18; Robert M. Eisberg and Lawrence S. Lerner, *Physics. Foundations and Applications* (McGraw-Hill, New York, 1981), Chap. 16; Raymond A. Serway, *Physics for Scientists and Engineers*, 3rd revised ed. (Saunders, New York, 1992), Chap. 15.

[7]John F. Douglas, J. M. Gasiorek, and J. A. Swaffield, *Fluid Mechanics*, 2nd ed. (Logman Scientific & Technical, London, 1985), Chap. 17.