Équations aux dérivées partielles/Partial Differential Equations

Minimizing Total Variation Flow

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Abstract - We prove existence and uniqueness of weak solutions for the minimizing Total Variation flow with initial data in L^1 under Neumann boundary conditions. We prove that the H^{N-1} measure of the boundaries of level sets of the solution decreases with time, as one would expect. We also prove that local maxima (minima) strictly decrease (increase) their level with time. We shall also consider the Dirichlet problem which presents some particular difficulties for general initial data in L^1.

Sur le flot qui minimise la variation totale

Résumé - On montre l'existence et l'unicité de solutions faibles du flot qui minimise la variation totale pour des données initiales dans L^1 et des conditions au bord du type Neumann. On montre que la mesure H^{N-1} des surfaces de niveau décroît au cours de l'évolution, de même, le niveau des maxima (minima) locaux décroît (croît) instantanément avec le temps. On démontre aussi des résultats d'existence et d'unicité pour le problème de Dirichlet avec des données initiales dans L^1.

Version française abrégée - Soit Ω un ouvert borné dans $I\!R^N$ de frontière $\partial\Omega$ Lipschitzienne. On considère le problème d'évolution

$$\begin{aligned} &\frac{\partial u}{\partial t} = div(\frac{Du}{|Du|}) \qquad \text{sur } Q = (0,\infty)\times\Omega \\ &u(0,x) = u_0(x), \qquad x \in \Omega \end{aligned} \tag{1}$$

où $u_0 \in L^1(\Omega)$, avec des conditions au bord du type Neumann

$$\frac{\partial u}{\partial \eta} = 0 \qquad \text{sur } S = (0,\infty)\times\partial\Omega \tag{2}$$

ou Dirichlet

$$u(t,x) = \varphi(x) \qquad \text{sur } S = (0,\infty)\times\partial\Omega \tag{3}$$

avec $\varphi \in L^1(\partial\Omega)$. L'équation (1), (2) est associée au problème de minimisation de la Variation Totale

$$\Phi(u) = \int_\Omega |\nabla u|, \tag{4}$$

méthode qui a été proposée en traitement d'images par L. Rudin, S. Osher et E. Fatemi [6] pour le débruitage et la reconstruction d'images. Dans ce contexte, la fonctionnelle (4) est minimisée avec des contraintes qui modélisent le processus d'acquisition de l'image, en particulier, le bruit et le flou (voir [6]).

Le résultats principaux sont les suivants. La notion de solution utilisé dans chaque cas sera precisée dans les sections qui suivent.

Théorème 1 *Soit* $u_0 \in L^1(\Omega)$. *Alors, pour tout* $T > 0$, *il existe une unique solution faible* $u(t, x)$ *de (1), (2) sur* $(0, T) \times \Omega$ *avec* $u(0) = u_0$. *Si* $u(t), \hat{u}(t)$ *sont les solutions faibles correspondant aux données initiales* u_0 *et* $\hat{u}_0$, *respectivement, on a*

$$\|(u(t) - \hat{u}(t))^+\|_1 \leq \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{et} \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1, \tag{5}$$

pour tout $t \geq 0$. *En plus,* $\|u(t) - \overline{u_0}\|_1 \to 0$ *quand* $t \to \infty$, *où* $\overline{u_0} = \frac{1}{\mu(\Omega)} \int_\Omega u_0(x)\, dx$.

Théorème 2 *Soient* $u_0 \in L^1(\Omega)$ *et* $\varphi \in L^1(\partial\Omega)$. *Alors, pour tout* $T > 0$, *il existe une unique solution entropique* $u(t, x)$ *de (1), (3) sur* $(0, T) \times \Omega$ *avec* $u(0) = u_0$. *Si* $u(t), \hat{u}(t)$ *sont les solutions entropiques correspondant aux données initiales* u_0 *et* $\hat{u}_0$, *respectivement, on a*

$$\|(u(t) - \hat{u}(t))^+\|_1 \leq \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{et} \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1 \tag{6}$$

pour tout $t \geq 0$.

Pour démontrer les deux Théorèmes on fait appel aux techniques des opérateurs complètement accrétifs [4] et au Théorème de Crandall-Liggett. L'accrétivité des opérateurs associés a l'équation (1) avec des conditions au bord du type Neumann (2) ou Dirichlet (3) est une conséquence de la formule d'intégration par parties établie dans [3]. Dans le cas des conditions au bord du type Neumann, on démontre la régularité en temps des solutions, comme conséquence de l'homogénéité de l'opérateur [4]. Dans le cas des conditions au bord du type Dirichlet la situation est, en général, différente et on aura aussi besoin d'une approche différente. La démonstration des résultats présentés dans cette Note est contenue dans [1, 2].

1 The Neumann problem

We consider Ω an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary. We are interested in problem (1-2). $BV(\Omega)$ will denote the space of functions of bounded variation. By $L^1_w(0, T, BV(\Omega))$ we denote the space of functions $w : [0, T] \to BV(\Omega)$ such that $w \in L^1((0, T) \times \Omega)$, the maps $t \in [0, T] \to < Dw(t), \phi >$ are measurable for every $\phi \in C^1_0(\Omega, \mathbb{R}^N)$ and $\int_0^T \|Dw(t)\| < \infty$. It is not difficult to see that the conditions on w imply the measurability of the map $t \in [0, T] \to \|Dw(t)\|$. We shall use the truncature functions defined by $T_k(r) = k \wedge (r \vee (-k))$, $k \geq 0$, $r \in \mathbb{R}$. The notion of weak solution (solution faible) required to prove Theorem 1 is the following.

Definition 1 *A measurable function $u : (0,T) \times \Omega \to I\!R$ is a weak solution of (1), (2) in $(0,T) \times \Omega$ if $u \in C([0,T], L^1(\Omega)) \cap W^{1,1}_{loc}((0,T), L^1(\Omega))$, $T_k(u) \in L^1_w([0,T], BV(\Omega))$ for all $k > 0$ and there exists $z \in L^\infty((0,T) \times \Omega; I\!R^N)$ with $\|z\|_\infty \le 1$, $u_t = div(z)$ in $\mathcal{D}'((0,T) \times \Omega)$ such that*

$$\int_\Omega (T_k(u(t)) - w) u_t(t) \le \int_\Omega z(t) \cdot \nabla w - \|DT_k(u(t))\| \qquad (7)$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, for every $k > 0$, and a.e. on $[0,T]$.

To prove Theorem 1 we use the techniques of completely accretive operators [4] and the Crandall-Liggett's semigroup generation Theorem. For that, we introduce the following operator $\mathcal{A}$ in $L^1(\Omega)$.

$$(u,v) \in \mathcal{A} \quad \text{if and only if } u, v \in L^1(\Omega),\ T_k(u) \in BV(\Omega) \text{ for all } k > 0 \text{ and}$$
$$\text{there exists } z \in X(\Omega) := \{z \in L^\infty(\Omega, I\!R^N) : div(z) \in L^1(\Omega)\}$$
$$\text{with } \|z\|_\infty \le 1,\ v = -div(z) \text{ in } \mathcal{D}'(\Omega) \text{ such that}$$
$$\int_\Omega (w - T_k(u)) v \le \int_\Omega z \cdot \nabla w dx - \|DT_k(u)\|, \quad \forall w \in W^{1,1}(\Omega) \cap L^\infty(\Omega),\ \forall k > 0.$$

This operator is an extension to $L^1(\Omega)$ of the subdifferential of the convex functional $\Phi(u)$ defined by (4) if $u \in L^2(\Omega) \cap BV(\Omega)$, $\Phi(u) = +\infty$ if $u \in L^2(\Omega) \setminus BV(\Omega)$. The use of the subdifferental of Φ in $L^2(\Omega)$ permits to prove existence and uniqueness of strong solutions in the sense of semigroups when the initial condition is in $L^2(\Omega)$. In this paper, we consider initial conditions in $L^1(\Omega)$ and, in some sense, we give a distributional characterization of the corresponding operator. As raised by the referee, a natural question is if there may be L^1-L^∞ or L^1-L^2 regularizing effect. The answer is negative. There is neither an L^1-L^∞ nor L^1-L^2 regularizing effect. For instance, $v(t,x) = \frac{1}{\|x\|^{N/2}} - \frac{t}{\|x\|}$ solves (1) in $(0,1) \times B(0,1)$ with initial datum $v_0(x) = \frac{1}{\|x\|^{N/2}}$. Observe that $v(t) \in L^1(\Omega) \setminus L^2(\Omega)$, $0 \le t < 1$. Obviously, this solution does not satisfy Neumann boundary conditions but it may be used together with a comparison principle to build a solution $u(t,x)$ of (1-2) which is in $L^1(\Omega) \setminus L^2(\Omega)$.

The accretivity of the operator $\mathcal{A}$ is proved using the integration by parts formula given in [3]. For that, we need first to prove that we can use test functions in $BV(\Omega) \cap L^\infty(\Omega)$ in the definition of $\mathcal{A}$.

Lemma 1 *We have the following characterization of the operator $\mathcal{A}$,*

$$(u,v) \in \mathcal{A} \quad \textit{if and only if } u, v \in L^1(\Omega),\ T_k(u) \in BV(\Omega) \textit{ for all } k > 0 \textit{ and}$$
$$\textit{there exists } z \in X(\Omega) \textit{ with } \|z\|_\infty \le 1,\ v = -div(z) \textit{ in } \mathcal{D}'(\Omega) \textit{ such that}$$
$$\int_\Omega (w - T_k(u)) v \le \int_\Omega (z, Dw) - \|DT_k(u)\|, \quad \forall w \in BV(\Omega) \cap L^\infty(\Omega),\ \forall k > 0. \qquad (8)$$

Moreover, we have that i) $\int_\Omega (z, DT_k(u)) = \|DT_k(u)\|$, for all $k > 0$, and ii) $\int_\Omega vT_k(u) = \|DT_k(u)\|$, for all $k > 0$.

Using this Lemma, we prove the accretivity of $\mathcal{A}$. Then Crandall-Liggett's semigroup generation Theorem proves the existence of a semigroup solution (also called mild solution) of (1-2). The proof of both existence and uniqueness of weak solutions in the sense of Definition 1 requires the regularity in time of the solution. This is proved using the regularizing effect due to the homogeneity of the operator [4] in the case of Neumann boundary conditions.

Let us mention an interesting geometric feature of the equation: the H^{N-1} measure of the boundaries of the level sets of the solution decreases with time.

Proposition 1 *Let $u_0 \in L^1(\Omega)$. Let $u(t,x)$ be the weak solution of (1), (2). Then, for almost all $\lambda \in I\!R$,*

$$\|D\chi_{\{u(t)>\lambda\}}\| \leq \|D\chi_{\{u(s)>\lambda\}}\| \tag{9}$$

a.e. in $s, t \in (0,\infty)$, $t > s > 0$.

Note that $\|D\chi_{\{u(t)>\lambda\}}\| = H^{N-1}(\partial^*\{u(t) > \lambda\})$, where H^{N-1} is the $(N-1)$-dimensional Hausdorff measure and $\partial^*\{u(t) > \lambda\}$ is the reduced boundary of the set $\{x \in \Omega : u(t) > \lambda\}$. Thus, the above proposition says that the length of the boundaries of the level sets decreases with time.

Next, we prove that flat zones which are local maxima (minima) immediately decrease (respectively, increase) with time.

Proposition 2 *Let Ω be a cube in $I\!R^N$. Let $u_0 \in C(\overline{\Omega})$, $0 \leq u_0 \leq 1$. Suppose that $\{x \in \overline{\Omega} : u_0(x) = 1\} = K \subseteq B \subset\subset \Omega$ for some ball B. Let u be the weak solution of (1), (2). Then $u(t,x) < 1$, for all $t > 0$, $x \in \Omega$.*

This result is proved by comparison with an explicit function satisfying the same property.

2 The Dirichlet problem

Let E^* be the dual of the Banach space E. For technical reasons related to measurability we need to introduce the Banach space $BV(\Omega)_2 = BV(\Omega) \cap L^2(\Omega)$ (observe that if $N = 2$, then $BV(\Omega)_2 = BV(\Omega)$).

We also need to introduce, as in [3], a weak trace on $\partial\Omega$ of the normal component of certain vector fields in Ω. We define

$$Z(\Omega) := \{(z,\xi) \in L^\infty(\Omega, I\!R^N) \times BV(\Omega)^* \ : \ <\xi, \varphi> + \int_\Omega z \cdot \nabla\varphi = 0, \forall\varphi \in C_0^1(\Omega)\}.$$

Let $R(\Omega) := W^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$. For $(z,\xi) \in Z(\Omega)$ and $w \in R(\Omega)$ we define

$$\langle (z,\xi), w\rangle_{\partial\Omega} := \langle \xi, w\rangle_{BV(\Omega)^*, BV(\Omega)} + \int_\Omega z \cdot \nabla w.$$

Working as in the proof of Theorem 1.1. in [3], we obtain that if $(z,\xi) \in Z(\Omega)$ and $w, v \in R(\Omega)$ with $w = v$ on $\partial\Omega$, then $\langle (z,\xi), w\rangle_{\partial\Omega} = \langle (z,\xi), v\rangle_{\partial\Omega}$ for all $(z,\xi) \in Z(\Omega)$. As a consequence, we can give the following definition: Given $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $(z,\xi) \in Z(\Omega)$, we define $\langle (z,\xi), u\rangle_{\partial\Omega}$ by setting

$$\langle (z,\xi), u\rangle_{\partial\Omega} := \langle (z,\xi), w\rangle_{\partial\Omega}$$

where w is any function in $R(\Omega)$ such that $w = u$ on $\partial\Omega$. Again, working as in the proof of Theorem 1.1. of [3], we can prove that there exists a linear operator $\gamma : Z(\Omega) \to L^\infty(\partial\Omega)$, with $\gamma(z,\xi) := \gamma_{z,\xi}$, satisfying

$$\langle (z,\xi), w\rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma_{z,\xi}(x) w(x)\; dH^{N-1} \qquad \forall\; w \in BV(\Omega) \cap L^\infty(\Omega).$$

In case $z \in C^1(\overline{\Omega}, I\!R^N)$, we have $\gamma_z(x) = z(x)\cdot\nu(x)$ for all $x \in \partial\Omega$. Hence, the function $\gamma_{z,\xi}(x)$ is the weak trace of the normal component of (z,ξ). For simplicity of the notation, we shall denote $\gamma_{z,\xi}(x)$ by $[z,\nu](x)$.

Definition 2 Let $\Psi \in L^1(0,T,BV(\Omega))$. We say Ψ admits a *weak derivative* in the space $L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$ if there is a function $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$ such that $\Psi(t) = \int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral.

Definition 3 Let $\xi \in (L^1(0,T,BV(\Omega)_2)^*$. We say that ξ is *the time derivative* in the space $(L^1(0,T,BV(\Omega)_2)^*$ of a function $u \in L^1((0,T)\times\Omega)$ if

$$\int_0^T < \xi(t), \Psi(t) > dt = -\int_0^T \int_\Omega u(t,x)\Theta(t,x)dxdt$$

for all test functions $\Psi \in L^1(0,T,BV(\Omega))$ with compact support in time which admit a weak derivative $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$.

Observe that if $w \in L^1(0,T,BV(\Omega)) \cap L^\infty(Q_T)$ and $z \in L^\infty(Q_T, I\!R^N)$ such that there exists $\xi \in (L^1(0,T,BV(\Omega)_2)^*$ with $div(z) = \xi$ in $\mathcal{D}(Q_T)$, we can define, asociated to the pair (z,ξ), the distribution (z,Dw) in Q_T by

$$\langle (z,Dw),\phi\rangle := -\int_0^T \langle \xi(t), w(t)\phi(t)\rangle - \int_0^T\int_\Omega z(t,x)w(t,x)\nabla_x\phi(t,x). \tag{10}$$

for all $\phi \in \mathcal{D}(Q_T)$.

Definition 4 Let $\xi \in (L^1(0,T,BV(\Omega)_2)^*$, $z \in L^\infty(Q_T, I\!R^N)$. We say that $\xi = div(z)$ in $(L^1(0,T,BV(\Omega)_2)^*$ if (z,Dw) is a Radon measure in Q_T with normal boundary values $[z,\nu] \in L^\infty((0,T)\times\partial\Omega)$, such that

$$\int_{Q_T}(z,Dw) + \int_0^T < \xi(t), w(t) > dt = \int_0^T \int_{\partial\Omega} [z(t,x),\nu] w(t,x) dH^{N-1}dt,$$

for all $w \in L^1(0,T,BV(\Omega)) \cap L^\infty(Q_T)$.

We consider the set $\mathcal{T} = \{T_k, T_k^+, T_k^- \ : \ k > 0\}$.

Definition 5 A measurable function $u : (0,T) \times \Omega \to I\!R$ is an *entropy solution* of (1), (3) in $Q_T = (0,T) \times \Omega$ if $u \in C([0,T]; L^1(\Omega))$, $p(u(\cdot)) \in L^1_w(0,T,BV(\Omega))$ $\forall\, p \in \mathcal{T}$ and there exist $(z(t), \xi(t)) \in Z(\Omega)$ with $\|z(t)\|_\infty \leq 1$, and $\xi \in (L^1(0,T,BV(\Omega)_2)^*$ such that ξ is the time derivative of u in $(L^1(0,T,BV(\Omega)_2)^*$, $\xi = div(z)$ in $L^\infty_{w^*}(0,T,BV(\Omega)^*)$ and $[z(t),\nu] \in \mathrm{sign}(p(\varphi) - p(u(t)))$ a.e. in $t \in [0,T]$, satisfying

$$-\int_0^T \int_\Omega j(u(t)-l)\eta_t + \int_0^T \int_\Omega \eta(t)\|Dp(u(t)-l)\| + z(t)\cdot D\eta(t)p(u(t)-l) \leq$$
$$\leq \int_0^T \int_{\partial\Omega} [z(t),\nu]\eta(t)p(u(t)-l),$$

for all $l \in I\!R$, for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geq 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$, and $p \in \mathcal{T}$, where $j(r) = \displaystyle\int_0^r p(s)\, ds$.

This is the notion of solution required in Theorem 2. To prove the existence we approximate u_0 in L^1 norm by $u_{0n} \in L^2(\Omega)$ and use the semigroup solutions in $L^2(\Omega)$ with initial condition u_{0n}. The uniqueness of entropy solutions is proved by means of Kruzhkov's method of doubling variables [5].

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