MEAN CURVATURE FLOW OF GRAPHS IN WARPED PRODUCTS

ALEXANDER A. BORISENKO AND VICENTE MIQUEL

Dedicated to Professor Antonio M. Naveira on the occasion of his 70th birthday.

Abstract. Let M be a complete Riemannian manifold which either is compact or has a pole, and let φ be a positive smooth function on M. In the warped product $M \times_\varphi \mathbb{R}$, we study the flow by the mean curvature of a locally Lipschitz continuous graph on M and prove that the flow exists for all time and that the evolving hypersurface is C^∞ for $t > 0$ and is a graph for all t. Moreover, under certain conditions, the flow has a well defined limit.

1. Introduction

Let M be a n-dimensional manifold, $(\overline{M}, \overline{g})$ a $n+1$ dimensional Riemannian manifold. A map $F : M \times [0, T[\longrightarrow \overline{M}$ such that every $F_t := F(\cdot, t) : M \longrightarrow \overline{M}$ is an immersion is called the mean curvature flow (MCF for short) of F_0 if it is a solution of the equation

$$\frac{\partial F}{\partial t} = \vec{H} \tag{1.1}$$

where $\vec{H}(\cdot, t)$ is the mean curvature vector of the immersion F_t.

We shall use the following *convention signs for the mean curvature H, the Weingarten map A and the second fundamental form (h for the scalar valued version and α for its vector valued version)*. For a chosen unit normal vector N, they are:

$AX = -\overline{\nabla}_X N$, $\alpha(X,Y) = \left\langle \overline{\nabla}_X Y, N \right\rangle N = \langle AX, Y \rangle N$, $h(X,Y) = \langle \alpha(XY), N \rangle$ and $H = \operatorname{tr} A = \sum_{i=1}^n h(E_i, E_i)$, for a local orthonormal frame $E_1, ..., E_n$ of the submanifold, and $\vec{H} = \sum_{i=1}^n \alpha(E_i, E_i) = H\ N$.

Along the rest of the paper, by M_t we shall denote both the immersion $F_t : M \longrightarrow \overline{M}$ and the image $F_t(M)$, as well as the Riemannian manifold (M, g_t) with the metric g_t induced by the immersion. Analogous notation will be used when we have a single immersion $F : M \longrightarrow \overline{M}$. Notice that, since in this paper we shall deal with graphs, all the immersions that will appear evolving by mean curvature flow will be, in fact, embeddings.

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In two fundamental papers [3] and [4], Ecker and Huisken studied the evolution of non-compact hypersurfaces in the Euclidean Space. In [3] they studied the evolution of a graph in $\mathbb{R}^{n+1}$ and showed that

(A) If F_0 a "locally Lipschitz" continuous graph and has linear growth rate for its height, then (1.1) with initial condition F_0 has a smooth solution for all time.

(B) If, moreover, F_0 is "straight" at the infinity, F_t asymptotically approaches a selfsimilar solution of (1.1). They give an example showing that the condition cannot be weakened.

In [4], Ecker and Huisken obtained some interior estimates and applied them to prove that the hypothesis of linear growth in (A) is not necessary, that is:

(A') If F_0 a "locally Lipschitz" continuous graph, then (1.1) with initial condition F_0 has a smooth solution for all time.

In [12] and [13], Unterberger extended result (A') to the Hyperbolic Space $\mathcal{H}^{n+1}$ and gave a result of type (B). In this space, the first problem to face with is the choosing of the right concept of "graph". A natural one is to say that a hypersurface M of $\mathcal{H}^{n+1}$ is a graph over a totally geodesic hypersurface $\mathcal{H}^n$ if all the geodesics orthogonal to $\mathcal{H}^n$ cut M once and transversally (we shall call it a *geodesic graph*). But, in his thesis [12], Unterberger found an example of hypersurface which is a geodesic graph but loses this property when it evolves under (1.1). Then he considered another concept of graph. Let $p \in \mathcal{H}^n$ be a fixed point, Γ a geodesic through p orthogonal to $\mathcal{H}^n$. We shall call *equidistant* curves all the curves which are at constant distance from Γ. Then we say that M is an *equidistant graph* over $\mathcal{H}^n$ if it cuts once and transversally all the equidistant curves. Unterberger proved the exact analog of (A') for equidistant graphs. As a result of type (B), he proved that if $F_0 : M \longrightarrow \mathcal{H}^{n+1}$ is a "locally Lipschitz" equidistant graph and is at bounded distance from $\mathcal{H}^n$, then it converges asymptotically to $\mathcal{H}^n$.

After $\mathcal{H}^{n+1}$, other ambient spaces natural for trying to extend the results of Ecker and Huisken are products $M \times \mathbb{R}$ or, more generally, warped products $\mathbb{R} \times_\varphi M$ or $M \times_\varphi \mathbb{R}$. As usual, by a warped product $\mathcal{M} \times_\varphi \mathcal{N}$ of two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ we understand the riemannian manifold $(\mathcal{M} \times \mathcal{N}, g + \varphi^2 h)$, being $\varphi : \mathcal{M} \longrightarrow \mathbb{R}$ a positive smooth map. For these spaces it is natural to say that *a hypersurface of $M \times_\varphi \mathbb{R}$ (or $\mathbb{R} \times_\varphi M$) is a graph* if it is a graph of a function $u : M \longrightarrow \mathbb{R}$ and $u(M)$ cuts transversally the curves $s \mapsto (x, s)$. The interest in the last years for studying minimal and constant mean curvature surfaces in these ambient spaces produces also a natural interest on the study of MCF on them.

When $\varphi(s) = \cosh s$ and $M = \mathcal{H}^n$, $\mathbb{R} \times_\varphi M$ is $\mathcal{H}^{n+1}$ and the concept of being a graph coincides with that of geodesic graph in $\mathcal{H}^{n+1}$. Then the counterexample of Unterberger gives us few hope of getting general results. Computations of the evolution of the gradient of u (see the appendix) give some analytic reasons of the failing of the preservation of geodesic graphs under MCF. Also the last paragraph in Remark 4 and the pictures in the appendix can help to get some gometric insight on this fact.

When $M = \mathcal{H}^n$, $x_0 \in M$ and $\varphi(x) = \cosh(dist(x_0, x))$, $M \times_\varphi \mathbb{R}$ is $\mathcal{H}^{n+1}$ and the concept of being a graph coincides with that of equidistant graph in $\mathcal{H}^{n+1}$. Then, general $M \times_\varphi \mathbb{R}$ seem to be good general ambient spaces where to extend results of type (A') and (B). In this paper we show that this is in fact the case, proving an extension of (A') for general $M \times_\varphi \mathbb{R}$ (theorems 9, 11 and 13), where the only conditions to be satisfied by M and φ are:

the quotients $\frac{|\widehat{\nabla}^m \varphi|}{\varphi}$ are bounded, the curvature of M and all its covariant derivatives are bounded and, when M is non-compact, M *has a pole* (that is, a point with empty cut-locus). May be the last condition can be weakened with some stronger analytic tools. On the other hand, like in [4], the only condition for M_0 is to be a "locally Lipschitz" graph. We also get a result of type (B) (theorems 15 and 16) imposing on M_0 the condition that its distance to M is bounded on it and on M a pinching condition on its sectional curvature related to the pinching of the hessian of φ.

We use mainly the methods in [4] and [13], where we have to introduce necessary technical tricks to choose the right functions to get estimates having into account the complications introduced by the terms containing φ and the curvature of M. We also needed to use the comparison theory of Riemannian Geometry to bound some functions of the distance to a point or to a hypersurface. Moreover we cannot use barriers as it is done in [13] because we are not in a model space where we know the evolution of some hypersurfaces and, at some points, we need to substitute barriers arguments for others (see the end of the proofs of Theorem 10 and Lemma 14).

Somewhat surprising for us has been the fact that the qualitative results of type (A') (not the estimates) do not depend on the curvature of M.

In contrast, the curvature of M plays an essential role in the theorem of convergence.

The paper is organized as follows: in section 2 we state the notation and recall some lemmas that will be used later. In section 3 we collect the properties of the ambient spaces and their hypersurfaces that we shall need. Section 4 is a short comment about short time existence when M is compact and also a useful description of the evolution of some geometric quantities under an equivalent flow in the direction of the last coordinate u in $M \times_\varphi \mathbb{R}$. In section 5 we give the gradient estimate, which, when M is compact, is enough to conclude the preservation of the property of being a graph under MCF . In section 6 we obtain the higher order estimates which give rise to the long time existence (and conclude -module the existence theorems below- with the main theorems of the paper when the initial condition is smooth). In section 7 we complete the discussion of the existence when M is non-compact, and, in section 8 we discuss the existence theorems for Lipschitz initial conditions. It is more usual to give the theorems for smooth and Lipschitz initial conditions simultaneously but, for the sake of non expert readers, we preferred to do it separately. In this way appears more clearly why in the Lipschitz case we cannot use the initial conditions to bound the second derivatives and beyond, which forces us to introduce bounds depending on t although these are becoming worse when t goes to 0. Finally, in section 9 we give a case where the flow has a limit and determine the limit.

2. Preliminaries

In this paper we shall consider a Riemannian manifold $(M, \widehat{g})$ and an immersion $F : M \longrightarrow \overline{M}$ into another Riemannian manifold $(\overline{M}, \overline{g})$, and we shall denote by g the metric induced on M by the immersion F.

We shall use the notation $|X|$ or $\langle X, Y\rangle$ for indicating the $\overline{g}$-norm, $\widehat{g}$-norm or g-norm of X or the $\overline{g}$-product, $\widehat{g}$-product or g-product of X and Y if X and Y are, respectively, tangent to $\overline{M}$, M or $F(M)$.

For any vector $X \in T_{F(x)}\overline{M}$, we shall denote by $X^\top$ the component of X tangent to $F(M)$.

We shall use $\overline{\nabla}$, $\widehat{\nabla}$ and ∇ to denote the covariant derivative and the gradient in $(\overline{M}, \overline{g})$, $(M, \widehat{g})$ and $(F(M), g)$ respectively. By $\overline{\Delta}$, $\widehat{\Delta}$ and Δ we shall denote the corresponding laplacians. The convention sign for the laplacians will be: $\overline{\Delta} f = \mathrm{tr}\overline{\nabla}^2 f$.
For any $\lambda \in \mathbb{R}$, we shall use the notation:

$$\mathrm{s}_\lambda(t) \text{ is the solution of the equation } s'' + \lambda s = 0 \text{ satisfying } s(0) = 0\ , \quad \mathrm{c}_\lambda(t) = \mathrm{s}_\lambda{}'(t)$$

These functions satisfy the following computational rules:

$$c_\lambda^2 + \lambda\ s_\lambda^2 = 1, \quad c_{4\lambda} = c_\lambda^2 - \lambda\ s_\lambda^2, \quad s_{4\lambda} = s_\lambda c_\lambda. \tag{2.1}$$

When $\lambda < 0$ one has $\mathrm{s}_\lambda(t) = \sinh(\sqrt{|\lambda|}\ t)/\sqrt{|\lambda|}$ and $\mathrm{c}_\lambda(t) = \cosh(\sqrt{|\lambda|}\ t)$.

Given a point x_0 in M, we shall denote by $\widehat{r}$ the $\widehat{g}$-distance to x_0 in M. We shall denote by $\partial_{\widehat{r}}^\perp$ at $x \in M$ the vector space orthogonal to $\partial_{\widehat{r}} := \widehat{\nabla}\widehat{r}$ in T_xM. Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(\widehat{r})$ will mean $f \circ \widehat{r}$. The comparison theorem for the hessian of the distance function $\widehat{r}$ says

Lemma 1 (cfr [6] or [11]). *If $\widehat{Sec} \geq \mu$, then, at points of M between x_0 and its cutlocus, one has*

$$\widehat{\nabla}^2\widehat{r}\Big|_{\partial_{\widehat{r}}^\perp} \leq \frac{\mathrm{c}_\mu}{\mathrm{s}_\mu}(\widehat{r})\ \widehat{g}, \qquad \widehat{\Delta}\widehat{r} \leq (n-1)\ \frac{\mathrm{c}_\mu}{\mathrm{s}_\mu}(\widehat{r}). \tag{2.2}$$

This will be essential for the gradient estimates we shall obtain in Theorem 7. The next comparison theorem will be used for proving the convergence of the solution of (1.1) in certain cases.

Lemma 2 (cf. [10]). *Let $\overline{M}$ be a complete riemannian manifold, with sectional curvature $\overline{Sec} \leq k < 0$, let M be a complete totally geodesic hypersurface of $\overline{M}$. Let ℓ be the $\overline{g}$-distance to M, $\partial_\ell := \overline{\nabla}\ell$. Let us denote by S_ℓ the Weingarten map (associated to ∂_ℓ) of the hypersurface $\tau_\ell M$ at distance ℓ of M (that is, $S_\ell X = -\overline{\nabla}_X \partial_\ell$ for every $X \in T\tau_\ell M$). At every point between M and the cut locus of M in $\overline{M}$ one has*

$$\langle S_\ell X, X\rangle \leq k\frac{\mathrm{s_k}}{\mathrm{c_k}}|X|^2 \text{ for every } X \in T\tau_\ell M. \tag{2.3}$$

Also in the proof of convergence we shall use the following maximum principle for non-compact manifolds due to Ecker and Huisken:

Lemma 3 ([4] Th. 4.3). *Let M be a manifold with a family of Riemannian metrics g_t satisfying, for some x_0,*

$$\mathrm{vol}_{g_t}(B_r^{g_t}(x_0)) \leq e^{k(1+r^2)},$$

where $B_r^{g_t}(x_0)$ is the g_t-ball of radius r centered at x_0, Let $f : M \times [0,T] \longrightarrow \mathbb{R}$ be a function which is smooth on $M \times]0,T]$ and continuous on $M \times [0,T]$, satisfying:

(i) $\dfrac{\partial f}{\partial t} \leq \Delta^{g_t} f + \langle \vec{a}, \nabla^{g_t} f\rangle + b\ f$, *with $|\vec{a}|$ and $|b|$ bounded on $M \times [0,T]$*
(ii) $f(\cdot, 0) \leq 0$
(iii) $\displaystyle\int_0^T \left(\int_M e^{\alpha r^2}|\nabla^{g_t} f|^2 d\mu_t\right) dt < \infty$ *for some $\alpha > 0$*
(iv) $\sup_{M\times[0,T]} \left|\dfrac{dg}{dt}\right| \leq \beta$ *for some $\beta < \infty$.*

Then $f \leq 0$ on $M \times [0,T]$.

3. The geometric setting

3.1. **The ambient space.** Given $(M,\widehat{g})$ a n-dimensional riemannian manifold and $\varphi : M \longrightarrow \mathbb{R}^+$ a $C^\infty function$, our ambient space will be $\overline{M} = (M\times\mathbb{R},\overline{g})$ with $\overline{g} = \widehat{g}+\varphi(x)^2\, du^2$ (usually denoted as $M \times_\varphi \mathbb{R}$).

If $\widehat{X}$ (resp. $\widehat{\theta}$) is a vector field (resp. a differential form) on M, we shall denote by the same letter the vector fields (resp. forms) on $\overline{M}$ induced by the family of embeddings $j_s : M \longrightarrow M \times \{s\} \subset M \times \mathbb{R}$ defined by $j_s(x) = (x,s)$.

In a $\widehat{g}$-orthonormal local frame $\widehat{e}_1,...,\widehat{e}_n$ of $(M,\widehat{g})$, if $\widehat{\theta}^1,...,\widehat{\theta}^n$ is the dual frame, $\widehat{g} = \widehat{\theta}^{1^2} + ... + \widehat{\theta}^{n^2}$. In the extension of this frame to $\overline{M}$ completed with du, $\overline{g}$ is written as $\overline{g} = \varphi^2du^2 + (\widehat{\theta}^1)^2 + ... + (\widehat{\theta}^n)^2$. Then, if we denote $\overline{\theta}^0 = \varphi du$, $\overline{\theta}^1 = \ \widehat{\theta}^1$, ... ,$\overline{\theta}^n = \ \widehat{\theta}^n$, one has

$$\overline{g} = (\overline{\theta}^0)^2 + (\overline{\theta}^1)^2 + ... + \ ... \ ,+(\overline{\theta}^n)^2$$

and its dual frame $\overline{e}_0,\ \overline{e}_1,...,\overline{e}_n$ in $\overline{M}$ is a $\overline{g}$-orthonormal frame related with the $\widehat{e}_i$ by $\overline{e}_i = \widehat{e}_i$, and $\overline{e}_0 = \dfrac{1}{\varphi}\dfrac{\partial}{\partial u} =: \dfrac{1}{\varphi}\partial_u$

From now on $i,j = 1,...,n$ and $a,b = 0,1,...,n$. In this subsection we shall use the Einstein convention of summing repeated indices when one is a subindex and the other a super-index.

3.1.1. *The Levi-Civita connection of* $\overline{M}$. In the frames given above, we compute the Cartan connection forms $\overline{\omega}_a^b$ defined by

$$d\overline{\theta}^b = -\sum_{a=0}^{n} \overline{\omega}_a^b \wedge \overline{\theta}^a \tag{3.1}$$

Differentiating the formulae for $\overline{\theta}^a$ and comparing with (3.1), we have

$$d\overline{\theta}^0 = d\varphi\wedge du = \widehat{e}_i(\varphi)\widehat{\theta}^i \wedge \frac{1}{\varphi}\overline{\theta}^0 = -\frac{1}{\varphi}\widehat{e}_i(\varphi)\overline{\theta}^0\wedge\overline{\theta}^i = -\overline{\omega}_j^0\wedge\overline{\theta}^j$$
$$d\overline{\theta}^i = d\widehat{\theta}^i = -\widehat{\omega}_j^i\wedge\widehat{\theta}^j = -\overline{\omega}_j^i\wedge\overline{\theta}^j - \overline{\omega}_0^i\wedge\overline{\theta}^0$$

and the solution of these two equations is, denoting $\varphi_i := \widehat{e}_i(\varphi)$,

$$\overline{\omega}_0^i = -\frac{\varphi_i}{\varphi}\ \overline{\theta}^0 = -\varphi_i\ du, \quad \overline{\omega}_j^i = \widehat{\omega}_j^i. \tag{3.2}$$

The relation between $\overline{\omega}$ and $\overline{\nabla}$ is

$$\overline{\nabla}_X Y = X(Y^a)\overline{e}_a + Y^a\overline{\omega}_a^b(X)\overline{e}_b,$$

then we have

$$\overline{\nabla}_{\overline{e}_0}\overline{e}_0 = \overline{\omega}_0^i(\overline{e}_0)\overline{e}_i = -\frac{1}{\varphi}\widehat{\nabla}\varphi, \quad \overline{\nabla}_{\overline{e}_0}\overline{e}_i = \overline{\omega}_i^j(\overline{e}_0)\overline{e}_j + \overline{\omega}_i^0(\overline{e}_0)\overline{e}_0 = \frac{\varphi_i}{\varphi}\overline{e}_0 = \frac{\varphi_i}{\varphi^2}\partial_u \tag{3.3}$$
$$\overline{\nabla}_{\overline{e}_i}\overline{e}_0 = \overline{\omega}_0^j(\overline{e}_i)\overline{e}_j = 0, \quad \overline{\nabla}_{\overline{e}_i}\overline{e}_j = \overline{\omega}_j^k(\overline{e}_i)\overline{e}_k + \overline{\omega}_j^0(\overline{e}_i)\overline{e}_0 = \widehat{\nabla}_{\widehat{e}_i}\widehat{e}_j.$$

3.1.2. *The curvature of $\overline{M}$.* In the frame $\overline{e}_a$, the curvature tensor $\overline{R}$ and the curvature 2-forms $\overline{\Omega}$ are related by

$$\overline{R}_{abcd} = \overline{\Omega}_c^d(\overline{e}_a, \overline{e}_b)$$

and the Cartan equations say

$$d\overline{\omega}_b^a = -\overline{\omega}_c^a \wedge \overline{\omega}_b^c - \overline{\Omega}_b^a. \tag{3.4}$$

Differentiating in (3.2) and applying equations (3.4),

$$\overline{\Omega}_j^i = \widehat{\Omega}_j^i - \overline{\omega}_0^i \wedge \overline{\omega}_j^0 = \widehat{\Omega}_j^i, \qquad \overline{\Omega}_0^i = \frac{\varphi_{ij}}{\varphi}\overline{\theta}^j \wedge \overline{\theta}^0 - \frac{\varphi_j}{\varphi}\widehat{\omega}_j^i \wedge \overline{\theta}^0. \tag{3.5}$$

where $\widehat{\Omega}_j^i$ are the curvature 2-forms of $(M, \widehat{g})$.

From the expressions for the curvature 2-forms we obtain for the components of the curvature tensor:

$$\begin{aligned}
\overline{R}_{0ijk} &= \overline{\Omega}_j^k(\overline{e}_0, \overline{e}_i) = \widehat{\Omega}_j^k(\overline{e}_0, \overline{e}_i) = 0\\
\overline{R}_{0i0k} &= \overline{\Omega}_0^k(\overline{e}_0, \overline{e}_i) = \left(\frac{\varphi_{kj}}{\varphi}\overline{\theta}^j \wedge \overline{\theta}^0 - \frac{\varphi_j}{\varphi}\widehat{\omega}_j^k \wedge \overline{\theta}^0\right)(\overline{e}_0, \overline{e}_i)\\
&= -\frac{\varphi_{ki}}{\varphi} + \frac{\varphi_j}{\varphi}\widehat{\theta}^j\left(\widehat{\nabla}_{\widehat{e}_i}\widehat{e}_k\right) = -\frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{e}_i, \widehat{e}_k)\\
\overline{R}_{ijk\ell} &= \overline{\Omega}_k^\ell(\overline{e}_i, \overline{e}_j) = \widehat{\Omega}_k^\ell(\overline{e}_i, \overline{e}_j) = \widehat{R}_{ijkl}.
\end{aligned} \tag{3.6}$$

Remark 1. *From the formulae for $\overline{\omega}$ and $\overline{R}$ given above it follows that the norms (in the metric induced by $\overline{g}$) of $\overline{R}$ and $\overline{\nabla}^i\overline{R}$ $(i = 0, 1, 2, ...)$ are bounded if the norms (in the metric induced by $\widehat{g}$) of $\widehat{R}$, $\widehat{\nabla}\widehat{R}$, $\dfrac{\widehat{\nabla}^i\varphi}{\varphi}$, $(i = 1, 2, ...)$ are bounded.*

The **general setting** *in this paper is that the geometry of $\overline{M}$ is bounded, that is, $|\overline{R}|$, $|\overline{\nabla}^i\overline{R}|$ $(i = 0, 1, 2, ...)$ are bounded.*

Remark 2. *From the formulae for the covariant derivatives it follows that*

The hypersurfaces $u =$ constant are totally geodesic in $\overline{M}$.

Fixed a $x_0 \in M$, if $\widehat{\nabla}\varphi(x_0) = 0$, the curve $u \mapsto (x_0, u)$ is a geodesic and the curves $u \mapsto (x, u)$ have constant geodesic curvature $\dfrac{|\widehat{\nabla}\varphi|}{\varphi}$, and all the points in $u \mapsto (x, u)$ are at constant distance from the curve $u \mapsto (x_0, u)$ (are equidistant curves).

Now we use the expression of the curvature components to obtain the sectional curvatures:

$$\overline{S}_{i0} = \overline{R}_{i0i0} = -\frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{e}_i, \widehat{e}_i), \qquad \overline{S}_{ij} = \overline{R}_{ijij} = \widehat{S}_{ij}, \tag{3.7}$$

Remark 3. *When M is the simply connected space of constant sectional curvature λ, M_λ^n and $\varphi(x) = \mathrm{c}_\lambda(\widehat{r}(x))$ ($\widehat{r} : M \longrightarrow \mathbb{R}$ defined by $\widehat{r}(x) = \mathrm{dist}_{\widehat{g}}(x_0, x)$ for some x_0 fixed), then $\overline{M}$ must be M_λ^{n+1}), the simply connected space of constant sectional curvature λ, as can be checked from the above formulae for curvatures.*

3.2. **The submanifold.**

3.2.1. *The vectors and quantities N, ∇u, σ and v.* We consider an embedding $F : M \longrightarrow \overline{M}$ given by the graph $F(x) = (x, u(x))$ of a function $u : M \longrightarrow \mathbb{R}$.

The frame $\widehat{e}_i$ of M induces a frame (on the submanifold $F(M)$) $e_i = F_*\widehat{e}_i = \widehat{e}_i + u_i\partial_u = \overline{e}_i + u_i\varphi\overline{e}_0$, where $u_i = \widehat{e}_i(u)$. In this frame, the matrix of the metric g of the submanifold and its inverse, and the dual frame θ^i are given by:

$$\begin{aligned} g_{ij} &= \varphi^2\, u_i\, u_j + \delta_{ij}, \\ g^{ij} &= \delta^{ij} - \frac{\varphi^2 u_i u_j}{1+\varphi^2|\widehat{\nabla} u|^2} \end{aligned} \tag{3.8}$$

A unit normal vector to $F(M)$ can be found using $\xi = a\widehat{\nabla}u + b\partial_u$, imposing the condition $\langle \xi, e_i\rangle = 0$ and dividing by the $\overline{g}$-norm. We choose

$$N = \frac{-\varphi^2\widehat{\nabla}u + \partial_u}{\varphi\sqrt{\varphi^2|\widehat{\nabla}u|^2+1}} = \frac{-\varphi\widehat{\nabla}u + \overline{e}_0}{\sqrt{\varphi^2|\widehat{\nabla}u|^2+1}}.$$

$$\text{Then } \langle N, \partial_u\rangle = \frac{\varphi}{\sqrt{\varphi^2|\widehat{\nabla}u|^2+1}} \text{ and } \langle N, \overline{e}_0\rangle = \frac{1}{\sqrt{\varphi^2|\widehat{\nabla}u|^2+1}}$$

The gradient of u in $F(M)$ can be computed using

$$\overline{\nabla}u = \frac{\overline{e}_0}{\varphi}, \quad \nabla u = \overline{\nabla}u^\top = \frac{1}{\varphi}\left(\overline{e}_0 - \langle \overline{e}_0, N\rangle N\right) =: \frac{1}{\varphi}\overline{e}_0^\top \tag{3.9}$$

or $\nabla u = g^{ij}u_i e_j$. In both cases we obtain:

$$\nabla u = \frac{\widehat{\nabla}u + |\widehat{\nabla}u|^2\partial_u}{1+\varphi^2|\widehat{\nabla}u|^2} \tag{3.10}$$

From (3.9) or (3.10) one gets

$$|\nabla u|^2 = \frac{1}{\varphi^2}\left(1 - \langle N, \overline{e}_0\rangle^2\right) = \frac{|\widehat{\nabla}u|^2}{1+\varphi^2|\widehat{\nabla}u|^2} \tag{3.11}$$

and

$$|\widehat{\nabla}u|^2 = \frac{|\nabla u|^2}{1-\varphi^2|\nabla u|^2} = \frac{1-\langle N, \overline{e}_0\rangle^2}{\varphi^2\,\langle N, \overline{e}_0\rangle^2} \tag{3.12}$$

If we define $\sigma = \langle N, \overline{e}_0\rangle$ and $v = \dfrac{1}{\sigma}$, the above formulae read

$$|\nabla u|^2 = \frac{1}{\varphi^2}\left(1 - \frac{1}{v^2}\right), \qquad |\widehat{\nabla}u|^2 = \frac{v^2-1}{\varphi^2}. \tag{3.13}$$

From the above formulae we also get

$$N = \frac{-\varphi\,\widehat{\nabla}u + \overline{e}_0}{v}, \qquad \widehat{\nabla}u = \frac{-vN + \overline{e}_0}{\varphi} \qquad \widehat{\nabla}u = \frac{\varphi^2 v^2\nabla u - (v^2-1)\partial_u.}{\varphi^2}$$

3.2.2. *Relations of ∇v and $\nabla^2 u$ with the second fundamental form.* For computations, on $F(M)$ we shall consider another local frame E_i orthonormal and satisfying $(\nabla_{E_i} E_j)_p = 0$ at the point p where we are doing the computations. Using it we compute the hessian of u. First:

$$\overline{\nabla}_{E_i}\overline{e}_0 = \langle E_i, \overline{e}_0\rangle \overline{\nabla}_{\overline{e}_0}\overline{e}_0 = -\langle E_i, \overline{e}_0\rangle \frac{\widehat{\nabla}\varphi}{\varphi}.$$

$$(\nabla^2 u)(E_i, E_j) = \langle \nabla_{E_i}\nabla u, E_j\rangle = \left\langle \nabla_{E_i}\frac{1}{\varphi}(\overline{e}_0 - \langle \overline{e}_0, N\rangle N), E_j\right\rangle$$

$$= -\frac{1}{\varphi}\langle E_i, \overline{e}_0\rangle \left\langle \frac{\nabla\varphi}{\varphi}, E_j\right\rangle - \frac{1}{\varphi}\langle E_j, \overline{e}_0\rangle \left\langle \frac{\nabla\varphi}{\varphi}, E_i\right\rangle + \langle \overline{e}_0, N\rangle \frac{1}{\varphi} h(E_i, E_j) \tag{3.14}$$

its trace

$$\Delta u = -\frac{2}{\varphi^2}\left\langle \overline{e}_0^\top, \nabla\varphi\right\rangle + \frac{1}{\varphi v} H. \tag{3.15}$$

and its norm

$$|\nabla^2 u|^2 = \frac{4}{\varphi^4}\left\langle \overline{e}_0^\top, \nabla\varphi\right\rangle^2 + \frac{1}{\varphi^2}\sigma^2 |A|^2 - 4\,\frac{1}{\varphi^3}\,\sigma\; h(\overline{e}_0^\top, \nabla\varphi), \tag{3.16}$$

where $\overline{e}_0^\top := \overline{e}_0 - \langle \overline{e}_0, N\rangle N = \varphi\nabla u$.

For ∇v we have, for every X tangent to $F(M)$,

$$\langle \nabla v, X\rangle = -\frac{1}{\sigma^2}\langle \nabla\sigma, X\rangle = -\frac{1}{\sigma^2}\left(\langle \overline{\nabla}_X \overline{e}_0, N\rangle + \langle \overline{e}_0, \overline{\nabla}_X N\rangle\right)$$

$$= -\frac{1}{\sigma^2}\left(-\langle X, \overline{e}_0\rangle \left\langle \frac{\widehat{\nabla}\varphi}{\varphi}, N\right\rangle - \langle \overline{e}_0, AX\rangle\right) = \frac{1}{\sigma^2}\left(\left\langle \left\langle \frac{\widehat{\nabla}\varphi}{\varphi}, N\right\rangle \overline{e}_0 + A\overline{e}_0^\top, X\right\rangle\right),$$

then

$$\nabla v = v^2\left(\left\langle \frac{\widehat{\nabla}\varphi}{\varphi}, N\right\rangle \overline{e}_0^\top + A\overline{e}_0^\top\right) \tag{3.17}$$

and

$$\langle \nabla v, \nabla\varphi\rangle = \left\langle \nabla v, \overline{\nabla}\varphi\right\rangle = v^2\left\langle \left\langle \frac{\widehat{\nabla}\varphi}{\varphi}, N\right\rangle \overline{e}_0^\top + A\overline{e}_0^\top, \overline{\nabla}\varphi\right\rangle$$

$$= v^2 h(\overline{e}_0^\top, \nabla\varphi) + \frac{v^2}{\varphi}\left\langle \widehat{\nabla}\varphi, N\right\rangle \left\langle \nabla\varphi, \overline{e}_0^\top\right\rangle. \tag{3.18}$$

4. Evolution of u up to tangential diffeomorphisms and short time existence when M is compact.

As it is well known, the equation (1.1) is not parabolic, but it is also known (see [5], or also [1]) that a solution $F(p,t)$ of (1.1) can be written (as far as it remains a graph over M) as the composition $F(p,t) = \overline{F}(\Phi(p,t),t)$, where $\Phi(\cdot,t)$ is a family of diffeomorphismd depending on t and $\overline{F}$ satisfies

$$\left\langle \frac{\partial\overline{F}}{\partial t}, N\right\rangle = H. \tag{4.1}$$

and can be written, for each t, as a graph $\overline{F}(x,t) = (x, u(x,t))$ over M. Using this parametrization of $\overline{F}$ the equation (4.1) becomes

$$\langle \partial_u, N\rangle \frac{\partial u}{\partial t} = H, \quad \text{that is} \quad \frac{\partial u}{\partial t} = \frac{v}{\varphi} H \tag{4.2}$$

In order to apply the standard theory of P.D.E., we compute H in terms of $u : M \longrightarrow \mathbb{R}$ and $\widehat{\nabla}$. We shall use the vectors e_i and the expression for N given in section 3.2. The second fundamental form can be computed using the expressions (3.3) for $\overline{\nabla}$, which gives:

$$\begin{aligned}\langle Ae_i, e_j\rangle &= -\left\langle \overline{\nabla}_{e_i} N, e_j\right\rangle = \frac{1}{\sqrt{\varphi^2|\widehat{\nabla} u|^2+1}}\left\langle -\overline{\nabla}_{\widehat{e}_i + u_i\partial_u}(-\varphi\widehat{\nabla} u + \overline{e}_0), \widehat{e}_j + u_j\varphi\overline{e}_0\right\rangle \\ &= \frac{1}{v}\left(\varphi_i\ u_j + \varphi\left\langle \widehat{\nabla}_{\widehat{e}_i}\widehat{\nabla} u, \widehat{e}_j\right\rangle + \varphi^2 u_i u_j\ \left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle + u_i\varphi_j\right)\end{aligned} \tag{4.3}$$

From (4.3) and (3.8),

$$\begin{aligned} H = g^{ij}\langle Ae_i, e_j\rangle &= \left(\delta^{ij} - \frac{\varphi^2 u_i u_j}{v^2}\right)\frac{1}{v}\left(\varphi_i\ u_j + \varphi\left\langle \overline{\nabla}_{\widehat{e}_i}\widehat{\nabla} u, \widehat{e}_j\right\rangle\right. \\ &\qquad\qquad\left. + \varphi^2 u_i u_j\ \left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle + u_i\varphi_j\right) \\ &= \frac{1}{v}\left(2\left\langle \widehat{\nabla}\varphi, \widehat{\nabla} u\right\rangle + \varphi\widehat{\Delta} u + \varphi^2|\widehat{\nabla} u|^2\left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle\right) \\ &\quad - \frac{\varphi^2}{v^3}\left(2|\widehat{\nabla} u|^2\left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle + \varphi\left\langle \widehat{\nabla}_{\widehat{\nabla} u}\widehat{\nabla} u, \widehat{\nabla} u\right\rangle + \varphi^2|\widehat{\nabla} u|^4\left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle\right) \\ &= \frac{\varphi}{v}\left(\widehat{\Delta} u - \frac{\varphi^2}{v^2}\left\langle \widehat{\nabla}_{\widehat{\nabla} u}\widehat{\nabla} u, \widehat{\nabla} u\right\rangle\right) + \frac{1}{v}\left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle\frac{v^2+1}{v^2}\end{aligned} \tag{4.4}$$

By substitution of this expression in (4.2),

$$\frac{\partial u}{\partial t} = \widehat{\Delta} u - \frac{v^2-1}{v^2}\widehat{\nabla}^2_{\widehat{1}\widehat{1}} u + \frac{1}{\varphi}\left\langle \widehat{\nabla} u, \widehat{\nabla}\varphi\right\rangle\frac{v^2+1}{v^2} \tag{4.5}$$

which is a parabolic equation whereas v remains bounded.

Then, when M is compact, the existence of solution of (4.5) for a small interval of time and smooth initial condition follows directly from the theory of parabolic equations. When M is non-compact, we postpone the discussion to section 7.

For sections 7 and 8 it will be useful to know the evolution under (4.1) of N, v and H. We get them now.

Using the parametrization $\overline{F}(x,t) = (x, u(x,t))$ and (3.3),

$$\begin{aligned}\frac{\overline{\nabla} N}{\partial t} &= \sum_{ij} g^{ij}\left\langle \overline{\nabla}_{\frac{\partial \overline{F}}{\partial t}} N, e_i\right\rangle e_j = -\sum_{ij} g^{ij}\left\langle N, \overline{\nabla}_{\frac{\partial \overline{F}}{\partial t}} e_i\right\rangle e_j = -\frac{\partial u}{\partial t}\varphi\sum_{ij} g^{ij}\langle N, \overline{\nabla}_{\overline{e}_0} e_i\rangle e_j \\ &= -\frac{\partial u}{\partial t}\sum_{ij} g^{ij}\langle N, \overline{\nabla}_{\overline{e}_0}(\widehat{e}_i + u_i\varphi\overline{e}_0)\rangle e_j = \frac{\partial u}{\partial t}\left(-\langle N, \overline{e}_0\rangle\frac{\nabla\varphi}{\varphi} + \left\langle N, \widehat{\nabla}\varphi\right\rangle\nabla u\right)\end{aligned}$$

From the above evolution equation and the definition of v,

$$\frac{\partial v}{\partial t} = -v^2 \left\langle \frac{\overline{\nabla} N}{\partial t}, \overline{e}_0 \right\rangle = v^2 \frac{\partial u}{\partial t} \left(\langle N, \overline{e}_0 \rangle \left\langle \frac{\nabla \varphi}{\varphi}, \overline{e}_0 \right\rangle - \left\langle N, \widehat{\nabla} \varphi \right\rangle \langle \nabla u, \overline{e}_0 \rangle \right), \text{ then}$$

$$\left| \frac{\partial v}{\partial t} \right| \leq v^2 \left| \frac{\partial u}{\partial t} \right| \left(\left| \frac{\widehat{\nabla} \varphi}{\varphi} \right| + |\widehat{\nabla} \varphi| \right) \tag{4.6}$$

From (4.4) we get the following evolution for H

$$\begin{aligned} \frac{\partial H}{\partial t} =& -\frac{1}{v^2} \left(\varphi \left(\widehat{\Delta} u - \frac{\varphi^2}{v^2} \left\langle \widehat{\nabla}_{\widehat{\nabla} u} \widehat{\nabla} u, \widehat{\nabla} u \right\rangle \right) + \left\langle \widehat{\nabla} u, \widehat{\nabla} \varphi \right\rangle \frac{v^2+1}{v^2} \right) \frac{\partial v}{\partial t} \\ &+ \frac{\varphi}{v} \left(\widehat{\Delta}(\frac{v}{\varphi} H) + 2 \frac{\varphi^2}{v^3} \left\langle \widehat{\nabla}_{\widehat{\nabla} u} \widehat{\nabla} u, \widehat{\nabla} u \right\rangle \frac{\partial v}{\partial t} \right. \\ &\quad - \frac{\varphi^2}{v^2} \left(\left\langle \widehat{\nabla}_{\widehat{\nabla}\left(\frac{v}{\varphi} H\right)} \widehat{\nabla} u, \widehat{\nabla} u \right\rangle + \left\langle \widehat{\nabla}_{\widehat{\nabla} u} \widehat{\nabla}(\frac{v}{\varphi} H), \widehat{\nabla} u \right\rangle \right. \\ &\qquad \left.\left. + \left\langle \widehat{\nabla}_{\widehat{\nabla} u} \widehat{\nabla} u, \widehat{\nabla} \left(\frac{v}{\varphi} H \right) \right\rangle \right) \right) \\ &+ \frac{1}{v} \left(\left\langle \widehat{\nabla} \left(\frac{v}{\varphi} H \right), \widehat{\nabla} \varphi \right\rangle \frac{v^2+1}{v^2} - \left\langle \widehat{\nabla} u, \widehat{\nabla} \varphi \right\rangle \frac{2}{v^3} \frac{\partial v}{\partial t} \right) \end{aligned} \tag{4.7}$$

The following observation will also be useful

Proposition 4. *Let f be a C^2 function defined on M_t, $\widehat{f} = f \circ \overline{F}(\cdot, t)$, then $|\widehat{\nabla} \widehat{f}|$ is bounded if $|\nabla f|$ and $|\widehat{\nabla} u|$ are bounded, and $|\widehat{\nabla}^2 \widehat{f}|$ is bounded if $|\nabla^2 f|$, $|\nabla f|$, $|A|$, $|\widehat{\nabla} u|$ and $|\widehat{\nabla}^2 u|$ are bounded*

Proof Let $\widehat{X}$ be a vector field over M, since $X := \overline{F}(\cdot, t)_* \widehat{X} = \widehat{X} + \left\langle \widehat{\nabla} u, \widehat{X} \right\rangle \overline{e}_0$,

$$\begin{aligned} \left\langle \widehat{\nabla} \widehat{f}, \widehat{X} \right\rangle &= df(\overline{F}(\cdot, t)_* \widehat{X}) = \left\langle \nabla f, \overline{F}(\cdot, t)_* \widehat{X} \right\rangle = \left\langle \nabla f, \widehat{X} + \left\langle \widehat{\nabla} u, \widehat{X} \right\rangle \overline{e}_0 \right\rangle \\ &= \left\langle \nabla f, \widehat{X} \right\rangle + \langle \nabla f, \overline{e}_0 \rangle \left\langle \widehat{\nabla} u, \widehat{X} \right\rangle, \quad \text{then} \end{aligned}$$

$$\widehat{\nabla} \widehat{f} = \nabla f - \langle \nabla f, \overline{e}_0 \rangle \overline{e}_0 + \langle \nabla f, \overline{e}_0 \rangle \widehat{\nabla} u \tag{4.8}$$

Using the above expressions for X and $\widehat{\nabla}\widehat{f}$ and the formulae (3.3) for $\overline{\nabla}$,

$$\begin{aligned}\widehat{\nabla}_{\widehat{X}}\widehat{\nabla}\widehat{f} &= \overline{\nabla}_{X-\langle\widehat{X},\widehat{\nabla}u\rangle\overline{e}_0}\widehat{\nabla}\widehat{f} = \overline{\nabla}_X\widehat{\nabla}\widehat{f} - \left\langle \widehat{X},\widehat{\nabla}u\right\rangle \overline{\nabla}_{\overline{e}_0}\widehat{\nabla}\widehat{f}\\ &= \nabla_X\nabla f + \alpha(X,\nabla f) - \langle \nabla_X\nabla f + \alpha(X,\nabla f),\overline{e}_0\rangle\,\overline{e}_0\\ &\quad + \left\langle \nabla f, \left\langle \widehat{X}, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \overline{e}_0 - \left\langle \widehat{\nabla}u,\widehat{X}\right\rangle \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle (-\overline{e}_0 + \widehat{\nabla}u)\\ &\quad + \langle \nabla_X\nabla f + \alpha(X,\nabla f),\overline{e}_0\rangle\,\widehat{\nabla}u\\ &\quad + \langle \nabla f,\overline{e}_0\rangle \left(\widehat{\nabla}_X\widehat{\nabla}u + \left\langle \widehat{X},\widehat{\nabla}u\right\rangle \left\langle \widehat{\nabla}u, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \overline{e}_0\right)\\ &\quad - \left\langle \widehat{X},\widehat{\nabla}u\right\rangle \left\langle \widehat{\nabla}\widehat{f}, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \overline{e}_0. \end{aligned}\tag{4.9}$$

It follows from (4.8) that $|\widehat{\nabla}\widehat{f}|$ is bounded if $|\nabla f|$ and $|\widehat{\nabla}u|$ are bounded, and from (4.9) that $|\widehat{\nabla}^2\widehat{f}|$ is bounded if $|\nabla^2 f|$, $|\nabla f|$, $|A|$, $|\widehat{\nabla}u|$ and $|\widehat{\nabla}^2 u|$ are bounded. □

5. Preserving the property of being a graph

The condition "cuts transversaly the curves $s \mapsto (x,s)$" in the definition of a graph means that $\sigma := \langle N,\overline{e}_0\rangle > 0$, which is equivalent to say $1 \le v = \dfrac{1}{\sigma} < \infty$. Therefore, our first goal is to obtain an upper bound for v. To achieve this, we need the evolution equation for v under (1.1).

Lemma 5. *Under* (1.1), *v evolves as*

$$\begin{aligned}\frac{\partial v}{\partial t} &= \Delta v - \frac{2}{v}|\nabla v|^2 + \frac{2}{\varphi}\langle \nabla v,\nabla\varphi\rangle - v|A|^2\\ &\quad - v\left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{R}ic_{\widehat{1}\widehat{1}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2} - \frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{1},\widehat{1})\right)\end{aligned}\tag{5.1}$$

where "$\widehat{1}$" is the unit vector in the direction $\widehat{\nabla}u$.

Proof First we compute $\Delta\sigma$. To do so, we shall use an orthonormal frame of M of the form $E_1,...,E_n$ as was introduced before.

$$\Delta\sigma = E_iE_i\langle N,\overline{e}_0\rangle = E_i\left(\left\langle \overline{\nabla}_{E_i}N,\overline{e}_0\right\rangle + \left\langle N,\overline{\nabla}_{E_i}\overline{e}_0\right\rangle\right)\tag{5.2}$$

$$\begin{aligned}
E_i \left\langle \overline{\nabla}_{E_i} N, \overline{e}_0 \right\rangle &= -E_i(h(E_i, \overline{e}_0^\top)) = -\nabla_{E_i}(h)(E_i, \overline{e}_0^\top) - h(E_i, \nabla_{E_i}\overline{e}_0^\top) \\
&= -\nabla_{\overline{e}_0^\top}(h)(E_i, E_i) + \overline{R}_{E_i \overline{e}_0^\top E_i N} - \left\langle AE_i, \overline{\nabla}_{E_i}(\overline{e}_0 - \langle \overline{e}_0, N\rangle N)\right\rangle \\
&= -\overline{e}_0^\top(H) + \overline{R}ic(\overline{e}_0^\top, N) + \left\langle AE_i, \langle E_i, \overline{e}_0\rangle \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \langle AE_i, \langle \overline{e}_0, N\rangle AE_i\rangle \\
&= -\overline{e}_0^\top(H) + \overline{R}ic(\overline{e}_0^\top, N) + \left\langle A\overline{e}_0^\top, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \langle \overline{e}_0, N\rangle |A|^2 \qquad (5.3)
\end{aligned}$$

$$\begin{aligned}
E_i \left\langle N, \overline{\nabla}_{E_i}\overline{e}_0 \right\rangle &= E_i \left\langle N, -\langle E_i, \overline{e}_0\rangle \frac{\widehat{\nabla}\varphi}{\varphi})\right\rangle \\
&= -\overline{e}_0^\top \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \left(\left\langle \overline{\nabla}_{E_i} E_i, \overline{e}_0\right\rangle + \left\langle E_i, \overline{\nabla}_{E_i}\overline{e}_o\right\rangle\right) \\
&= -\overline{e}_0^\top \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \left(H\langle N, \overline{e}_0\rangle - \left\langle E_i, \langle E_i, \overline{e}_0\rangle \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle\right) \\
&= -\overline{e}_0^\top \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \langle N, \overline{e}_0\rangle H + \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \left\langle \overline{e}_0^\top, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle. \qquad (5.4)
\end{aligned}$$

Then

$$\begin{aligned}
\Delta\sigma = &-\overline{e}_0^\top(H) + \overline{R}ic(\overline{e}_0^\top, N) + \left\langle A\overline{e}_0^\top, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \langle \overline{e}_0, N\rangle |A|^2 \\
&- \overline{e}_0^\top \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \langle N, \overline{e}_0\rangle H + \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \left\langle \overline{e}_0^\top, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle. \qquad (5.5)
\end{aligned}$$

Now, let us compute

$$\begin{aligned}
\overline{R}ic(\overline{e}_0^\top, N) &= \overline{R}ic(\overline{e}_0, N) - \langle \overline{e}_0, N\rangle \overline{R}ic(N, N) \\
&= \overline{R}ic\left(\overline{e}_0, \frac{-\varphi\, \widehat{\nabla}u + \overline{e}_0}{v}\right) - \frac{1}{v}\overline{R}ic\left(\frac{-\varphi\, \widehat{\nabla}u + \overline{e}_0}{v}, \frac{-\varphi\, \widehat{\nabla}u + \overline{e}_0}{v}\right) \\
&= \frac{1}{v}\overline{R}ic(\overline{e}_0, \overline{e}_0) - \frac{\varphi}{v}\overline{R}ic(\overline{e}_0, \widehat{\nabla}u) - \frac{\varphi^2}{v^3}\overline{R}ic(\widehat{\nabla}u, \widehat{\nabla}u) + 2\frac{\varphi}{v^3}\overline{R}ic(\overline{e}_0, \widehat{\nabla}u) - \frac{1}{v^3}\overline{R}ic(\overline{e}_0, \overline{e}_0) \\
&= \frac{1}{v}\left(1 - \frac{1}{v^2}\right)\overline{R}ic(\overline{e}_0, \overline{e}_0) - \frac{\varphi}{v}\left(1 - \frac{2}{v^2}\right)\overline{R}ic(\overline{e}_0, \widehat{\nabla}u) - \frac{\varphi^2}{v^3}\overline{R}ic(\widehat{\nabla}u, \widehat{\nabla}u).
\end{aligned}$$

From the expressions (3.6) of the components of $\overline{R}$, it follows that

$$\begin{aligned}
\overline{R}ic(\overline{e}_0, \overline{e}_0) &= \overline{R}(\overline{e}_0, \overline{e}_i, \overline{e}_0, \overline{e}_i) = -\frac{\widehat{\Delta}\varphi}{\varphi} \\
\overline{R}ic(\overline{e}_0, \widehat{\nabla}u) &= 0 \\
\overline{R}ic(\widehat{\nabla}u, \widehat{\nabla}u) &= \widehat{R}ic(\widehat{\nabla}u, \widehat{\nabla}u) = \frac{v^2 - 1}{\varphi^2}\widehat{R}ic_{\widehat{1}\widehat{1}}.
\end{aligned}$$

then

$$\Delta\sigma = -\bar{e}_0^\top(H) - \frac{1}{v}\left(1-\frac{1}{v^2}\right)\frac{\widehat{\Delta}\varphi}{\varphi} - \frac{1}{v}\left(1-\frac{1}{v^2}\right)\widehat{R}ic_{\widehat{1}\widehat{1}}. + \left\langle A\bar{e}_0^\top, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle$$
$$- \frac{1}{v}|A|^2 - \bar{e}_0^\top\left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle \langle N, \bar{e}_0\rangle H + \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle\left\langle \bar{e}_0^\top, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle. \tag{5.6}$$

$$-\bar{e}_0^\top\left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle = -\frac{1}{\varphi}\overline{\nabla}^2\varphi(\bar{e}_0^\top, N) - \frac{1}{\varphi}\left\langle\overline{\nabla}\varphi, \overline{\nabla}_{\bar{e}_0^\top}N\right\rangle + \frac{1}{\varphi^2}\left\langle\bar{e}_0^\top, \overline{\nabla}\varphi\right\rangle\langle N, \overline{\nabla}\varphi\rangle$$
$$= -\frac{\overline{\nabla}^2\varphi}{\varphi}(\bar{e}_0^\top, N) + \frac{1}{\varphi}h(\bar{e}^\top, \nabla\varphi) + \frac{1}{\varphi^2}\left\langle\bar{e}_0^\top, \nabla\varphi\right\rangle\langle N, \overline{\nabla}\varphi\rangle \tag{5.7}$$

which gives

$$\Delta\sigma = -\bar{e}_0^\top(H) - \frac{1}{v}\left(1-\frac{1}{v^2}\right)\frac{\widehat{\Delta}\varphi}{\varphi} - \frac{1}{v}\left(1-\frac{1}{v^2}\right)\widehat{R}ic_{\widehat{1}\widehat{1}}. - \frac{1}{v}|A|^2 + \frac{2}{\varphi}h(\bar{e}^\top, \nabla\varphi)$$
$$- \frac{\overline{\nabla}^2\varphi}{\varphi}(\bar{e}_0^\top, N) + \frac{2}{\varphi^2}\left\langle\bar{e}_0^\top, \nabla\varphi\right\rangle\langle N, \overline{\nabla}\varphi\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle\langle N, \bar{e}_0\rangle H, \text{ that is, using (3.18)}$$

$$\Delta\sigma = -\bar{e}_0^\top(H) - \frac{1}{v}\left(1-\frac{1}{v^2}\right)\frac{\widehat{\Delta}\varphi}{\varphi} - \frac{1}{v}\left(1-\frac{1}{v^2}\right)\widehat{R}ic_{\widehat{1}\widehat{1}} - \frac{\overline{\nabla}^2\varphi}{\varphi}(\bar{e}_0^\top, N)$$
$$- \frac{1}{v}|A|^2 + \frac{2}{\varphi\, v^2}\langle\nabla v, \nabla\varphi\rangle - \left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle\frac{H}{v}. \tag{5.8}$$

On the other hand:

$$\nabla v = -\frac{1}{\sigma^2}\nabla\sigma, \tag{5.9}$$

$$\Delta v = E_iE_iv = E_i\left(-\frac{1}{\sigma^2}E_i\sigma\right) = \frac{2}{\sigma^3}|\nabla\sigma|^2 - \frac{1}{\sigma^2}E_iE_i(\sigma) = \frac{2}{v}|\nabla v|^2 - v^2\Delta\sigma. \tag{5.10}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\sigma^2}\frac{\partial\sigma}{\partial t} = -v^2\left(\left\langle\frac{\overline{\nabla}N}{\partial t}, \bar{e}_0\right\rangle + \left\langle N, \frac{\overline{\nabla}\bar{e}_0}{\partial t}\right\rangle\right)$$
$$= -v^2\left(\langle -\nabla H, \bar{e}_0\rangle - \left\langle N, H\langle N, \bar{e}_0\rangle\frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle\right)$$
$$= v^2\bar{e}_0^\top(H) + vH\left\langle N, \frac{\widehat{\nabla}\varphi}{\varphi}\right\rangle. \tag{5.11}$$

Joining the expressions for Δv, $\Delta\sigma$ and $\dfrac{\partial v}{\partial t}$, we obtain

$$\frac{\partial v}{\partial t} = \Delta v - \frac{2}{v}|\nabla v|^2 - v\left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{Ric}_{\widehat{1}\widehat{1}}\right) - v^2\frac{\overline{\nabla}^2\varphi}{\varphi}(\overline{e}_0^\top, N) - v|A|^2 + \frac{2}{\varphi}\langle\nabla v, \nabla\varphi\rangle$$

Defining $N^h := N - \langle N, \overline{e}_0\rangle\,\overline{e}_0 = N - \sigma\overline{e}_0$,

$$\begin{aligned}\overline{\nabla}^2\varphi(\overline{e}_0^\top, N) &= \overline{\nabla}^2\varphi\left(\overline{e}_0 - \sigma(N^h+\sigma\overline{e}_0), N^h+\sigma\overline{e}_0\right)\\ &= \sigma\frac{|\widehat{\nabla}\varphi|^2}{\varphi} - \sigma\widehat{\nabla}^2\varphi(N^h,N^h) - \sigma^3\frac{|\widehat{\nabla}\varphi|^2}{\varphi}\\ &= \frac{1}{v}\left(1-\frac{1}{v^2}\right)\frac{|\widehat{\nabla}\varphi|^2}{\varphi} - \frac{1}{v}\widehat{\nabla}^2\varphi(N^h,N^h).\end{aligned} \tag{5.12}$$

and sustitution in the evolution equation for v gives

$$\frac{\partial v}{\partial t} = \Delta v - \frac{2}{v}|\nabla v|^2 - v\left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{Ric}_{\widehat{1}\widehat{1}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2} - \frac{\widehat{\nabla}^2\varphi}{\varphi}\left(\frac{N^h}{|N^h|},\frac{N^h}{|N^h|}\right)\right) - v|A|^2 + \frac{2}{\varphi}\langle\nabla v,\nabla\varphi\rangle$$

Let us remark that $N^h = \dfrac{-\varphi\widehat{\nabla}u}{\sqrt{\varphi^2|\widehat{\nabla}u|^2+1}}$, then $\widehat{1} = \dfrac{N^h}{|N^h|}$, and $\dfrac{\widehat{\Delta}\varphi}{\varphi} - \dfrac{\widehat{\nabla}^2\varphi}{\varphi}\left(\dfrac{N^h}{|N^h|},\dfrac{N^h}{|N^h|}\right)$ is $-\overline{Ric}(\overline{e}_0,\overline{e}_0) + \overline{R}_{0\widehat{1}0\widehat{1}}$. □

Notation 1. *A general hypothesis in this work is that $\overline{M}$ has bounded geometry, in particular that $\dfrac{|\widehat{\nabla}\varphi|}{\varphi}$, $\dfrac{|\widehat{\nabla}^2\varphi|}{\varphi}$ and $|\widehat{R}|$ are bounded. Then there are constants η, μ_1, μ_2 and μ satisfying*

$$\eta = \sup_{\overline{M}}\frac{|\widehat{\nabla}\varphi|}{\varphi}, \quad \mu_1\widehat{g} \le \frac{\widehat{\nabla}^2\varphi}{\varphi} \le \mu_2\widehat{g}, \quad \widehat{Sec} \ge \mu \tag{5.13}$$

and we define the constant ν by the formula

$$\mu = \frac{-n\mu_1+\mu_2}{n-1} - \nu \tag{5.14}$$

Theorem 6. *Let M be compact. Let $F(x,t) : M \longrightarrow \overline{M}$ be a solution of (1.1) defined on a time interval $[0,T[$. If $\overline{M} = M\times_\varphi\mathbb{R}$ and M_0 is a graph over M, then M_t is a graph over M for every $t\in[0,T[$. With more precision: $v(F(x,t) \le \max_{M_0} v\ e^{(n-1)\nu t}$.*

Proof Using the notation (5.13) and (5.14), we have

$$-\left(\widehat{Ric}_{\widehat{1}\widehat{1}} - \frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{1},\widehat{1}) + \frac{\widehat{\Delta}\varphi}{\varphi} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2}\right) \le (n-1)\nu. \tag{5.15}$$

From (5.1) and forgetting about the negative summands, we reach the inequality:

$$\frac{\partial v}{\partial t} < \Delta v - \frac{2}{v}|\nabla v|^2 + \frac{2}{\varphi}\langle \nabla v, \nabla \varphi\rangle + (n-1)\nu\, v. \tag{5.16}$$

By the maximum principle, v is bounded by the solution of $y'(t) = (n-1)\nu\, y(t)$ with the initial condition $y(0) = v_0 := \sup_{M_0} v$, then $v \leq v_0 e^{(n-1)\nu t}$, as claimed.

On the other hand, if M_t is a graph over a proper subset of M and M is compact, it cannot be homeomorphic to M, then if M_0 is a graph over all M it cannot move to M_t without producing singularities, which means $t \notin [0, T[$. From this and the estimate on v it follows that M_t remains to be a graph over M for all time in $[0, T[$. □

Theorem 7. *Let M be complete non compact with a pole x_0. Let M_t be a solution of* (1.1) *defined on a maximal time interval $[0, T[$. If $\overline{M} = M \times_\varphi \mathbb{R}$ and M_0 is a graph over M, then $v \circ F(x,t)$ remains bounded at each point $F(x,t)$ for every $t \in [0, T[$.*

Remark *The hypothesis M non compact also imposes $\mu \leq 0$ (cf.* (5.13)*). On the other hand, if $\widehat{S}ec \geq 0$, it is also true that $\widehat{S}ec > \mu'$ for every $\mu' < 0$, then we can suppose, without loose of generality, that $\mu < 0$.*

Proof Let us recall that $\widehat{r}$ denotes the $\widehat{g}$-distance from x_0. As in the compact case, we have the inequality (5.15). An standard procedure to apply the maximum principle to v in the non-compact case is to look at the evolution of the product ϕv of v times some cut-function ϕ suitably chosen. Inspired in [5] and [13], we define first a function

$$\zeta(\widehat{r}, t) = f(\widehat{r})\; e^{\beta n t},$$

where the function f and the constant β will be defined later, and

$$\phi : [0, \infty[\longrightarrow \mathbb{R}, \text{ by } \phi(\zeta) = \alpha\; (f(\rho) - \zeta)^2, \tag{5.17}$$

where, again, the constant $\alpha > 0$ will be chosen later. Let us observe that:

$$\phi' = -2\alpha\; (f(\rho) - \zeta) < 0 \text{ if } \zeta < f(\rho), \quad \text{and} \quad \phi'' = 2\alpha > 0 \tag{5.18}$$

In order to compute the evolution of ϕv we need first to compute the **evolution of** $\widehat{r}$. Since $\overline{\nabla}\widehat{r} = \widehat{\nabla}\widehat{r}$, which we shall denote $\partial_{\widehat{r}}$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\widehat{r} = HN(\widehat{r}) - \sum_{i=1}^n E_i E_i \widehat{r} = H\langle N, \partial_{\widehat{r}}\rangle - \sum_{i=1}^n \left\langle \overline{\nabla}_{E_i} E_i, \partial_{\widehat{r}}\right\rangle - \sum_{i=1}^n \left\langle E_i, \overline{\nabla}_{E_i}\partial_{\widehat{r}}\right\rangle$$
$$= -\sum_{i=1}^n \left\langle E_i, \overline{\nabla}_{E_i}\partial_{\widehat{r}}\right\rangle = -\sum_{i=1}^n \left\langle E_i, \overline{\nabla}_{E_i - \langle E_i, \partial_{\widehat{r}}\rangle \partial_{\widehat{r}} - \langle E_i, \overline{e}_0\rangle \overline{e}_0}\partial_{\widehat{r}}\right\rangle - \sum_{i=1}^n \langle E_i, \overline{e}_0\rangle \left\langle E_i, \overline{\nabla}_{\overline{e}_0}\partial_{\widehat{r}}\right\rangle.$$

Denoting $E_i^\perp := E_i - \langle E_i, \partial_{\widehat{r}}\rangle \partial_{\widehat{r}} - \langle E_i, \overline{e}_0\rangle \overline{e}_0$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\widehat{r} = -\sum_{i=1}^n \widehat{\nabla}^2\widehat{r}\left(E_i^\perp, E_i^\perp\right) - \sum_{i=1}^n \langle E_i, \overline{e}_0\rangle \frac{\partial_{\widehat{r}}\varphi}{\varphi}\langle \overline{e}_0, E_i\rangle$$

$$\underset{\text{by (2.2)}}{\geq} -\frac{c_\mu}{s_\mu}\sum_{i=1}^n \widehat{g}(E_i^\perp, E_i^\perp) - \frac{\partial_{\widehat{r}}\varphi}{\varphi}\left\langle \overline{e}_0^\top, \overline{e}_0^\top\right\rangle$$

$$= -\frac{c_\mu}{s_\mu}\left(n - |\partial_{\widehat{r}}|^2 - |\overline{e}_0^\top|^2\right) - \frac{\partial_{\widehat{r}}\varphi}{\varphi}|\overline{e}_0^\top|^2. \tag{5.19}$$

evolution of $f(\widehat{r}) := f \circ \widehat{r}$ for $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $f' > 0$

$$\left(\frac{\partial}{\partial t} - \Delta\right) f(\widehat{r}) = f'\left(\frac{\partial}{\partial t} - \Delta\right)\widehat{r} - f''\left|\partial_{\widehat{r}}^\top\right|^2$$

$$\underset{\text{by (2.2)}}{\geq} f'\left(-\frac{c_\mu}{s_\mu}\left(n - |\partial_{\widehat{r}}^\top|^2 - |\overline{e}_0^\top|^2\right) - \frac{\partial_{\widehat{r}}\varphi}{\varphi}|\overline{e}_0^\top|^2.\right) - f''\left|\partial_{\widehat{r}}^\top\right|^2$$

$$= \left(-f'' + \frac{c_\mu}{s_\mu}f'\right)\left|\partial_{\widehat{r}}^\top\right|^2 + f'\frac{c_\mu}{s_\mu}\left(|\overline{e}_0^\top|^2 - n\right) - f'\frac{\partial_{\widehat{r}}\varphi}{\varphi}|\overline{e}_0^\top|^2. \tag{5.20}$$

In order that $f' > 0$ and the first summand in the last expression be zero, we take $f = \dfrac{c_\mu}{-\mu}$ and the above expression gives

$$\left(\frac{\partial}{\partial t} - \Delta\right) f(\widehat{r}) \geq c_\mu\left(|\overline{e}_0^\top|^2 - n\right) - s_\mu\frac{\partial_{\widehat{r}}\varphi}{\varphi}|\overline{e}_0^\top|^2 \geq c_\mu\left(|\overline{e}_0^\top|^2 - n\right) - \eta\, s_\mu|\overline{e}_0^\top|^2, \tag{5.21}$$

where we have used $\left|\dfrac{\partial_{\widehat{r}}\varphi}{\varphi}\right| \leq \left|\dfrac{\widehat{\nabla}\varphi}{\varphi}\right| \leq \eta$ for the second inequality.

evolution of ζ

$$\left(\frac{\partial}{\partial t} - \Delta\right)\zeta = e^{\beta nt}\left(\frac{\partial}{\partial t} - \Delta\right) f(\widehat{r}) + \beta\, n\, e^{\beta nt}\, f$$

$$\underset{(5.21)}{\geq} e^{\beta nt}\left(c_\mu\left(|\overline{e}_0^\top|^2 - n\right) - \eta\, s_\mu|\overline{e}_0^\top|^2\right) + \beta\, n\, e^{\beta nt}\, \frac{c_\mu}{-\mu} \tag{5.22}$$

evolution of $\phi := \phi \circ \zeta$

First, let us observe that $\nabla\phi = \phi'\nabla\zeta$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi = \phi'\left(\frac{\partial}{\partial t} - \Delta\right)\zeta - \phi''\,|\nabla\zeta|^2 = \phi'\left(\frac{\partial}{\partial t} - \Delta\right)\zeta - \phi''\frac{|\nabla\phi|^2}{\phi'^2}$$

$$\underset{\phi'<0 \text{ and}(5.22)}{\leq} \phi'\left(e^{\beta nt}\left(c_\mu\left(|\overline{e}_0^\top|^2 - n\right) - \eta\, s_\mu|\overline{e}_0^\top|^2\right) + \beta\, n\, e^{\beta nt}\, \frac{c_\mu}{-\mu}\right) - \phi''\frac{|\nabla\phi|^2}{\phi'^2}$$

$$= \phi' e^{\beta nt}\left(c_\mu|\overline{e}_0^\top|^2 - nc_\mu - \eta\, s_\mu|\overline{e}_0^\top|^2 + \; n\; \frac{\beta}{-\mu}c_\mu\right) - \phi''\frac{|\nabla\phi|^2}{\phi'^2} \tag{5.23}$$

Now, we choose

$$\beta = -\mu\left(1 + \frac{\eta}{n}\right), \qquad \text{(notice that } \beta = \nu \text{ when } \varphi = 1) \tag{5.24}$$

By substitution of this value of β in (5.23), using $\mathrm{c}_\mu \geq \sqrt{|\mu|}\mathrm{s}_\mu$, we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi \leq \phi' e^{\beta n t}\mathrm{c}_\mu |\overline{e}_0^\top|^2 - \phi''\frac{|\nabla\phi|^2}{\phi'^2} \leq -\phi''\frac{|\nabla\phi|^2}{\phi'^2} \tag{5.25}$$

and, finaly, the **evolution of** ϕv

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(\phi\ v) &= v\left(\frac{\partial}{\partial t} - \Delta\right)\phi + \phi\left(\frac{\partial}{\partial t} - \Delta\right)v - 2\langle\nabla\phi, \nabla v\rangle \\ &\leq -v\ \phi''\frac{|\nabla\phi|^2}{\phi'^2} - 2\langle\nabla\phi,\nabla v\rangle \\ &\quad + \phi\left(-\frac{2}{v}|\nabla v|^2 + 2\left\langle\nabla v, \frac{\nabla\varphi}{\varphi}\right\rangle - v|A|^2 + v\left(1 - \frac{1}{v^2}\right)(n-1)\nu\right) \\ &= -2\alpha v\ \frac{|\nabla\phi|^2}{\phi'^2} + 2\left(\left\langle\nabla(\phi v), \frac{\nabla\varphi}{\varphi}\right\rangle - v\left\langle\nabla\phi, \frac{\nabla\varphi}{\varphi}\right\rangle\right) \\ &\quad - \phi v|A|^2 + \phi v\left(1 - \frac{1}{v^2}\right)(n-1)\nu - 2\left\langle\nabla(\phi v), \frac{\nabla v}{v}\right\rangle \\ &\leq 2\left\langle\nabla(\phi v), \frac{\nabla\varphi}{\varphi} - \frac{\nabla v}{v}\right\rangle - 2\alpha v\ \frac{|\nabla\phi|^2}{\phi'^2} + 2\eta v|\nabla\phi| + (n-1)\nu\phi v, \end{aligned} \tag{5.26}$$

where we have used $\nu \geq 0$ and $\dfrac{|\nabla\varphi|}{\varphi} \leq \dfrac{|\widehat{\nabla}\varphi|}{\varphi} \leq \eta$, for the last inequality. If $\nu < 0$, we can forget the last summand in (5.26). To consider both cases in the future we define

$$\epsilon\nu = \begin{cases}\nu \text{ if } \nu > 0\\ 0 \text{ if } \ \nu \leq 0\end{cases}. \tag{5.27}$$

Now, from the obvious inequality $(|\nabla\phi| - 2\phi'^2)^2 \geq 0$, we obtain

$$2\eta v|\nabla\phi| \leq \frac{\eta}{2}v\frac{|\nabla\phi|^2}{\phi'^2} + 2\eta\phi'^2 v \tag{5.28}$$

Then, if we take $\alpha = \left(1 + \dfrac{\eta}{4}\right)$, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(\phi\ v) &< 2\left\langle\nabla(\phi v), \frac{\nabla\varphi}{\varphi} - \frac{\nabla v}{v}\right\rangle + 2\ \eta\frac{\phi'^2}{\phi}\ \phi v + (n-1)\epsilon\nu\ \phi v \\ &= 2\left\langle\nabla(\phi v), \frac{\nabla\varphi}{\varphi} - \frac{\nabla v}{v}\right\rangle + \left(2\eta^2 + (n-1)\epsilon\nu\right)\phi v.\end{aligned} \tag{5.29}$$

Let us consider the sets

$$S_{\rho,t} = \{F(x,t)\ ;\ \frac{\mathrm{c}_\mu}{-\mu}(\rho) - \frac{\mathrm{c}_\mu}{-\mu}(\widehat{r}(F(x,t)))e^{\beta n t} \geq 0\}, \quad \mathbb{S}_{\rho,\tau} = \bigcup_{t\in[0,\tau)} S_{\rho,t},$$

(where ")" means "]" if $\tau < T$ and "[" if $\tau = T$). Observe that ϕv vanishes on the boundary of $S_{\rho,t}$ and that

$$S_{\rho,0} = \{F(x,0);\ \widehat{r}(F(x,0)) \leq \rho\} = \widehat{r}^{-1}(\rho)\cap M_0.$$

By the maximum principle applied to the function ϕv, on $\mathbb{S}_{\rho,T}$, $\phi v(F(x,t))$ is bounded from above by the solution of the equation $y' = (2\eta^2 + (n-1)\epsilon\nu)y$ with the initial conditions $y(0) = \sup_{S_{\rho,0}} \phi v$, that is,

$$\alpha \left(\frac{c_\mu}{-\mu}(\rho) - \frac{c_\mu}{-\mu}(\widehat{r}(F(x,t)))e^{\beta n t} \right)^2 v(F(x,t)) \leq \sup_{S_{\rho,0}} \phi v \; e^{(2\eta^2+(n-1)\epsilon\nu)t} \leq \alpha \left(\frac{c_\mu}{-\mu}(\rho) \right)^2 \sup_{S_{\rho,0}} v \; e^{(2\eta^2+(n-1)\epsilon\nu)t} \tag{5.30}$$

which, for every $F(x,t)$ in the slice M_t, taking ρ big enough, gives an upper bound of $v(F(x,t))$ (depending on t and ρ), which finishes the proof of the claim. □

Remark 4. *Let us observe that, unlike in the compact case, the conclusion of Theorem 7 is just a bound on v, with no conclusion about the preservation of the property of being a graph, which will be a consequence of the proof of the existence theorem in section 7 and the prolongation theorem 10. The reason of this is that, when M_t is not compact, it is not clear that v bounded at each point implies M_t is a graph.*

However, if M_t is complete and has $|A|$ bounded by a universal constant, it is true that v bounded at each point implies that M_t is a graph over M. In fact, let us suppose that M_t is a graph over a proper subset of M and has $|A|$ bounded, let $x \in M$ such that the line $L = \{x\} \times \mathbb{R}$ does not cut M_t. The distance d to L is a C^∞ function on M_t, and its infimum is the distance δ between M_t and L. Then, by Omori's Lemma (cf [9]*), there is a sequence of points $p_n \in M_t$ such that $d(p_n) - \delta < \frac{1}{n}$ and $|\nabla d|(p_n) < \frac{1}{n}$.*

For each p_n, let $q_n \in L$ such that $\operatorname{dist}(q_n, p_n) = d(p_n)$*. Let $\gamma_n(t)$ be the geodesic from q_n to p_n realizing $d(p_n)$. It is orthogonal to L at q_n and, since the hypersurfaces $u =$constant are totally geodesic (cf. Remark 2), γ_n is contained in such a hypersurface and orthogonal to the vector field $\overline{e}_0$. Moreover, it is an integral curve of $\overline{\nabla} d$ which is then, orthogonal to $\overline{e}_0$. On the other hand $\nabla d = \overline{\nabla} d - \langle \overline{\nabla} d, N \rangle N$, then $|\nabla d|(p_n) < \frac{1}{n}$ implies $\overline{\nabla} d(p_n)$ approach $N(p_n)$ as $n \to \infty$, then, as $n \to \infty$, $\langle \overline{e}_0, N \rangle_{p_n}$ approach* 0 *and v goes to ∞. But, taking $\rho >$* $\operatorname{dist}(x, x_0)$*, this is in contradiction with* (5.30)*. Then, on [0,T[, M_t remains to be a graph over all M.*

The property above gives a geometric difference between the concepts of graph in $M \times_\varphi \mathbb{R}$ and $\mathbb{R} \times_\varphi M$. In fact, for graphs in $\mathbb{R} \times_\varphi M$ and φ not constant, it is no longer true that the hypersurfaces $u =$constant are totally geodesic, then $\overline{\nabla} d$ is not orthogonal to $\overline{e}_0$ and the argument fails. It is not dificult, using this idea, to construct a hypersurface in the hyperbolic space $\mathcal{H}^{n+1}$ which is complete, a geodesic graph on an open set U of $\mathcal{H}^n$ ($U \neq \mathcal{H}^n$) and with v bounded. For instance, in the Poincaré's ball model, take a disc with boundary at the infinite and parallel to the equator.

Remark 5. *For use in the long time existence theorem, it is convenient to give a more explicit (although less precise) bound for v than that obtained in* (5.30)*. For this, first we consider a smaller set $S_{\rho,t,\gamma}$ than $S_{\rho,t}$, defined, for any positive $\gamma < 1$, by*

$$S_{\rho,t,\gamma} = \{F(x,t) \; ; \; \gamma \frac{c_\mu}{-\mu}(\rho) - \frac{c_\mu}{-\mu}(\widehat{r}(F(x,t)))e^{\beta n t} \geq 0\} \subset S_{\rho,t}, \quad \textit{and } \mathbb{S}_{\rho,\tau,\gamma} = \cup_{t\in[0,\tau)} S_{\rho,t,\gamma}.$$

From (5.30) *and some obvious inequalities one has, for every* $F(x,t) \in \mathbb{S}_{R,T,\gamma}$,

$$\alpha\left(\frac{\mathrm{c}_\mu}{-\mu}(R)\right)^2(1-\gamma)^2 v(F(x,t)) \le \alpha\left(\frac{\mathrm{c}_\mu}{-\mu}(R)\right)^2 \sup_{S_{R,0}} v\ e^{(2\eta^2+(n-1)\epsilon\nu)t}, \tag{5.31}$$

that is, for every $F(x,t) \in S_{R,t,\gamma}$

$$(1-\gamma)^2 v(F(x,t)) \le \sup_{S_{R,0}} v\ e^{(2\eta^2+(n-1)\epsilon\nu)T}. \tag{5.32}$$

Then, if v *is bounded on* M_0, *one has* $v(F(x,t)) \le (1-\gamma)^{-2}\left(\sup_{M_0} v\right)\ e^{(2\eta^2+(n-1)\epsilon\nu)T}$. *Since this formula is true for any* γ *between* 0 *and* 1, *we have*

$$v(F(x,t))\ \le\ \sup_{M_0} v\ \ e^{(2\eta^2+(n-1)\epsilon\nu)T}. \tag{5.33}$$

Notation 2. *For the following sections, it will be convenient to introduce the following notation: given any* $\rho' > 0$, $m \in \mathbb{N}$ *and* $0 < \gamma < 1$, *there is a* $\rho > 0$ *such that*

$$\gamma^{m+2}\frac{\mathrm{c}_\mu(\rho)}{-\mu} = \frac{\mathrm{c}_\mu(\rho')}{-\mu} \tag{5.34}$$

and we define ρ_i, $i = 1, 2, ..., m+2$ *by:*

$$\gamma^{i}\frac{\mathrm{c}_\mu(\rho)}{-\mu} = \frac{\mathrm{c}_\mu(\rho_i)}{-\mu}, \qquad \left(\text{that is } \gamma\frac{\mathrm{c}_\mu(\rho_{i-1})}{-\mu} = \frac{\mathrm{c}_\mu(\rho_i)}{-\mu}\right), \tag{5.35}$$

which gives $\rho' = \rho_{m+2} < \rho_{m+1} < ... < \rho_1 < \rho$. *It is important to remark that* $\mathbb{S}_{\rho_i,\tau,\gamma} = \mathbb{S}_{\rho_{i+1},\tau} \subset \mathbb{S}_{\rho_i,\tau}$.

6. Long time existence

First remember the evolution of $|A|^2$, which we take from [2] (erasing the terms with $\overline{H}$) because the notation used there is more similar to that of this paper.

$$\begin{aligned}\frac{\partial |A|^2}{\partial t} &= \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2\left(|A|^2 + \overline{R}ic(N,N)\right)\\ &-4\sum_{i,j}\left(\overline{R}_{AE_iE_jAE_iE_j} - \overline{R}_{AE_iE_jE_iAE_j}\right) - 2\sum_{i,j}\left(\overline{\nabla}_{AE_i}\overline{R}_{NE_jE_iE_j} + \overline{\nabla}_{E_j}\overline{R}_{NE_iAE_iE_j}\right).\end{aligned} \tag{6.1}$$

In our case, using the orthonormal local frame $N, E_1, E_2, ..., E_n$ such that $AE_i = k_i E_i$, we get

$$\begin{aligned}
\frac{\partial |A|^2}{\partial t} &= \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2|A|^2 \overline{R}ic(N,N) \\
&\quad - 4\sum_{i,j} \left(k_i^2 - k_i k_j\right) \overline{R}_{E_iE_jE_iE_j} \\
&\quad - 2\sum_{i,j} \; k_i \left(\overline{\nabla}_{E_i}\overline{R}_{NE_jE_iE_j} + \overline{\nabla}_{E_j}\overline{R}_{NE_iE_iE_j}\right) \\
&= \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2|A|^2 \overline{R}ic(N,N) \\
&\quad - 4\left(\sum_{i<j} \left(k_i^2 - k_i k_j\right) \overline{R}_{E_iE_jE_iE_j} + \sum_{i<j} \left(k_j^2 - k_j k_i\right) \overline{R}_{E_jE_iE_jE_i}\right) \\
&\quad - 2\sum_{i} k_i \left(\tilde{\delta}\overline{R}_N(E_i,E_i)\right) \\
&= \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 + 2|A|^2 \overline{R}ic(N,N) \qquad (6.2)\\
&\quad - 4\sum_{i<j} \left(k_i - k_j\right)^2 \overline{R}_{E_iE_jE_iE_j} - 2\left\langle \alpha, \tilde{\delta}\overline{R}_N\right\rangle .
\end{aligned}$$

where α is the second fundamental form of M_t and $\tilde{\delta}\overline{R}_N(X,Y) := \sum_j \left(\overline{\nabla}_X\overline{R}_{NE_jYE_j} + \overline{\nabla}_{E_j}\overline{R}_{NYXE_j}\right)$.

We cannot deduce any bound for $|A|$ directly from (6.2). Then, as in [4] and many other places (in fact, working exactly as in [2]) we shall study the evolution of $\mathfrak{g} = (\psi \circ v)|A|^2$, where

$$\psi(v) := \frac{v^2}{1-\delta v^2}, \qquad (6.3)$$

$$\delta := \begin{cases} \dfrac{1}{2(\sup_{M_0} v^2) e^{2((n-1)\nu)T}} & \text{if } M \text{ is compact} \\ \dfrac{1}{2(\sup_{M_0} v^2) e^{2(2\eta^2+(n-1)\epsilon\nu)T}} & \begin{array}{c}\text{if } M \text{ is complete non-compact} \\ \text{and } v \text{ is bounded on } M_0\end{array} \\ \dfrac{(1-\gamma)^4}{2(\sup_{S_{\rho,0}} v^2) e^{2(2\eta^2+(n-1)\epsilon\nu)T}} & \begin{array}{c}\text{if } M \text{ is complete non-compact} \\ \text{and } v \text{ is not bounded on } M_0\end{array} \end{cases}. \qquad (6.4)$$

From these definitions of ψ and δ it follows that, for $F(x,t) \in M_t$, $t \in [0,T[$, when v is bounded on M_0 and for $F(x,t) \in \mathbb{S}_{\rho,T,\gamma}$ in the other case,

$$\frac{1}{1-\delta} \leq \psi \leq \frac{1}{\delta} \; , \; \left(-\frac{2}{v} + \delta\right)\frac{1}{\psi'} - \frac{\psi''}{(\psi')^2} + \frac{3}{2\psi} < 0 \; , \; -v\frac{\psi'}{\psi^2} + 2\frac{1}{\psi} = -2\delta < 0, \qquad (6.5)$$

(6.6)

$$\delta \leq \frac{1}{2} \; , \; \frac{\sqrt{2}}{(1-\delta)\sqrt{\delta}} \leq \frac{\psi'}{\psi} \leq 4 \; , \; \frac{\sqrt{2}}{(1-\delta)\delta^{3/2}} \leq \frac{1}{\delta}\frac{\psi'}{\psi} \leq \frac{4}{\delta} \text{ and } 0 \leq \frac{v^2-1}{v}\frac{\psi'}{\psi} \leq 4(1-2\delta).$$

Lemma 8. *For every $F(x,t) \in F(M \times [0,T[)$ if v is universally bounded on M_0, and for every $F(x,t) \in \mathbb{S}_{\rho_1,T}$ in other case (with ρ and ρ_1 determined by an arbitrary $\rho' > 0$ by (5.34)* *and* (5.35)*), one has:*

$$
(6.7)\qquad
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\mathfrak{g} \le -\frac{1}{\psi}\langle\nabla\mathfrak{g},\nabla\psi\rangle - 2\delta\mathfrak{g}^2 + \mathfrak{B}\ \mathfrak{g} + \mathfrak{C}\ \sqrt{\mathfrak{g}}, \qquad \textit{where}\\
&\mathfrak{B} = \frac{1}{\delta}\frac{\psi'}{\psi}\left(\frac{|\widehat{\nabla}\varphi|}{\varphi}\right)^2 - \frac{v^2-1}{v}\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \overline{R}ic_{\widehat{1}\widehat{1}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2}\right)\ \frac{\psi'}{\psi}\\
&\qquad + 2\overline{R}ic(N,N) + 8|\overline{Scal} - \overline{R}ic(N,N)|,\\
&\mathfrak{C} = 2\sqrt{\psi}\ |\tilde{\delta}\overline{R}_N|.
\end{aligned}
$$

Proof The evolution of $\mathfrak{g}$ is given by those of ψ and $|A|^2$ according to the formula:

$$
(6.8)\qquad
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\mathfrak{g} &= |A|^2\left(\frac{\partial}{\partial t}-\Delta\right)\psi + \psi\left(\frac{\partial}{\partial t}-\Delta\right)|A|^2 - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle\\
&=|A|^2\psi'\left(\frac{\partial}{\partial t}-\Delta\right)v - |A|^2\psi''\ |\nabla v|^2 + \psi\left(\frac{\partial}{\partial t}-\Delta\right)|A|^2 - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle.
\end{aligned}
$$

For the last summand in (6.8), we use the following inequality (cf. [4], page 555, but be aware that in [4] is used $\varphi(v^2)$ instead of $\psi(v)$.)

$$
(6.9)\qquad -2\left\langle\nabla\psi,\nabla|A|^2\right\rangle \le -\frac{1}{\psi}\langle\nabla\mathfrak{g},\nabla\psi\rangle + 2\psi|\nabla|A||^2 + \frac{3}{2\psi}|A|^2|\nabla\psi|^2
$$

and Kato's inequality

$$
(6.10)\qquad |\nabla|A||^2 \le |\nabla A|^2.(\text{ equivalently }\left|\nabla|A|^2\right|^2 \le 4|A|^2|\nabla A|^2)
$$

From (6.8), (6.9) and (6.10),

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\mathfrak{g} \le\ & |A|^2\psi'\left(-\frac{2}{v}|\nabla v|^2 + \frac{2}{\varphi}\langle\nabla v,\nabla\varphi\rangle - v|A|^2\right.\\
&\qquad\left. -v\left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{R}ic_{\widehat{1}\widehat{1}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2} - \frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{1},\widehat{1})\right)\right)\\
&- |A|^2\psi''\ |\nabla v|^2\\
&+ \psi\left(-2|\nabla A|^2 + 2|A|^4 + 2|A|^2\overline{R}ic(N,N) - 4\sum_{i<j}(k_i-k_j)^2\,\overline{R}_{E_iE_jE_iE_j} - 2\left\langle\alpha,\tilde{\delta}\overline{R}_N\right\rangle\right)\\
&- \frac{1}{\psi}\langle\nabla\mathfrak{g},\nabla\psi\rangle + 2\ \psi\ |\nabla A|^2 + \frac{3}{2\psi}|A|^2|\nabla\psi|^2.
\end{aligned}
$$

Since $\psi' > 0$, using Young's inequality $xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon}y^2$ with $x = 2\frac{|\overline{\nabla}\varphi|}{\varphi}$, $y = |\nabla v|$ and $\varepsilon := \dfrac{1}{4\delta}$, we conclude that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\mathfrak{g} \leq |A|^2\psi' \Bigg(&-\frac{2}{v}|\nabla v|^2 + \frac{1}{\delta}\left(\frac{|\overline{\nabla}\varphi|}{\varphi}\right)^2 + \delta|\nabla v|^2 - v|A|^2 \\
&-v\left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{R}ic_{\widehat{\mathbf{1}}\widehat{\mathbf{1}}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2} - \frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{\mathbf{1}},\widehat{\mathbf{1}})\right)\Bigg) \\
&- |A|^2\psi''\,|\nabla v|^2 \\
&+ 2\,\psi\,|A|^4 + 2\,\psi\,|A|^2\overline{R}ic(N,N) - 4\,\psi\sum_{i<j}(k_i-k_j)^2\,\overline{R}_{E_iE_jE_iE_j} \\
&+ 2\,\psi\,|A||\tilde{\delta}\overline{R}_N| - \frac{1}{\psi}\langle\nabla\mathfrak{g},\nabla\psi\rangle + \frac{3}{2\psi}|A|^2|\nabla\psi|^2. \qquad (6.11)
\end{aligned}$$

Next, let us bound and/or rearrange the different terms in (6.11).

$$\begin{aligned}
&\left(-\frac{2}{v}+\delta\right)|A|^2\psi'|\nabla v|^2 - |A|^2\psi''|\nabla v|^2 + \frac{3}{2\psi}|A|^2|\nabla\psi|^2 \qquad (6.12)\\
&\quad = \left(\left(-\frac{2}{v}+\delta\right)\frac{1}{\psi'} - \frac{\psi''}{(\psi')^2} + \frac{3}{2\psi}\right)|A|^2|\nabla\psi|^2 \leq 0 \text{ by (6.5)}
\end{aligned}$$

$$\begin{aligned}
&|A|^2\psi'\frac{1}{\delta}\left(\frac{|\widehat{\nabla}\varphi|}{\varphi}\right)^2 - \left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{R}ic_{\widehat{\mathbf{1}}\widehat{\mathbf{1}}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2} - \frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{\mathbf{1}},\widehat{\mathbf{1}})\right) v\,|A|^2\psi' \\
&- |A|^4 v\psi' + 2\psi|A|^4 + 2\,\psi|A|^2\overline{R}ic(N,N) - 4\psi\sum_{i<j}(k_i-k_j)^2\,\overline{R}_{E_iE_jE_iE_j} + 2\psi|A|\,|\tilde{\delta}\overline{R}_N| \\
&\leq \left(\frac{\psi'}{\psi}\,\frac{1}{\delta}\left(\frac{|\widehat{\nabla}\varphi|}{\varphi}\right)^2 - \left(1-\frac{1}{v^2}\right)\left(\frac{\widehat{\Delta}\varphi}{\varphi} + \widehat{R}ic_{\widehat{\mathbf{1}}\widehat{\mathbf{1}}} + \frac{|\widehat{\nabla}\varphi|^2}{\varphi^2} - \frac{\widehat{\nabla}^2\varphi}{\varphi}(\widehat{\mathbf{1}},\widehat{\mathbf{1}})\right) v\,\frac{\psi'}{\psi}\right)\mathfrak{g} \\
&+ \left(-v\frac{\psi'}{\psi^2} + 2\frac{1}{\psi}\right)\mathfrak{g}^2 \; + 2\overline{R}ic(N,N)\mathfrak{g} + 8|\overline{Scal} - \overline{R}ic(N,N)|\mathfrak{g} + 2\sqrt{\psi}\;|\tilde{\delta}\overline{R}_N|\sqrt{\mathfrak{g}}.
\end{aligned}$$

because $-4\psi\sum_{i<j}(k_i-k_j)^2\,\overline{R}_{E_iE_jE_iE_j} \leq 4\psi 4|A|^2\sum_{i<j}\overline{R}_{E_iE_jE_iE_j} \leq 2\psi 4|A|^2|\overline{Scal}-\overline{R}ic(N,N)|$. And the formula of the theorem follows putting all this together and using the relation between $\widehat{R}$ and $\overline{R}$ computed in section 3.1.2. □

Theorem 9. *Let M be compact. If $\overline{M} = M \times_\varphi \mathbb{R}$ and M_0 is a C^∞ graph over M, then* (1.1) *has a solution whith initial condition M_0, defined on a maximal time interval $[0,\infty[$ and which is a graph for all t. More, when $\nu \leq 0$, $|A|$ is bounded on $[0,\infty[$ with a bound which does not depend on t.*

Proof From (6.5) and (6.6) we obtain the following inequalities and define the constants K and C by

$$\mathfrak{B} \leq \frac{4}{\delta}\eta^2 + 4(1-2\delta)\ \epsilon\nu + (2+8n)\sup_{\overline{M}} |\overline{R}ic| =: K \tag{6.13}$$

$$\mathfrak{C} \leq \frac{4}{\sqrt{\delta}}\ \sup_{\overline{M}} |\overline{\nabla R}| =: C. \tag{6.14}$$

From these definitions of K and C and Lemma 4 it follows

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathfrak{g} \leq -\frac{1}{\psi}\left\langle \nabla\mathfrak{g}, \nabla\psi\right\rangle - 2\delta\mathfrak{g}^2 + K\mathfrak{g} + C\sqrt{\mathfrak{g}}. \tag{6.15}$$

We remark that K and C depend on T only through δ, δ depends on $\sup v$ and, according to Theorem 6, when M is compact and $\nu \leq 0$, $\sup v$ does not increase with time.

As we remarked in section 4, when M is compact, there is a solution of (1.1) for a maximal interval $[0, T[$. Let $t_1 \in [0, T[$. Let us suppose that (x_0, t_0) is the point where $\mathfrak{g}$ attains its maximum for $t \leq t_1$ and $0 < t_0 < t_1$.

If $\mathfrak{g}_0 := \mathfrak{g}(x_0, t_0) > 1$, then it has to satisfy $0 \leq -2\ \delta(t_0)\mathfrak{g}_0^2 + (K+C)(t_0)\ \mathfrak{g}_0 \leq -2\ \delta\mathfrak{g}_0^2 + (K+C)\ \mathfrak{g}_0$, so

$$\mathfrak{g}_0 \leq \max\{\max_{M_0}\mathfrak{g}, \frac{K+C}{2\delta}, 1\}. \tag{6.16}$$

Since ψ and $\mathfrak{g}$ are bounded, $|A|^2$ is bounded on $[0, T[$ with a bound which does not depends on T when $\nu \leq 0$, because in this case neither K, C and δ depend on t.

Once we achieve the upper bound for $|A|^2$, it follows, like in [7] and [8], that $|\nabla^j A|$ is bounded for every $j \geq 1$. If $T < \infty$, these bounds imply (cf. [7] pages 257, ff.) that X_t converges (as $t \to T$, in the C^∞-topology) to a unique smooth limit X_T. Now we can apply the short time existence theorem to continue the solution after T, contradicting the maximality of $[0, T[$. □

The next is a long time existence theorem modulo the existence theorem that will be proved in section 7. For this reason, in its hypotheses appears the existence of a short time existence theorem.

Theorem 10. *Let M be complete non compact with a pole x_0. Let $\overline{M} = M \times_\varphi \mathbb{R}$. Let us suppose that for every graph $\mathcal{M}_0$ over M there is a solution of* (1.1) *with initial condition $\mathcal{M}_0$ that is a graph and defined on some time interval. Let M_t be a solution of* (1.1) *defined on a maximal time interval $[0, T[$. If M_0 is a C^∞ graph over M, then $T = \infty$.*

Proof Let η, ν and $\epsilon\nu$ be as defined by (5.13), (5.14) and (5.27). For the terms $\mathfrak{B}$ and $\mathfrak{C}$ in Lemma 8 we have the same bounds that in Theorem 9, so we define K and C again by (6.13) and (6.14) respectively. Given any $\rho' > 0$, $m \in \mathbb{N}$ and $0 < \gamma < 1$, we consider the ρ and ρ_i defined in Notation 2. Now we shall use the functions ϕ and ζ defined in Theorem 7 and compute the evolution of $\phi\mathfrak{g}$ for points in $S_{\rho_1,T}$.

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)(\phi\ \mathfrak{g}) &= \mathfrak{g}\left(\frac{\partial}{\partial t}-\Delta\right)\phi+\phi\left(\frac{\partial}{\partial t}-\Delta\right)\mathfrak{g}-2\left\langle\nabla\phi,\nabla\mathfrak{g}\right\rangle\\ &\leq -2\alpha\mathfrak{g}\frac{|\nabla\phi|^2}{\phi'^2}+\phi\left(-\frac{1}{\psi}\left\langle\nabla\mathfrak{g}.\nabla\psi\right\rangle-2\delta\mathfrak{g}^2+K\mathfrak{g}+C\sqrt{\mathfrak{g}}\right)-2\left\langle\nabla\phi,\nabla\mathfrak{g}\right\rangle\\ &= -2\alpha\mathfrak{g}\frac{|\nabla\phi|^2}{\phi'^2}-\left\langle\nabla(\phi\mathfrak{g})-\mathfrak{g}\nabla\phi,\frac{\nabla\psi}{\psi}\right\rangle-2\frac{\delta}{\phi}(\phi\mathfrak{g})^2+K\phi\mathfrak{g}\\ &\quad +C\sqrt{\phi}\sqrt{\phi\mathfrak{g}}-2\left\langle\frac{\nabla\phi}{\phi},\nabla(\phi\mathfrak{g})-\mathfrak{g}\nabla\phi\right\rangle \end{aligned}\tag{6.17}$$

but $\phi'^2=4\alpha\phi$, then $-2\alpha\mathfrak{g}\frac{|\nabla\phi|^2}{\phi'^2}+2\mathfrak{g}\frac{|\nabla\phi|^2}{\phi}=6\alpha\frac{|\nabla\phi|^2}{\phi'^2}\mathfrak{g}=6\alpha|\nabla\zeta|^2\mathfrak{g}=6\alpha\mathfrak{g}{\rm s}_\mu{}^2|\partial_{\widehat{r}}^\top|^2e^{2\beta nt}$. By substitution in the evolution of $\phi\mathfrak{g}$, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)(\phi\ \mathfrak{g}) &\leq 6\alpha\mathfrak{g}{\rm s}_\mu{}^2e^{2\beta nt}-\left\langle\nabla(\phi\mathfrak{g})-\mathfrak{g}\nabla\phi,\frac{\nabla\psi}{\psi}\right\rangle-2\delta\ \phi\ \mathfrak{g}^2+K\phi\mathfrak{g}\\ &\quad +C\sqrt{\phi}\sqrt{\phi\mathfrak{g}}-2\left\langle\frac{\nabla\phi}{\phi},\nabla(\phi\mathfrak{g})\right\rangle\\ &= 6\alpha\mathfrak{g}{\rm s}_\mu{}^2e^{2\beta nt}-\left\langle\nabla(\phi\mathfrak{g}),\frac{\nabla\psi}{\psi}+2\frac{\nabla\phi}{\phi}\right\rangle+\left\langle\mathfrak{g}\nabla\phi,,\frac{\nabla\psi}{\psi}\right\rangle\\ &\quad -2\delta\ \phi\ \mathfrak{g}^2+K\phi\mathfrak{g}+C\sqrt{\phi}\sqrt{\phi\mathfrak{g}}.\end{aligned}\tag{6.18}$$

Using $\nabla\psi=\psi'\nabla v$ and the expression (3.17) of ∇v,

$$\frac{\nabla\psi}{\psi}=\frac{1-\delta^2v^4}{v^2}\nabla v=(1-\delta^2v^4)\left(\left\langle\frac{\widehat{\nabla}\varphi}{\varphi},N\right\rangle\overline{e}_0^\top+A\overline{e}_0^\top\right),\ \text{then}\tag{6.19}$$

$$\left|\frac{\nabla\psi}{\psi}\right|\leq\eta+|A|=\eta+\frac{\sqrt{\mathfrak{g}}}{\sqrt{\psi}}.\tag{6.20}$$

On the other hand

$${\rm s}_\mu{}^2e^{2\beta nt}\leq\left({\rm c}_\mu e^{\beta nt}\right)^2\leq{\rm c}_\mu(\rho_1)^2\text{ on }\mathbb{S}_{\rho_1,T}.\tag{6.21}$$

Pluging these inequalitues in (6.18),

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)(\phi\ \mathfrak{g}) &\leq 6\alpha{\rm c}_\mu(\rho_1)^2\mathfrak{g}-\left\langle\nabla(\phi\mathfrak{g}),\frac{\nabla\psi}{\psi}+2\frac{\nabla\phi}{\phi}\right\rangle+|\nabla\phi|\left(\eta+\frac{\sqrt{\mathfrak{g}}}{\sqrt{\psi}}\right)\mathfrak{g}\\ &\quad -2\delta\ \phi\ \mathfrak{g}^2+K\phi\mathfrak{g}+C\sqrt{\phi}\sqrt{\phi\mathfrak{g}}.\\ &= -\left\langle\nabla(\phi\mathfrak{g}),\frac{\nabla\psi}{\psi}+2\frac{\nabla\phi}{\phi}\right\rangle-2\delta\ \phi\ \mathfrak{g}^2\\ &\quad +\frac{\nabla\phi}{\sqrt{\psi}}\mathfrak{g}\sqrt{\mathfrak{g}}+\left(6\alpha{\rm c}_\mu(\rho_1)^2+\eta|\nabla\phi|+K\phi\right)\mathfrak{g}+C\sqrt{\phi}\sqrt{\phi\mathfrak{g}}\end{aligned}\tag{6.22}$$

At a point where a maximum for $\phi\mathfrak{g}$ is attained it follows from the above inequality that, if this happens when $t \neq 0$,

$$2\delta\ \phi\ \mathfrak{g}^2 \leq \frac{\nabla\phi}{\sqrt{\psi}}\mathfrak{g}\sqrt{\mathfrak{g}} + \left(6\alpha c_\mu(\rho_1)^2 + \eta|\nabla\phi| + K\phi\right)\mathfrak{g} + C\sqrt{\phi}\sqrt{\phi\mathfrak{g}},$$

and, multiplying by $\dfrac{\sqrt{\phi}}{\sqrt{\mathfrak{g}}}$,

$$2\delta\left(\sqrt{\phi\mathfrak{g}}\right)^3 \leq \frac{\nabla\phi}{\sqrt{\psi}\sqrt{\phi}}\phi\mathfrak{g} + \left(6\alpha c_\mu(\rho_1)^2 + \eta|\nabla\phi| + K\phi\right)\sqrt{\phi\mathfrak{g}} + C\phi\sqrt{\phi}. \tag{6.23}$$

having into account that

$$\nabla\phi = -2\sqrt{\alpha\phi}\ s_\mu(\widehat{r})\partial_{\widehat{r}}^\top, \quad \phi \leq \alpha\frac{c_\mu(\rho_1)^2}{\mu^2}, \qquad \frac{1}{\sqrt{\psi}} \leq \sqrt{1-\delta}, \tag{6.24}$$

we have

$$\begin{aligned}2\delta\left(\sqrt{\phi\ \mathfrak{g}}\right)^3 &\leq 2\sqrt{\alpha}s_\mu(\rho_1)\sqrt{1-\delta}\left(\sqrt{\phi\ \mathfrak{g}}\right)^2 \\ &\quad + \left(\left(6+\frac{K}{\mu^2}\right)\alpha\ c_\mu(\rho_1)^2 + \eta 2\alpha\frac{c_\mu(\rho_1)}{-\mu}s_\mu(\rho_1)\right)\sqrt{\phi\ \mathfrak{g}} + C\alpha^{3/2}\frac{c_\mu(\rho_1)^3}{-\mu^3}.\end{aligned}$$

Having into account that $s_\mu(\rho_1) \leq c_\mu(\rho_1)$ and dividing by $c_\mu(\rho_1)^3$, we get

(6.25)
$$2\delta\left(\frac{\sqrt{\phi\ \mathfrak{g}}}{c_\mu(\rho_1)}\right)^3 - 2\sqrt{\alpha}\sqrt{1-\delta}\left(\frac{\sqrt{\phi\ \mathfrak{g}}}{c_\mu(\rho_1)}\right)^2 - \left(\left(6+\frac{K}{\mu^2}\right)\alpha\ + \eta 2\alpha\frac{1}{-\mu}\right)\frac{\sqrt{\phi\ \mathfrak{g}}}{c_\mu(\rho_1)} - C\alpha^{3/2}\frac{1}{-\mu^3} \leq 0.$$

As the coefficient of $\left(\dfrac{\sqrt{\phi\ \mathfrak{g}}}{c_\mu(\rho_1)}\right)$ is positive, the above inequality implies that

$$\left(\frac{\sqrt{\phi\ \mathfrak{g}}}{c_\mu(\rho_1)}\right) \leq D = \max\{\mathcal{D}, \max_{S_{\rho_1,0}}\sqrt{\phi\mathfrak{g}}\}, \tag{6.26}$$

where $\mathcal{D}$ is the biggest solution of the third order polynomial equation given by (6.25) when we change the inequality by an equality. $\mathcal{D}$ depends on the coefficients of the equation, then it only depends on the geometry of $\overline{M}$ and δ. From the above inequality we have that, every $F(x,t) \in \mathbb{S}_{\rho_1,T,\gamma}$ one has

$$\left(\frac{\sqrt{\alpha}(1-\gamma)}{-\mu}\right)\sqrt{\mathfrak{g}} \leq D,$$

then, remembering that $\sqrt{\mathfrak{g}} = \sqrt{\psi}\ |A|$ and the last inequality (6.24),

$$|A| \leq \frac{-\mu\sqrt{(1-\delta)}D}{\sqrt{\alpha}(1-\gamma)}, \tag{6.27}$$

This shows that, on $\mathbb{S}_{\rho_2,T} = \mathbb{S}_{\rho_1,T,\gamma}$, $|A|$ is bounded by a bound $C(0,\rho',\gamma,T)$ depending T, γ and ρ' (only through δ and ρ) and on M_0.

From here, a variant of the procedure used by Ecker, Huisken and Unterberger works. Let us give some details of these computations. Let us suppose, by induction, that, on $\mathbb{S}_{\rho_{2+k},T}$,

$$|\nabla^k A| \leq C(k,\rho',\gamma,T) \quad \text{for} \quad k = 0,...,m-1 \tag{6.28}$$

where $C(k,\rho',\gamma,T)$ depends on ρ', γ and k (through ρ_{2+k}), the bounds of $|\nabla^i A|$ for $0 \le i \le k-1$ and the geometry of $\overline{M}$ (the bounds on $|\overline{\nabla}^j\overline{R}|$, $j=0,...,k$).

Following the same procedure of [7] and [8], we can find a constant D_1 depending on m, n, the geometry of $\overline{M}$ (that is, the bounds on $|\overline{\nabla}^k\overline{R}|$, $k=0,...,m$) and the bounds of $|\nabla^k A|$, $k=0,...,m-1$ such that

$$\frac{\partial}{\partial t}|\nabla^m A|^2 \le \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + D_1(|\nabla^m A|^2+1) \tag{6.29}$$

Next, we define

$$f := |\nabla^m A|^2 + \xi|\nabla^{m-1}A|^2, \tag{6.30}$$

being ξ a constant to be specified later. For the time derivative of f, (6.29) yields

$$\frac{\partial f}{\partial t} \le \Delta|\nabla^m A|^2 + D_1\left(|\nabla^m A|^2+1\right) + \xi\frac{\partial}{\partial t}|\nabla^{m-1}A|^2 \tag{6.31}$$

Observe that the last addend on the right hand side of (6.31) can be estimated using again (6.29). In fact,

$$\frac{\partial|\nabla^{m-1}A|^2}{\partial t} = \Delta|\nabla^{m-1}A|^2 - 2|\nabla^m A|^2 + D_2,$$

being D_2 a constant depending on $C(m-1,\rho',\gamma,T)$.

Substituting this in (6.31), we get

$$\frac{\partial f}{\partial t} \le \Delta f + (D_1 - 2\xi)|\nabla^m A|^2 + D_1 + \xi D_2.$$

If we choose $\xi \ge D_1$ and $D_3 := D_1 + \xi D_2$, then

$$\begin{aligned}\frac{\partial f}{\partial t} &\le \Delta f - \xi|\nabla^m A|^2 + D_3 \underset{(6.30)}{=} \Delta f - \xi f + \xi^2|\nabla^{m-1}A|^2 + D_3\\ &\le \Delta f - \xi f + \xi^2 C_{m-1} + D_3.\end{aligned}$$

Considering the same function ϕ as before, the same computational rules and the evolution of ϕ, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\phi\ f) \le -2\alpha f\frac{|\nabla\phi|^2}{\phi'^2} - \xi\ \phi f + (\xi^2 C_{m-1} + D_3)\phi - 2\left\langle\frac{\nabla\phi}{\phi}, \nabla(\phi f) - f\nabla\phi\right\rangle \tag{6.32}$$

From (6.32), the same computations used to obtain (6.22) from (6.17) give now

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\phi\ f) \le 6\alpha \mathrm{c}_\mu(\rho_{m+1})^2 f - \left\langle\nabla(\phi f), 2\frac{\nabla\phi}{\phi}\right\rangle - \xi\ \phi f + (\xi^2 C_{m-1} + D_3)\phi \tag{6.33}$$

At a point of $\mathbb{S}_{\rho_{m+1},T}$ where a maximum for ϕf is attained it follows from the above inequality that

$$(\xi\ \phi - 6\alpha\mathrm{c}_\mu(\rho_{m+1})^2)\phi f \le (\xi^2 C_{m-1} + D_3)\phi^2 \le (\xi^2 C_{m-1} + D_3)\frac{\alpha^2}{\mu^4}\mathrm{c}_\mu(\rho_{m+1})^4, \tag{6.34}$$

then, at any point in $\mathbb{S}_{\rho_{2+m},T} = \mathbb{S}_{\rho_{m+1},T,\gamma}$, since, on these points, $\phi \geq \frac{\alpha}{\mu^2}\mathrm{c}_\mu(\rho_{m+1})^2(1-\gamma)^2$,

$$(\xi \frac{\alpha}{\mu^2}\mathrm{c}_\mu(\rho_{m+1})^2(1-\gamma)^2 - 6\alpha\mathrm{c}_\mu(\rho_{m+1})^2)\phi f \leq (\xi^2 C_{m-1} + D_3)\frac{\alpha^2}{\mu^4}\mathrm{c}_\mu(\rho_{m+1})^4,$$

If we take $\xi = \max\{D_1, \frac{7\mu^2}{(1-\gamma)^2}\}$, we have, for points in $\mathbb{S}_{\rho_1,T,\gamma}$,

$$\alpha\mathrm{c}_\mu(\rho_{m+1})^2\frac{\alpha}{\mu^2}\mathrm{c}_\mu(\rho_{m+1})^2(1-\gamma)^2 f \leq (\xi^2 C_{m-1} + D_3)\frac{\alpha^2}{\mu^4}\mathrm{c}_\mu(\rho_{m+1})^4, \tag{6.35}$$

that is

$$f \leq (1-\gamma)^{-2}(\xi^2 C_{m-1} + D_3)\frac{1}{\mu^2} =: C(m,\rho,\gamma,T), \tag{6.36}$$

Finally, by (6.30), $|\nabla^m A|^2 \leq f \leq C(m,\rho,\gamma,T)$ on $\mathbb{S}_{\rho_{2+m},T} = \mathbb{S}_{\rho',T}$.

Now the final argument cannot be exactly like in the compact case because there, to bound $\mathrm{dist}(F(p,t),x_0)$, one uses integration along t, but here it could happen that $F(p,t) \in S_{\rho_{2+m},T}$ but $F(p,s) \notin S_{\rho_{2+m},T}$ for $s < t$, then we do not know anything about the bounds of $|A|$ in $F(p,t)$. To avoid this problem, we have to use the parametrization $\overline{F}(\cdot,t)$ of M_t and the equation (4.1) or its equivalent (4.5). From the bounds on v obtained in the section 5 and (3.13) we get that the gradient of u is bounded on $S_{\rho_1,T}$. From this, formula (4.3) and the above bounds on $|\nabla^m A|$ we get that the higher order derivatives of u are bounded. And also from the bounds on v and $|A|$ and formula (4.2) it follows that $|u(t)| \leq |u(0)| + \int_0^T \frac{v}{\varphi}n|A|$ is bounded on $S_{\rho_2,T}$. Once we have these bounds, we have, on each compact $\left\{x \in M_t; \frac{\mathrm{c}_\mu}{-\mu}(\rho') - \frac{\mathrm{c}_\mu}{-\mu}(\widehat{r}(x))e^{\beta n t} \geq 0\right\}$ a well defined limit of M_t when $t \to T$, which allows to continue the flow after T. Then $T = \infty$. □

7. Existence of solution when M is non-compact

When M is complete non-compact and posses a pole x_0, the existence of solution follows from the parabolic theory using the estimates of section 5 and 6 in the following way.

Given $\rho_0 > 0$, let us define $\rho = \rho(\rho_0)$ by the expression

$$\gamma^{m+2}\frac{\mathrm{c}_\mu(\rho)}{-\mu}e^{-\beta n T} = \frac{\mathrm{c}_\mu(\rho_{2+m})}{-\mu}e^{-\beta n T} = \frac{\mathrm{c}_\mu(\rho_0)}{-\mu}. \tag{7.1}$$

For any $\rho'' \geq \rho$, by the theory of parabolic equations, there is a unique solution $u_{\rho''}$ of (4.5) on $\widehat{B}(x_0,\rho'') \times [0,T]$ satisfying the conditions:

$$u_{\rho''}(\cdot,0) = u_0|_{\widehat{B}(x_0,\rho'')} \quad \text{and} \quad u_{\rho''}(x,t) = u_0(x) \text{ for } (x,t) \in \partial\widehat{B}(x_0,\rho'') \times [0,T]. \tag{7.2}$$

From (7.1) one has that if $(x,t) \in \widehat{B}(x_0,\rho_0) \times [0,T]$, then $(x,u_{\rho''}(x,t)) \in S_{\rho_{2+m},T}$ and, from the estimates (5.32), (6.27) and (6.36) we get that there are constants $C(m,\rho_0,\gamma,T)$ such that $|\widehat{\nabla}^m u_{\rho''}| \leq C(m,\rho_0,\gamma,T)$. From these bounds one gets also an estimate for $|u''_\rho|$ in the same way that was done at the end of the proof of Theorem 10.

Now, observe that, thanks to (4.2) and (6.28), $\left|\frac{\partial u}{\partial t}\right|$ is bounded by a constant depending only on m, ρ_0 γ and T. From (4.2), $\left|\frac{\partial^2 u}{\partial t^2}\right|$ is bounded if v, $|H|$, $\left|\frac{\partial v}{\partial t}\right|$ and $\left|\frac{\partial H}{\partial t}\right|$ are bounded. We proved before that the first two quantities are bounded. The bounds on the other two follow from (4.6), (4.7), Proposition 4, formula (3.17) and the fact (proven before) that v, $|A|$, $|\nabla A|$, $|\nabla^2 A|$, $|\widehat{\nabla} u|$ and $|\widehat{\nabla}^2 u|$ are bounded.

As a consequence, for every $\rho_0 > 0$, the family of C^∞ functions $\{u_{\rho''}\}_{\rho''\geq\rho}$ converges to a smooth function $\widehat{u}_{\rho_0}$ on $\widehat{B}(x_0,\rho_0)$ which is at least C^1 on t and a solution of (4.5) on $\widehat{B}(x_0,\rho_0)\times[0,T]$ (then, by parabolic theory, it is also C^∞ on t). Given a sequence $\rho_0^1 < \rho_0^2 < \cdots \to \infty$, for $j > i$ the families $\{u_{\rho''}\}_{\rho''\geq\rho(\rho_0^i)}$ and $\{u_{\rho''}\}_{\rho''\geq\rho(\rho_0^j)}$, coincide for $\rho'' \geq \rho(\rho_0^j)$, then their limits $\widehat{u}_{\rho^i}$ satisfy the property that $\widehat{u}_{\rho^j}|_{\widehat{B}(x_0,\rho^i)} = \widehat{u}_{\rho^i}$, then they define a smooth function $\widehat{u}$ on M which is the C^∞ limit on the compacts of the family $u_{\rho''}$ when $\rho''\to\infty$, and it is a solution of (4.5). Then, joining these arguments with those of the proof of Theorem 10 we have

Theorem 11. *Let M be complete non compact with a pole x_0. If $\overline{M} = M\times_\varphi\mathbb{R}$ and M_0 is a C^∞ graph over M, then there is a solution M_t of* (1.1) *with initial condition M_0 which is a graph over M and is defined on* $[0,\infty[$.

8. Existence of solution for a Lipschitz initial condition

The existence of solution of (1.1) when M_0 is a graph over M given by a Lipschitz continuous function follows by approximating M_0 by a sequence of smooth graphs M_n and applying the existence theorems 9 and 11 to these approximations. But, in order to show that the solutions of (1.1) with initial conditions M_n converge to a smooth solution, we need to get bounds of $|\nabla^m A|$ that do not depend on the bounds on the initial condition, because these could go to ∞ as $n\to\infty$ because the limit M_0 of M_n is only Lipschitz. In this section we shall obtain these estimates.

Lemma 12. *Given a smooth solution of* (1.1) *defined on* $[0,T[$, *for every* $m = 0,1,2,...$ *there is a constant α_m such that*

$$|\nabla^m A|^2 \leq \alpha_m\left(\frac{t+1}{t}\right)^{m+1} \tag{8.1}$$

on M_t if M is compact or in $\mathbb{S}_{\rho_{2+m},T}$ if M is non-compact. Moreover, the constant α_m depends on m, the geometry of $\overline{M}$ and T in the first case, and also on ρ and γ in the second.

Proof First, let us consider the case M compact. For obtaining the bound when $m=0$ we start using the inequality (6.15) to obtain the following inequation for the evolution of $t\mathfrak{g}$

$$\left(\frac{\partial}{\partial t}-\Delta\right)(t\mathfrak{g}) \leq -\frac{1}{\psi}\left\langle\nabla(t\mathfrak{g}),\nabla\psi\right\rangle - 2\ \delta\ \mathfrak{g}^2\ t + K\ \mathfrak{g}\ t + C\ \sqrt{\mathfrak{g}}\ t + \mathfrak{g}. \tag{8.2}$$

Given $t\in]0,T[$, let t_0 be the time when $t\mathfrak{g}$ attains it maximum value $t_0\mathfrak{g}_0$ in $[0,t]\times M$ (then $\mathfrak{g}_0 = \max_{x\in M_{t_0}}\mathfrak{g}(x)$). By the maximum principle, from (8.2) we get

$$2\ \delta\ \mathfrak{g}_0^2\ t_0 \leq (1+K\ t_0)\mathfrak{g}_0 + C\ \sqrt{\mathfrak{g}_0}\ t_0. \tag{8.3}$$

If $\mathfrak{g}_0 \leq 1$, then, by definition of maximum, $t\,\mathfrak{g} \leq t_0\mathfrak{g}_0 \leq t_0 \leq t$, then $\mathfrak{g} \leq 1$ on M_t, and, by the definition of $\mathfrak{g}$ and (6.5), $|A|^2 = \dfrac{1}{\psi}\mathfrak{g} \leq (1-\delta)\mathfrak{g} \leq 1-\delta \leq \dfrac{1+t}{t}$.

If $\mathfrak{g}_0 \geq 1$, then $t_0\sqrt{\mathfrak{g}_0} \leq t_0\mathfrak{g}_0$ and it follow from (8.3) and the definition of maximum that

$$2\delta\mathfrak{g}t \leq 2\delta\mathfrak{g}_0 t_0 \leq (1+(K+C)t_0) \leq (1+(K+C)t), \tag{8.4}$$

then, using again (6.5),

$$|A|^2 = \frac{1}{\psi}\mathfrak{g} \leq (1-\delta)\frac{1+(K+C)t}{2\delta t} \leq \alpha_0\frac{1+t}{t}, \tag{8.5}$$

with $\alpha_0 = (1-\delta)(\max\{K+C,1\})/(2\delta)$.

Now, let us suppose that (8.1) holds for values of m between 0 and $m-1$. Let us show that it is true also for m. We start with the well known formulae (cf. [8]):

$$\begin{aligned}(8.6)\quad &\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^m A|^2 \leq -2|\nabla^{m+1}A|^2 \\ &+ D_m\left(\sum_{i+j+k=m}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| + \sum_{i+j=m}|\nabla^i A||\nabla^j \overline{R}||\nabla^m A| + |\nabla^m\overline{\nabla R}||\nabla^m A|\right),\end{aligned}$$

where D is a constant which depends only on m and n. Also, by repetitive use of the Gauss formula for a submanifold, for the restriction of any tensor field $\overline{B}$ on $\overline{M}$ to M_t one has

$$\nabla^k\overline{B} = \overline{\nabla}^k\overline{B} + \sum_{j=1}^{k}\sum_{i_0+i_1+\ldots+i_j=k-j}\overline{\nabla}^{i_0}\overline{B} * \nabla^{i_1}A * \cdots * \nabla^{i_j}A, \tag{8.7}$$

where " $*$ " has the same meaning that in [8].

If, for every k, $|\overline{\nabla}^k\overline{B}|$ is bounded in $\overline{M}$ by some constant $\bar{b}_k$, from (8.7) and the induction hypothesis we obtain, renaming the constants each time we need,

$$\begin{aligned}&|\nabla^k\overline{B}| \leq \bar{b}_k + c(n,k)\sum_{j=1}^{k}\sum_{i_1+\ldots+i_j\leq k-j}\bar{b}_{k-j-(i_1+\ldots+i_j)}\left(\alpha_{i_1}\ldots\alpha_{i_j}\right)^{1/2}\left(\frac{1+t}{t}\right)^{(i_1+\cdots+i_j+j)/2} \\ (8.8)\quad &\leq \bar{b}_k + \bar{c}(n,k,\overline{M},T)\left(\frac{1+t}{t}\right)^{k/2} \leq b_k\left(\frac{1+t}{t}\right)^{k/2},\end{aligned}$$

where $b_k = \max\{\bar{b}_k, \bar{c}(n,k,\overline{M},T)\}$.

Like in [3], we consider the function $f_m = t^{m+1}|\nabla^m A|^2 + \xi t^m|\nabla^{m-1}A|^2$, where ξ is some constant that will be defined later, and estimate its evolution with time under (1.1). Using

(8.6), we get

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) f_m \leq\ & (m+1) t^m |\nabla^m A|^2 + \\
& + t^{m+1} \left(-2|\nabla^{m+1} A|^2 + D_m \left(\sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| \right.\right. \\
& \left.\left. + \sum_{i+j=m} |\nabla^i A||\nabla^j \overline{R}||\nabla^m A| + |\nabla^m \overline{\nabla R}||\nabla^m A| \right)\right) + m\, t^{m-1} \xi |\nabla^{m-1} A|^2 \\
& + t^m \xi \left(-2|\nabla^m A|^2 + D_{m-1} \left(\sum_{i+j+k=m-1} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^{m-1} A| \right.\right. \\
& \left.\left. + \sum_{i+j=m-1} |\nabla^i A||\nabla^j \overline{R}||\nabla^{m-1} A| + |\nabla^{m-1} \overline{\nabla R}||\nabla^{m-1} A| \right)\right) \qquad (8.9)
\end{aligned}
$$

From the induction hypothesis, and because $0 \leq (t^{k/2}|\nabla^k A| - t^{m/2}|\nabla^m A|)^2 = t^k|\nabla^k A|^2 + t^m|\nabla^m A|^2 - 2t^{k/2}|\nabla^k A| t^{m/2}|\nabla^m A|$,

$$
\begin{aligned}
& t^{m+1} \sum_{i+j+k=m} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| \\
& \leq t^{m+1} \sum_{i+j+k=m} (\alpha_i \alpha_j)^{1/2} \frac{(1+t)^{\frac{i+j}{2}+1}}{t^{\frac{i+j}{2}+1}} |\nabla^k A||\nabla^m A| \\
& = c \sum_{k=0}^{m} (1+t)^{\frac{m-k}{2}+1} t^{k/2}\ |\nabla^k A| t^{m/2} |\nabla^m A| \\
& \leq (c/2)(1+t)^{m+1} \sum_{k=0}^{m} \left(\frac{t^k}{(1+t)^k} |\nabla^k A|^2 + \frac{t^m}{(1+t)^m} |\nabla^m A|^2 \right) \\
& = (c/2)(1+t)^{m+1} \left(\sum_{k=0}^{m} \frac{t^k}{(1+t)^k} |\nabla^k A|^2 + (m+1) \frac{t^m}{(1+t)^m} |\nabla^m A|^2 \right) \\
& \leq (m+2)(c/2)(1+t)^{m+1} \sum_{k=0}^{m} \frac{t^k}{(1+t)^k} |\nabla^k A|^2 \\
& \leq c_m (1+t)^{m+1} \sum_{k=0}^{m} \frac{t^k}{(1+t)^k} |\nabla^k A|^2 \qquad (8.10)
\end{aligned}
$$

For the second summand in (8.9), denoting by r_j the analog of b_j in (8.8) when $\overline{B}$ is $\overline{R}$, using (8.8) and the induction hypothesis

$$t^{m+1}\sum_{i+j=m}|\nabla^i A||\nabla^j\overline{R}||\nabla^m A| \le t^{m+1}\sum_{i+j=m}\alpha_i^{1/2}\frac{(1+t)^{(i+j+1)/2}}{t^{(i+j+1)/2}}r_j|\nabla^m A|$$

$$\le (1+t)^{m+1}\left(\sum_{i+j=m}\alpha_i^{1/2}r_j\right)\left(1+\frac{t^{m+1}}{(1+t)^{m+1}}|\nabla^m A|^2\right)$$

$$= (1+t)^{m+1}d_m + d_m t^{m+1}|\nabla^m A|^2. \tag{8.11}$$

Analogously, denoting by $\tilde{r}_j$ the analog of b_j in (8.8) when $\overline{B}$ is $\overline{\nabla R}$,

$$t^{m+1}|\nabla^m\overline{\nabla R}||\nabla^m A| \le (1+t)^{m/2}\ t^{(m/2)+1}\ \tilde{r}_m|\nabla^m A| = (1+t)^m\ t\ \tilde{r}_m\ \frac{t^{m/2}}{(1+t)^{m/2}}|\nabla^m A|$$

$$\le (1+t)^m\ t\ \tilde{r}_m\left(1+\frac{t^m}{(1+t)^m}|\nabla^m A|^2\right) \tag{8.12}$$

and

$$t^m\sum_{i+j+k=m-1}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^{m-1}A| \le c_{m-1}(1+t)^m\sum_{k=0}^{m-1}\frac{t^k}{(1+t)^k}|\nabla^k A|^2, \tag{8.13}$$

$$t^m\sum_{i+j=m-1}|\nabla^i A||\nabla^j\overline{R}||\nabla^{m-1}A| \le (1+t)^m d_{m-1} + d_{m-1}(1+t)^m\frac{t^m}{(1+t)^m}|\nabla^{m-1}A|^2, \tag{8.14}$$

$$t^m|\nabla^{m-1}\overline{\nabla R}||\nabla^{m-1}A| \le (1+t)^{m-1}\ t\ \tilde{r}_{m-1}\ \left(1+\frac{t^{m-1}}{(1+t)^{m-1}}|\nabla^{m-1}A|^2\right). \tag{8.15}$$

By substitution of all these inequalitues in (8.9) we get

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)f_m \le\ & (m+1)t^m|\nabla^m A|^2 + D_m c_m(1+t)^{m+1}\sum_{k=0}^{m}\frac{t^k}{(1+t)^k}|\nabla^k A|^2\\ &+ D_m(1+t)^{m+1}d_m + D_m d_m t^{m+1}|\nabla^m A|^2\\ &+ D_m(1+t)^m\ t\ \tilde{r}_m\left(1+\frac{t^m}{(1+t)^m}|\nabla^m A|^2\right)\\ &+ \xi\ m\ t^{m-1}|\nabla^{m-1}A|^2\\ &- 2t^m\xi|\nabla^m A|^2 + D_{m-1}\xi c_{m-1}(1+t)^m\sum_{k=0}^{m-1}\frac{t^k}{(1+t)^k}|\nabla^k A|^2\\ &+ \xi D_{m-1}(1+t)^m d_{m-1} + \xi D_{m-1}d_{m-1}(1+t)^m\frac{t^m}{(1+t)^m}|\nabla^{m-1}A|^2\\ &+ \xi D_{m-1}(1+t)^{m-1}\ t\ \tilde{r}_{m-1}\left(1+\frac{t^{m-1}}{(1+t)^{m-1}}|\nabla^{m-1}A|^2\right)\end{aligned} \tag{8.16}$$

Using again the induction hypothesis, grouping terms and renaming constants,

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right) f_m \leq &\left((m+1)\frac{1}{t}+D_m c_m \frac{1+t}{t}+D_m d_m+D_m\ \tilde{r}_m-\ \frac{2}{t}\xi\right) t^{m+1}\ |\nabla^m A|^2\\ &+D_m c_m \frac{(1+t)^{m+2}}{t}\sum_{k=0}^{m-1}\alpha_k+D_m(1+t)^{m+1} d_m+D_m(1+t)^m\ t\ \tilde{r}_m\\ &+\xi\ m\ \frac{(1+t)^m}{t}\alpha_{m-1}+D_{m-1}\xi c_{m-1}\frac{(1+t)^{m+1}}{t}\sum_{k=0}^{m-1}\alpha_k\\ &+\xi D_{m-1}(1+t)^m\ d_{m-1}+\xi\ D_{m-1} d_{m-1}(1+t)^m\ \alpha_{m-1}\\ &+\xi D_{m-1}(1+t)^{m-1}\ t\ \tilde{r}_{m-1}+\xi D_{m-1}(1+t)^m\ \tilde{r}_{m-1}\alpha_{m-1}\end{aligned} \tag{8.17}$$

Let us rename $D_m c_m = C_m$, $D_m(d_m+\tilde{r}_m) = E_m$, and let us define ξ by

$$2\xi = m+1+C_m(1+2T)+E_m 2T \tag{8.18}$$

With this choosing of ξ, the coefficient of $t^{m+1}|\nabla^m A|^2$ in (8.17) is bounded from above by

$$\left(C_m\frac{t-2T}{t}+E_m\left(1-2\frac{T}{t}\right)\right) = F_m\frac{t-2T}{t} < 0. \tag{8.19}$$

From (8.6) to (8.19) the computations are valid for compact and non-compact cases. When M is compact, let $t \in]0,T[$. Let $t_0 \in [0,t]$ where f_m attains its maximum value. At the point (p,t_0) where this happens, it follows from (8.17), (8.18), (8.19) and the maximum principle, renaming again all the constants, that

$$\begin{aligned}t_0^{m+1}|\nabla^m A|^2 \leq &\left(\mathcal{A}\frac{(1+t_0)}{2T-t_0}+\mathcal{B}\frac{t_0}{2T-t_0}+\mathcal{C}\frac{t_0^2}{(1+t_0)(2T-t_0)}\right.\\ &+\mathcal{D}(2T-t_0)+\mathcal{E}\frac{t_0}{(1+t_0)(2T-t_0)}+\mathcal{F}\frac{1}{(1+t_0)(2T-t_0)}\\ &\left.+\mathcal{G}\frac{t_0^2}{(1+t_0)^2(2T-t_0)}\right)(1+t_0)^{m+1}\end{aligned} \tag{8.20}$$

Then, by the definition of maximum and because $t_0 \leq t < T$

$$\begin{aligned}t^{m+1}|\nabla^m A|^2 &\leq f_m(t_0) = t_0^{m+1}|\nabla^m A|^2+\xi t_0^m|\nabla^{m-1}A|^2\\ &\leq \left(C_1 T+C_2+C_3\frac{1}{T}\right)(1+t)^{m+1}+K(1+t)^{m+1}\\ &\leq \mathcal{K}(1+t)^{m+1}.\end{aligned} \tag{8.21}$$

which finishes the proof by induction of the formula (8.1) when M is compact.

If M is non-compact, we start using the inequality (6.22) to obtain the following inequation for the evolution of $t\phi\mathfrak{g}$

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)(t\phi\mathfrak{g}) \leq &-\left\langle\nabla(t\phi\mathfrak{g}),\frac{\nabla\psi}{\psi}+2\frac{\nabla\phi}{\phi}\right\rangle-2\delta\ t\ \phi\ \mathfrak{g}^2+\frac{\nabla\phi}{\sqrt{\psi}}\ t\ \mathfrak{g}\sqrt{\mathfrak{g}}\\ &+\left(6\alpha c_\mu(R)^2+\eta|\nabla\phi|+K\phi\right) t\ \mathfrak{g}+C\ t\ \sqrt{\phi}\sqrt{\phi\mathfrak{g}}+\phi\ \mathfrak{g}.\end{aligned} \tag{8.22}$$

Given $t \in]0,T[$, let $t_0 \in [0,t]$ be the time when $t\phi\mathfrak{g}$ attains it maximum value $t_0\phi\mathfrak{g}_0$ in $S_{\rho_1,t}$ (then $\phi\mathfrak{g}_0 = \max_{x\in S_{\rho_1,t_0}} \phi\mathfrak{g}(x)$ and $t_0 \neq 0$). By the maximum principle, from (8.22) and, multiplying by $\frac{\sqrt{t_0\phi}}{\sqrt{\mathfrak{g}_0}}$, we get

$$2\delta\left(\sqrt{t_0\phi\mathfrak{g}_0}\right)^3 \leq \frac{\nabla\phi}{\sqrt{\psi}\sqrt{\phi}}\sqrt{t_0}\; t_0\phi\mathfrak{g}_0 + \left(6\alpha \mathrm{c}_\mu(R)^2 + \eta|\nabla\phi| + K\phi\right) t_0\sqrt{t_0\phi\mathfrak{g}_0}$$
$$+ C\left(\sqrt{t_0}\right)^3\left(\sqrt{\phi}\right)^3 + \phi\sqrt{t_0\phi g_0}. \tag{8.23}$$

Having into account (6.24), that $\mathrm{s}_\mu(\rho_1) \leq \mathrm{c}_\mu(\rho_1)$ and dividing by $\mathrm{c}_\mu(\rho_1)^3$, we get

$$2\delta\left(\frac{\sqrt{t_0\phi}\,\mathfrak{g}_0}{\mathrm{c}_\mu(\rho_1)}\right)^3 - 2\sqrt{\alpha}\,T\sqrt{1-\delta}\left(\frac{\sqrt{t_0\phi}\,\mathfrak{g}_0}{\mathrm{c}_\mu(\rho_1)}\right)^2$$
$$-\left(\left(\left(6+\frac{K}{\mu^2}\right)\alpha\; + \frac{\eta 2\alpha}{-\mu}\right)T + \frac{\alpha^{3/2}}{-\mu^3}\right)\frac{\sqrt{t_0\phi}\,\mathfrak{g}_0}{\mathrm{c}_\mu(\rho_1)} - C\alpha^{3/2}\frac{1}{-\mu^3}T \leq 0. \tag{8.24}$$

From now, arguing like after (6.25), since now $t\phi\mathfrak{g}$ is 0 at $S_{\rho_1,0}$,

$$\left(\frac{\sqrt{t\phi}\,\mathfrak{g}}{\mathrm{c}_\mu(\rho_1)}\right) \leq \left(\frac{\sqrt{t_0\phi}\,\mathfrak{g}_0}{\mathrm{c}_\mu(\rho_1)}\right) \leq D, \qquad \text{and} \qquad t|A|^2 \leq \frac{\mu^2(1-\delta)D^2}{\alpha(1-\gamma)^2}, \tag{8.25}$$

This shows that, on $\mathbb{S}_{\rho_1,T,\gamma}$, $t|A|$ is bounded by a bound depending T and ρ_1 (only through δ) and on M_0 (only through the maximum of v in $\mathbb{S}_{\rho_1,T}$). Renaming the constants, and, since $1 \leq 1+t$,

$$t|A|^2 \leq \alpha_0(1+t) \text{ on } \mathbb{S}_{\rho_1,T,\gamma} = \mathbb{S}_{\rho_2,T}.$$

This proves (8.1) for M non-compact and $m = 0$. To prove it for every m we consider the evolution of ϕf_m. From (5.25) and (8.17), computing like we did for getting (6.17) and renaming some constants,

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\phi f_m) \leq \phi\left(\left(\frac{(m+1)}{t} + C_m\frac{1+t}{t} + E_m - \frac{2}{t}\xi\right)t^{m+1}|\nabla^m A|^2\right.$$
$$+ \mathcal{C}\frac{(1+t)^{m+2}}{t} + \mathcal{D}(1+t)^{m+1} + \mathcal{B}(1+t)^m\; t$$
$$\left.+\xi\;\left(\mathcal{E}\;\frac{(1+t)^m}{t} + \mathcal{F}\frac{(1+t)^{m+1}}{t} + \mathcal{G}(1+t)^m + \mathcal{H}(1+t)^{m-1}\;t\right)\right)$$
$$-\,2\alpha\; f_m\frac{|\nabla\phi|^2}{\phi'^2} - 2\left\langle\frac{\nabla\phi}{\phi}, \nabla(\phi f_m) - f_m\nabla\phi\right\rangle. \tag{8.26}$$

From here, reasoning like we did from equations (6.32) to (6.36), but taking, at the end, $\xi = m+1+C_m(1+2T)+E_m 2T + 3\frac{\mu^2\;T}{(1-\gamma)^2}$ (instead of (8.19)), we get $t^{m+1}|\nabla^m A|^2 \leq \alpha_m(1+t)^{m+1}$ on $\mathbb{S}_{\rho_{1+m},T,\gamma}$, which finishes the proof by induction for the non-compact case. □

Theorem 13. *Let M be complete (compact or not). If $\overline{M} = M \times_\varphi \mathbb{R}$ and M_0 is a Lipschitz continuous graph over M, then there is a solution M_t of* (1.1) *with initial condition M_0 which is a graph over M, is defined on $[0,\infty[$, and M_t is smooth for every $t \in]0,\infty[$.*

Proof We shall write the details for the case M non-compact. When M is compact the arguments are similar, simplified by the fact that we can take always M or M_t instead of $\widehat{\mathrm{B}}(x_0, \rho_0)$ or $\mathbb{S}_{\rho_k,t,\gamma}$ respectively.

Let M_k be a sequence of smooth manifolds given by smooth graphs u_k over M and converging to M_0. For each M_k, let M_{kt} be the smooth solution of (4.1) which has M_k as initial condition, which exists and is defined for $t \in [0, \infty[$ by theorems 9 and 11. Each M_{kt} is represented by the graph of a function $u_k(\cdot, t)$ which is a solution of (4.5) with the initial condition $u_k(\cdot, 0)$.

Given $\rho_0 > 0$ and $T > 0$, let us define ρ by (7.1). It follows from (5.32) and (3.13) that

$$(8.27) \quad |\widehat{\nabla} u_k| \leq c \text{ on } \mathbb{S}^k_{\rho,T,\gamma} = \{(x, u_k(x,t)); \frac{\mathrm{c}_\mu(\widehat{r}((x, u_k(x,t)))}{-\mu} \leq \gamma \frac{\mathrm{c}_\mu(\rho)}{-\mu} e^{-\beta n t}, t \in [0, T]\},$$

where c is a constant which depends only on ρ_0 (through ρ), γ, T, an upper bound of v on $\mathbb{S}^k_{\rho,0} \subset M_k$ and the bounds of φ. Since M_0 is Lipschitz, v is bounded on the corresponding $\mathbb{S}_{\rho,0} \subset M_0$. Since the M_k converge to M_0, the maxima of v on $\mathbb{S}^k_{\rho,0} \subset M_k$ will be near the maximum of v on $\mathbb{S}_{\rho,0} \subset M_0$ for k big enough. Then we can take the constant c in (8.27) independent of k. From the definition of ρ, if $(x, t) \in \widehat{\mathrm{B}}(x_0, \rho_0) \times [0, T]$ then $(x, u_k(x, t)) \in \mathbb{S}_{\rho,T,\gamma}$ and (8.27) holds on $\widehat{\mathrm{B}}(x_0, \rho_0) \times [0, T]$.

The same argument as given at the end of last paragraph shows that, by Lemma 12,

$$(8.28) \qquad |\nabla^m A|^2 \leq \alpha_m \left(\frac{t+1}{t}\right)^{m+1} \quad \text{on } \widehat{\mathrm{B}}(x_0, \rho_0) \times]0, T],$$

where the constants α_m depend only on m, ρ_0, γ and T.

Now, let us consider the functions $u_{k\rho_0}(x, t) = u_k|_{\widehat{\mathrm{B}}(x_0,\rho_0)\times[0,T]}(x, t)$. Through the relation between u and A, the bounds (8.27) and (8.28) give that for every $t \in]0, T]$, there are constants $\beta(m, t, \rho_0, T, \gamma)$ depending only on m, t, ρ_0, T and γ such that

$$(8.29) \qquad |\widehat{\nabla}^m u_{k\rho_0}(\cdot, t)| \leq \beta(m, t, \rho_0, T, \gamma).$$

then, by the Ascoli-Arzela lemma, for each t and each ρ_0, there is a subsequence $u_{k\rho_0}(\cdot, t)$ which converges to a C^∞ function $u_{0\rho_0}(\cdot, t)$.

If we consider the $u_{k\rho_0}(\cdot, t)$ and $u_{0\rho_0}(\cdot, t)$ as functions of t, the same arguments used in section 7 (now with bound (8.1)) show that $\left|\frac{\partial u}{\partial t}\right|$ and $\left|\frac{\partial^2 u}{\partial t^2}\right|$ are bounded on $\widehat{\mathrm{B}}(x_0, \rho_0)) \times [t_0, T]$ by a constant depending only on $t_0 > 0$, ρ_0, T and γ, but not on k. Once we know that, we can conclude that the limit $u = u_{0\rho_0}$ is, at least, of class C^1 on t and, since all $u_{k\rho_0}$ satisfy equation (4.2) on $\widehat{\mathrm{B}}(x_0, \rho_0)) \times [t_0, T]$, also does $u_{0\rho_0}$. Taking ρ_0 and T bigger, we have sequences of functions satisfying similar conditions and which coincide with the older ones for the older values of ρ_0 and T, then for these new values of ρ_0 and T we have a new limit function which coincides withe the older limit when restricted to the older values of ρ_0 and T. Letting ρ_0 an T go to infinity and t_0 to 0, this gives a function u_0 which is a solution of (4.2), satisfies the initial condition $u_0(\cdot, 0) =$ the function defining M_0 as a graph, and, for every $t \in]0, \infty[$, $u_0(\cdot, t)$ is C^∞ as a function of M. □

9. Some results on convergence

Lemma 14. *Let $\overline{M}$ be a complete (may be compact) riemannian manifold, with bounded sectional curvature $k_0 \leq \overline{S}ec \leq k < 0$ and having a complete totally geodesic hypersurface M. Let M_t be the evolution of a complete hypersurface M_0 at time t by (1.1). Let us suppose that, for every $x \in M_0$, the distance $\ell(x)$ from x to M is bounded from above by some constant ℓ_0. If M is non-compact, we add the hypothesis that the norm $|A_t|$ of the Weingarten map A_t of the hypersurface M_t is bounded by a constant (depending on t). Then, one has*

$$\mathrm{s_k}(\ell(F(x,t))) \leq \mathrm{s_k}(\ell_0)e^{knt} \tag{9.1}$$

Proof A calculation similar to that done before for u, but now for ℓ is

$$\begin{aligned}(\nabla^2\ell)(E_i,E_j) &= \langle \nabla_{E_i}\nabla\ell, E_j\rangle = \langle \nabla_{E_i}(\partial_\ell - \langle\partial_\ell,N\rangle N), E_j\rangle \\ &= \left\langle \overline{\nabla}_{E_i}\partial_\ell, E_j\right\rangle + \langle\partial_\ell,N\rangle\, h(E_i,E_j) \\ &= \left\langle \overline{\nabla}_{E_i-\langle E_i,\partial_\ell\rangle\partial_\ell}\partial_\ell, E_j\right\rangle + \langle\partial_\ell,N\rangle\, h(E_i,E_j)\end{aligned} \tag{9.2}$$

and, using (2.3), one gets

$$\begin{aligned}\Delta\ell &= \sum_i \left\langle \overline{\nabla}_{E_i-\langle E_i,\partial_\ell\rangle\partial_\ell}\partial_\ell, E_i - \langle E_i, \partial_e ll\rangle\partial_\ell\right\rangle + \langle\partial_\ell,N\rangle\, H \\ &\geq -k\frac{\mathrm{s_k}}{\mathrm{c_k}}\left(n - |\partial_\ell^\top|^2\right) + \langle\partial_\ell,N\rangle\, H.\end{aligned} \tag{9.3}$$

Then, for the evolution of $\mathrm{s_k}(\ell) := \mathrm{s_k}\circ\ell$ one has

$$\begin{aligned}\frac{\partial \mathrm{s_k}(\ell)}{\partial t} &= \mathrm{c_k}(\ell)\; H\,\langle\partial_\ell,N\rangle \leq \mathrm{c_k}(\ell)\;\Delta\ell + k\mathrm{s_k}(\ell)\left(n - |\partial_\ell^\top|^2\right) \\ &= \Delta\mathrm{s_k}(\ell) - (-k\mathrm{s_k}(\ell)|\partial_\ell^\top|^2) + k\;\mathrm{s_k}(\ell)\left(n - |\partial_\ell^\top|^2\right) \\ &= \Delta\mathrm{s_k}(\ell) + n\;k\;\mathrm{s_k}(\ell)\end{aligned} \tag{9.4}$$

If M is compact, then the maximum principle states that ℓ is bounded by the solution of the ODE $\mathrm{s_k}(\ell)' = n\;k\;\mathrm{s_k}(\ell)$ with the initial condition $\ell(0) = \ell_0$, which is $\mathrm{s_k}(\ell) = \mathrm{s_k}(\ell_0)e^{nkt}$, from which the statement of the theorem follows.

If M is non-compact, let us consider first the evolution of ℓ, which follows from (9.3) as above

$$\frac{\partial\ell}{\partial t} = H\,\langle\partial_\ell,N\rangle \leq \Delta\ell + k\frac{\mathrm{s_k}(\ell)}{\mathrm{c_k}(\ell)}\left(n - |\partial_\ell^\top|^2\right) \leq \Delta\ell + k\frac{\mathrm{s_k}(\ell)}{\mathrm{c_k}(\ell)}(n-1) \leq \Delta\ell, \tag{9.5}$$

we observe first that the added hypothesis implies that we can apply the maximum principle for noncompact manifolds given in Lemma 3 to $f = \ell - \ell_0$. In fact, our hypothesis, together with the hypothesis $k_0 \leq \overline{S}ec$ and the Gauss formulae give that the sectional curvature of each M_t is bounded from below, which shows that the hypothesis on the growing of volume of geodesic balls of that Lemma is satisfied. Conditions (i) and (ii) are obvious. Condition iv) follows from the evolution equation $\frac{\partial}{\partial t}g = -2H\alpha$ and the added hypothesis. Condition (iii) follows from $|\nabla\ell| \leq 1$ and the fact that $d\mu_t \leq \frac{{\mathrm{s}_\lambda}^n(r)}{r^n}d\mu_e$, where λ is determined by μ_0 and the upper bounds of $|A_t|$ and $d\mu_e$ is the euclidean measure. Then we can conclude, from the quoted theorem, that $\ell - \ell_0 \leq 0$ for all time.

Once we know that ℓ is bounded, we can study the evolution of $u = \mathrm{s_k}(\ell) - \mathrm{s_k}(\ell_0)e^{nkt}$, which, from (9.4) is

$$\frac{\partial u}{\partial t} = \Delta u + n\ k\ u, \tag{9.6}$$

and, again, this equation satisfies the conditions of Theorem 4.3 in [4], the only condition that has to be checked now is (iii), which follows because $|\nabla u| = \mathrm{c_k}(\ell)|\nabla \ell| \leq \mathrm{c_k}(\ell_0)$ and now we know that $\ell \leq \ell_0$. Then, as before, we can conclude that $u \leq 0$ and the theorem is proved. □

With the same notations (5.13) and (5.14), we have

Theorem 15. *Let M be compact. Let M_t be the solution of* (1.1) *defined on a maximal time interval* $[0, T[$. *If $\overline{M} = M \times_\varphi \mathbb{R}$, $0 > -\mu_1 > \widehat{S}ec \geq \dfrac{-n\mu_1 + \mu_2}{n-1}$ and M_0 is a graph over M, then M_t is a graph, $T = \infty$ and M_t converges in the C^∞ topology to $M \times \{0\}$ as $t \to \infty$.*

Proof Notice that the hypothesis on the lower bound of $\widehat{S}ec$ is equivalent to $\nu \leq 0$, then, from theorems 6 and 9 it follows that both v and $|\nabla^i A|$, $i = 0, 1, ...$ are bounded by some constants not depending on time, then, by Ascoli-Arzelà, the M_t converge to some limit M_∞. The hypotheses $-\mu_1 > \widehat{S}ec$ implies $-\mu_1 > \overline{S}ec$, then we can apply Lemma 14 to conclude that M_∞ must be at distance 0 from M, then it must be M. □

Theorem 16. *Let M be complete non compact with a pole x_0. Let M_t be the solution of* (1.1) *defined on a maximal time interval* $[0, T[$. *If $\overline{M} = M \times_\varphi \mathbb{R}$, $0 > -\mu_1 > \widehat{S}ec \geq \dfrac{-n\mu_1 + \mu_2}{n-1}$, M_0 is a graph over M with $|A|^2$ bounded, then M_t is a graph, $T = \infty$ and M_t converges in the C^∞ topology to $M \times \{0\}$ as $t \to \infty$.*

Proof Looking at (5.33) and at the proof of Theorem 10, one observes that, when v and $|\nabla^i A|$, $i = 0, 1, ...$, are bounded on M_0, then both v and $|\nabla^i A|$ are bounded by some constants depending on time, but not on ρ'. Then, we can apply Lemma 3 to the evolution equation (5.1) satisfied by v when $\nu \leq 0$ to conclude that v is bounded by a bound not depending on time. Using this fact in the proof of Theorem 10, we see that the bounds of $|\nabla^i A|$ do not depend on time, then, as above, we can apply Ascoli-Arzela to conclude that M_t has a limit M_∞, and, using Lemma 14 as before, we conclude that M_∞ must be M. □

10. Appendix: About the flow in $\mathbb{R} \times_\varphi M$

When we consider the ambient space $\overline{M} = \mathbb{R} \times_\varphi M$, that is $(\mathbb{R} \times M, \overline{g} = du^2 + \varphi(u)\widehat{g})$, where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$. If we consider the evolution under (1.1) of a hypersurface M_0 which is a graph over M we obtain for $v = \langle N, \partial_y \rangle^{-1}$ the evolution equation

$$\begin{aligned}\frac{\partial v}{\partial t} &= \Delta v - \frac{2}{v}|\nabla v|^2 + 2\frac{\varphi'}{\varphi}\langle \nabla v, \nabla u \rangle - \frac{\varphi'}{\varphi}2Hv^2 \\ &- \left(1 - \frac{1}{v^2}\right)\left(\frac{\widehat{Ric}_{\widehat{1}\widehat{1}}}{\varphi^2} + n\frac{\varphi''}{\varphi}\right)v - |A|^2 v - \left(\frac{\varphi'}{\varphi}\right)^2 n\frac{1}{v}.\end{aligned} \tag{10.1}$$

where $\widehat{Ric}_{\widehat{1}\widehat{1}}$ is the Ricci curvature in the direction $\widehat{\nabla} u$.

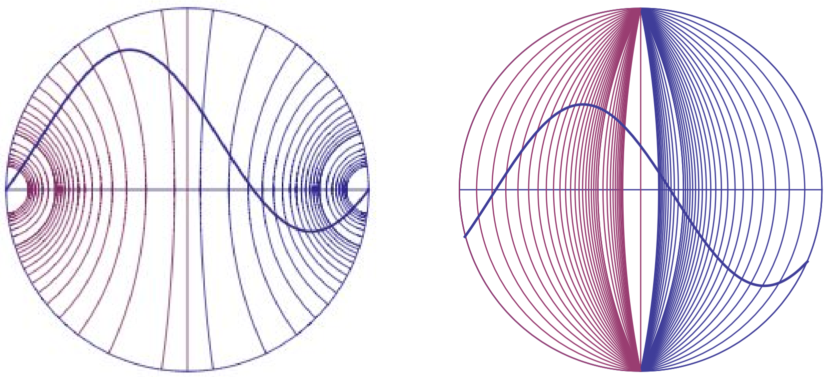
The term $-\frac{\varphi'}{\varphi}2Hv^2$ in this equation do not allow to apply the maximum principle as we did with (5.1) and, in fact, can be considered as the analytical reason why the property of being a graph is not preserved in general under the MCF in $\mathbb{R} \times_\varphi M$.

Looking again at (10.1) it is possible to think that we can get interesting results for hypersurfaces satisfying that the sign of H and the sign of φ' is the same. But this condition has many drawbacks :

(1) A computation shows that if M_0 is the graph of a function u,
$$\Delta u = \left(n - \left(1 - \frac{1}{v^2}\right)\right)\frac{\varphi'}{\varphi} + \frac{1}{v}H.$$
Then, when M is compact, it follows from this formula and the maximum principle that H has the same sign that φ' if and only if $u = 0$.

(2) When M is non-compact, the known proof (for the euclidean space) given in [4] that the sign of H is preserved uses the bounds of $|A|$, and, for them, bounds of v are also used. But in our case, are just the bounds of v which we do not know.

Another idea of why it is easy that the property of being a graph is preserved for $M \times_\varphi \mathbb{R}$ and not for $\mathbb{R} \times_\varphi M$ is given by the following pictures. In both $M = \mathcal{H}^n$ and $\varphi(u) = \cosh u$ for the first and $\varphi(x) = \cosh(\mathrm{dist}(x_0, x))$ for the second. Then $\overline{M} = \mathcal{H}^{n+1}$ in both, but a graph for the first one corresponds to the sense of graph in this appendix (a graph for geodesics) and the second one corresponds to a graph in the sense o the previous sections of this paper, that is a graph for "equidistant" curves.



a graphic for geodesics

a graphic for equidistants which is not a graphic for geodesics

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Kharkov National University, Mathematics Faculty. Geometry Department, Pl. Svobodi 4, 61077-Kharkov, Ukraine

E-mail address: borisenk@univer.kharkov.ua

Universidad de Valencia, Departamento de Geometría y Topología, Avda. Andrés Estellés 1, 46100-Burjassot (Valencia) Spain

E-mail address: miquel@uv.es