Upper bounds for the first Dirichlet eigenvalue of a tube around an algebraic complex curve of $\mathbb{C}P^n(\lambda)$

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Abstract

We give an upper bound for the first Dirichlet eigenvalue of a tube around a complex curve P of $\mathbb{C}P^n(\lambda)$ which depends only on the radius of the tube and the degrees of the polynomials defining P. The bound is sharp on a totally geodesic $\mathbb{C}P^1(\lambda)$ and gives a gap between the eigenvalue of a tube around $\mathbb{C}P^1(\lambda)$ and around other complex curves.

1 Introduction

Let $\mathbb{C}P^n(\lambda)$ be the complex projective space of holomorphic sectional curvature 4λ. We shall denote by $\wp : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}P^n(\lambda)$ the canonical projection. The classical Chow's Theorems states that every complete complex submanifold P of $\mathbb{C}P^n(\lambda)$ is algebraic, then, if P has complex dimension q it is the image by $\wp$ of the set of zeroes of $n-q$ homogeneous polynomials of degrees $a_{q+1}, ..., a_n$. For such a submanifold, we shall denote by P_ρ the tube of radius $\rho > 0$ around P and by ∂P_ρ its boundary. We shall always consider ρ lower than the cut distance from P. In this paper we shall consider the case $q = 1$ (complex curves) and we shall prove:

Theorem 1.1 *The first eigenvalue $\mu_1(P_\rho)$ of the Dirichlet eigenvalue problem*

$$\Delta f = \mu f, \qquad f|_{\partial P_\rho} = 0$$

satisfies the inequality

$$\mu_1(P_\rho) \leq \mu_1(\mathbb{C}P^1(\lambda)_\rho) - \frac{2\lambda\left(\displaystyle\sum_{s=2}^{n} a_s - (n-1)\right)}{1 - \lambda\left(\dfrac{1}{n-1}\displaystyle\sum_{s=2}^{n} a_s\right)\; C(\rho)} \tag{1}$$

where $\mathbb{C}P^1(\lambda)$ is embedded as a complex totally geodesic submanifold of $\mathbb{C}P^n(\lambda)$, $C(\rho)$ is a well defined constant which depends only on ρ and satisfies $\lambda\left(\dfrac{1}{n-1}\displaystyle\sum_{s=2}^{n} a_s\right) C(\rho) < 1$. Moreover the equality is attained if and only if $P = \mathbb{C}P^1(\lambda)$.

The interest of this bound lies on the fact that it relates the first Dirichlet eigenvalue of P_r with the degrees of the polynomials defining P (which, in turn, are related with the first Chern class of the normal bundle of P, as it is shown along the proof). Moreover, Theorem 1.1 shows a gap phenomenon for $\mu_1(P_\rho)$ between the case $P = \mathbb{C}P^1(\lambda)$ (which corresponds to $a_2 = ... = a_n = 1$) and the other complex curves P, and states that the gap is measured by the degrees a_s. A similar gap (not related with the "a_s") also appeared in the study of the first closed eigenvalue of P did by Bourguignon, Li and Yau in [1].

Our study was motivated by the work of A. Gray in [4] and [5] on the volume of P_ρ, where he showed that it is determined by ρ and and the degrees a_s. The previous work of Cheng ([2]), Lee ([8]), Giménez, Palmer and the last two authors of this paper (cf. [3], [9], [10] and [11]) on $\mu_1(P_\rho)$ has shown that there is a strong relation between volume(P_ρ) and $\mu_1(P_\rho)$, then it was natural to ask for some result relating the degrees a_s with $\mu_1(P_\rho)$. In this case it seems too strong to expect that the degrees of the polynomials determine the first Dirichlet eigenvalue, but what we have seen is that, at least, they allow to obtain a bound and to measure the gap between the first Dirichlet eigenvalue of a tube around a totally geodesic complex curve and of tubes around the other complex curves.

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2 Preliminaries

Given a complex curve P of $\mathbb{C}P^n(\lambda)$, we shall denote by r the distance to P in $\mathbb{C}P^n(\lambda)$. Let us denote by $\mathcal{N}P$ the normal bundle of P, by A_ξ the Weingarten map of P in the direction of $\xi \in \mathcal{N}P$, $|\xi| = 1$. Moreover, we shall use the notations s_λ and c_λ for

$$\mathrm{s}_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}, \qquad \mathrm{c}_\lambda(t) = \cos(\sqrt{\lambda}t),$$

which satisfy the computational rules $\mathrm{s}'_\lambda = \mathrm{c}_\lambda$ and $\mathrm{c}^2_\lambda + \lambda \mathrm{s}^2_\lambda = 1$.

Since P is a complex submanifold, it is compatible with $\mathbb{C}P^n(\lambda)$ (in the sense given by Gray in [6] page 95). Then, given $\xi \in \mathcal{N}P$, $|\xi| = 1$, there exists a holomorphic orthonormal frame $\{e_1, Je_1, ..., e_n, Je_n\}$ such that $e_2 = \xi$ and $\{e_1, Je_1\}$ is a basis of T_pP which diagonalizes the Weingarten map A_ξ, with eigenvalues $k(\xi), -k(\xi)$.

As, on the other hand, $\mathbb{C}P^n(\lambda)$ is locally symmetric, by Theorem 6.14 in [6] page 96, we can choose a parallel orthonormal J-frame $\{E_1(t), JE_1(t)\}$ along the normal geodesic $\gamma_\xi(t) = \exp_p t\xi$ which coincides with $\{e_1, Je_1\}$ at p and such that diagonalzes the Weingarten map $S_\lambda(t)$ of ∂P_t. In this J-frame the expression of $S_\lambda := S_\lambda(\rho)$ is (cf. [6], page 125):

$$
\begin{aligned}
S_\lambda E_1(\rho) &= \frac{\lambda \mathrm{s}_\lambda(\rho) + k(\xi)\mathrm{c}_\lambda(\rho)}{\mathrm{c}_\lambda(\rho) - k(\xi)\mathrm{s}_\lambda(\rho)}\, E_1(\rho) \\
S_\lambda JE_1(\rho) &= \frac{\lambda \mathrm{s}_\lambda(\rho) - k(\xi)\mathrm{c}_\lambda(\rho)}{\mathrm{c}_\lambda(\rho) + k(\xi)\mathrm{s}_\lambda(\rho)}\, JE_1(\rho) \\
S_\lambda JE_2(\rho) &= -\frac{\mathrm{c}_{4\lambda}(\rho)}{\mathrm{s}_{4\lambda}(\rho)} JE_2(\rho) \qquad (2) \\
S_\lambda E_j(\rho) &= -\frac{\mathrm{c}_\lambda(\rho)}{\mathrm{s}_\lambda(\rho)} E_j(\rho), \text{ and} \qquad (3) \\
S_\lambda JE_j(\rho) &= -\frac{\mathrm{c}_\lambda(\rho)}{\mathrm{s}_\lambda(\rho)} JE_j(\rho), \qquad j = 3, ..., n.
\end{aligned}
$$

From these expressions it follows

$$
\operatorname{tr} S_\lambda = \frac{2\mathrm{s}_\lambda(\rho)\mathrm{c}_\lambda(\rho)(\lambda + k^2)}{\mathrm{c}_\lambda^2(\rho) - k^2\mathrm{s}_\lambda^2(\rho)} - (2n-3)\frac{\mathrm{c}_\lambda(\rho)}{\mathrm{s}_\lambda(\rho)} + \lambda \frac{\mathrm{s}_\lambda(\rho)}{c_\lambda(\rho)} \tag{4}
$$

On the other hand, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞ function, one has (cf. [10] for instance):

$$
\Delta(f \circ r) = -f'' \circ r + \operatorname{tr} S_\lambda \; f' \circ r \tag{5}
$$

From now on we shall omit the writing of "$\circ\ r$", which should be understood by the context.

Since for $\mathbb{C}P^1(\lambda)$, $k = 0$, it follows from (4) and (5) that the eigenfunction associated to $\mu_1(\mathbb{C}P^1(\lambda)_\rho)$ is $f_\lambda \circ r$, where f_λ is the solution of the equation:

$$
\left.
\begin{aligned}
&-f'' + \left(3\lambda \frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda} - (2n-3)\frac{\mathrm{c}_\lambda}{\mathrm{s}_\lambda}\right) f' = \mu f \\
&f(\rho) = 0 \\
&f'(0) = 0
\end{aligned}
\right\} \tag{6}
$$

for the minimum $\mu > 0$ satisfying (6), which will be $\mu_1(\mathbb{C}P^1(\lambda)_\rho)$. This function satisfies the inequalities (cf. [9] or [10] for instance)

$$
f_\lambda > 0 \text{ on } [0, \rho[\quad \text{ and } \quad f'_\lambda < 0 \text{ on }]0, \rho]. \tag{7}
$$

3 Proof of Theorem 1.1

We shall divide the proof in three steps. In **the first one**, we shall obtain an upper bound of $\mu_1(P_\rho)$ which is some function of the integral along P of the norm of its second fundmental form. For this estimation we shall apply the Raileigh theorem

$$
\mu_1(P_\rho) = \inf_{\substack{f \in \mathcal{C}^\infty(P_\rho, \mathbb{R}) \\ f|_{\partial P_\rho} = 0}} \frac{\int_{P_\rho} (\Delta f) f}{\int_{P_\rho} f^2} \tag{8}
$$

and we'll use $f_\lambda \circ r$ (which many times we shall write as $f_\lambda(r)$ or just f_λ) as a test function, then

$$\mu_1(P_\rho) \leq \frac{\int_{P_\rho} f_\lambda(\Delta f_\lambda)}{\int_{P_\rho} f_\lambda^2} \tag{9}$$

Let us compute the right hand side of the above inequality. From (4) and (5), we get

$$\Delta(f_\lambda \circ r) = -f_\lambda'' \circ r \quad + \left(\frac{2\mathrm{s}_\lambda \circ r \ \mathrm{c}_\lambda \circ r \ (\lambda + k^2)}{\mathrm{c}_\lambda^2 \circ r \ - k^2 \ \mathrm{s}_\lambda^2 \circ r} - (2n-3)\frac{\mathrm{c}_\lambda \circ r}{\mathrm{s}_\lambda \circ r} + \lambda \frac{\mathrm{s}_\lambda \circ r}{\mathrm{c}_\lambda \circ r} \right) f_\lambda' \circ r$$

In order to compare with $\mu_1(\mathbb{C}P^1(\lambda))_\rho$, we add and substract $2\lambda \frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda}$ to the coefficient of f_λ' and we get

$$\Delta f_\lambda = - f_\lambda'' + \left(\frac{2k^2}{\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2} \frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda} + 3\lambda \frac{\mathrm{s}_\lambda}{c_\lambda} - (2n-3)\frac{\mathrm{c}_\lambda}{\mathrm{s}_\lambda} \right) f_\lambda' \tag{10}$$

Having into account that f_λ satisfies (6) with $\mu = \mu_1(\mathbb{C}P^1(\lambda)_\rho)$, we obtain from (10) that

$$\int_{P_\rho} f_\lambda(\Delta f_\lambda) = \int_{P_\rho} \mu_1(\mathbb{C}P^1(\lambda)_\rho) f_\lambda^2 + \int_{P_\rho} 2\frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda} \frac{k^2}{\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2} f_\lambda f_\lambda' \tag{11}$$

Taking under consideration (7) we have the inequality

$$(f_\lambda^2)' = 2 f_\lambda f_\lambda' \leq 0 \tag{12}$$

Then, it follows from (9), (11), (12) and the expression of the volume element of P_ρ in spherical Fermi coordinates (cf. [6], page 125) that

$$\begin{aligned}
\mu_1(P_\rho) &\leq \frac{\int_{P_\rho} f_\lambda(\Delta f_\lambda)}{\int_{P_\rho} f_\lambda^2} \leq \frac{\int_{P_\rho} \mu_1(\mathbb{C}P^1(\lambda)_\rho) f_\lambda^2}{\int_{P_\rho} f_\lambda^2} + \frac{\displaystyle\int_{P_\rho} \frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda} \frac{k^2}{\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2} (f_\lambda^2)'}{\int_{P_\rho} f_\lambda^2} \\
&= \mu_1(\mathbb{C}P^1(\lambda)_\rho) + \frac{\displaystyle\int_0^\rho \int_P \int_{S^{2n-3}} (f_\lambda^2)' \frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda} \frac{k^2}{\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2} \ \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda (\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2) d\xi \ dp \ dr}{\displaystyle\int_0^\rho \int_P \int_{S^{2n-3}} f_\lambda^2 \ \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda (\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2) \ d\xi \ dp \ dr} \\
&= \mu_1(\mathbb{C}P^1(\lambda)_\rho) + \frac{\displaystyle\int_0^\rho \int_P \int_{S^{2n-3}} (f_\lambda^2)' \mathrm{s}_\lambda^{2n-2} \ k^2 \ d\xi \ dp \ dr}{\displaystyle\int_0^\rho \int_P \int_{S^{2n-3}} f_\lambda^2 \ \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda (\mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2) \ d\xi \ dp \ dr},
\end{aligned} \tag{13}$$

where dp and $d\xi$ denote, respectively, the volume elements of P and of $S^{2n-2q-1}$.

In the **second step**, we shall relate $|A_\xi|$ with the first Chern number of $\mathcal{N}P$ and with the degrees "a_s". For it, first we shall write the first Chern form of the normal connection D on $\mathcal{N}P$ (which satisfies $DJ = 0 = D\langle , \rangle$) using the real formalism, like it is done, for instance, in [6] and in [7]. With it, for a local J-orthonormal frame $N_2, JN_2, ..., N_n, JN_n$ the first Chern form γ_1 of $\mathcal{N}P$ is given by

$$2\pi\ \gamma_1 = \sum_{s=2}^{n} < R^D(\cdot,\cdot)N_s, JN_s >,$$

where R^D is the curvature of D

Let us denote by $e(N_s)$ the eigenvector of A_{N_s} associated to the eigenvalue $k(N_s)$. Using the Ricci equations relating the curvature R^D with the curvature $\overline{R}$ of $\mathbb{C}P^n(\lambda)$ and A_{N_s}, the above expression for γ_1 gives:

$$2\pi\ \gamma_1(X,Y) = \sum_{s=2}^{n} \{< \overline{R}(X,Y)N_s, JN_s > - < [A_{N_s}, A_{JN_s}]X, Y >\}$$

Having into account the expression of $\overline{R}$

$$\begin{aligned}\overline{R}_{wxyz} = \lambda\ \{&\langle w,y\rangle\langle x,z\rangle - \langle w,z\rangle\langle x,y\rangle \\ &+\langle Jw,y\rangle\langle Jx,z\rangle - \langle Jw,z\rangle\langle Jx,y\rangle + 2\langle Jw,x\rangle\langle Jy,z\rangle\}\,. \end{aligned} \tag{14}$$

and the properties of the Weingarten map of a complex submanifold

$$A_{J\xi}X = JA_\xi X \text{ and } A_\xi JX = -JA_\xi X, \tag{15}$$

we obtain

$$2\pi\gamma_1(X,Y) = \sum_{s=2}^{n} 2\left(\lambda F(X,Y) - F(A_{N_s}X, A_{N_s}Y)\right), \tag{16}$$

where F is the Kähler form of P defined by $F(X,Y) = \langle JX, Y\rangle$.

On the other hand, it is well known that the first Chern class $[\gamma_1^P]$ of a complex curve P of $\mathbb{C}P^n(\lambda)$ given as the intersection of $n-1$ homogeneous polynomials of degrees $a_2, ..., a_n$is given by (cf. page 114 of [6])

$$\left[\gamma_1^P\right] = \left(n+1-\sum_{s=2}^{n} a_s\right)\left[\frac{\lambda}{\pi}F\right], \tag{17}$$

and the first Chern class of $\mathbb{C}P^n(\lambda)$ is (cf. page 105 of [6])

$$\left[\gamma_1^{\mathbb{C}P^n(\lambda)}\right] = (n+1)\left[\frac{\lambda}{\pi}F\right] \tag{18}$$

Given the canonical inclusion $i : P \longrightarrow \mathbb{C}P^n(\lambda)$, we have $i^*T\mathbb{C}P^n(\lambda) = TP \oplus \mathcal{N}P$, then $i^*[\gamma_1^{\mathbb{C}P^n(\lambda)}] = [\gamma_1^P] + [\gamma_1]$, then, from (17) and (18),

$$\begin{aligned}[\gamma_1] &= \left[\gamma_1^{\mathbb{C}P^n(\lambda)}\right] - \left[\gamma_1^P\right] \\ &= (n+1)\left[\frac{\lambda}{\pi}F\right] - \left(n+1-\sum_{s=2}^{n} a_s\right)\left[\frac{\lambda}{\pi}F\right] = \sum_{s=2}^{n} a_s\left[\frac{\lambda}{\pi}F\right]\end{aligned} \tag{19}$$

From (16) and (19) it follows that there is a 1-form θ such that

$$\gamma_1(X,Y) = \frac{1}{\pi}\sum_{s=2}^{n}(\lambda F(X,Y) - F(A_{N_s}X, A_{N_s}Y)) \tag{20}$$

$$= \frac{\lambda}{\pi}\sum_{s=2}^{n} a_s F(X,Y) + d\theta(X,Y) \tag{21}$$

Now, let us compute γ_1. First we do the computation using (20). For it, using an arbitrary local J-orthonormal frame $\{e_1, Je_1\}$ of TP and its dual frame $\{\theta^1, \theta^{1*}\}$ we compute first

$$\begin{aligned} &F(A_{N_s}\cdot, A_{N_s}\cdot) \\ &\quad = F(A_{N_s}\cdot, A_{N_s}\cdot)\ (e_1, Je_1)\ \theta^1 \wedge \theta^{1*} \\ &\quad = \langle JA_{N_s}e_1, A_{N_s}Je_1\rangle\ dp = -\langle JA_{N_s}e_1, JA_{N_s}e_1\rangle\ dp = -\frac{1}{2}|A_{N_s}|^2\ dp \end{aligned} \tag{22}$$

then, from (20) and (22)

$$\gamma_1 = (n-1)\frac{\lambda}{\pi}dp + \frac{1}{2\pi}\sum_{s=2}^{n}|A_{N_s}|^2\ dp \tag{23}$$

Now, let us compute using (21),

$$\gamma_1 = \frac{\lambda}{\pi}\left(\sum_{s=2}^{n} a_s\right)\ dp + d\theta \tag{24}$$

From (23) and (24), integrating along P, which is compact, and applying Stokes theorem, we obtain

$$\int_P \left((n-1)\lambda + \frac{1}{2}\sum_{s=2}^{n}|A_{N_s}|^2\right)\ dp = \lambda\left(\sum_{s=2}^{n} a_s\right)\int_P dp, \tag{25}$$

from which it follows that

$$\frac{1}{2(n-1)}\int_P\left(\sum_{s=2}^{n}|A_{N_s}|^2\right) dp = \lambda\left(\frac{1}{n-1}\sum_{s=2}^{n} a_s - 1\right)\mathrm{volume}(P) \tag{26}$$

In the **third step** we shall obtain (1) from (13) and (26). First, we consider, at each $p \in P$, the bilinear map $\Phi : \mathcal{N}_pP \times \mathcal{N}_pP \longrightarrow \mathbb{R}$ defined by $\Phi(\xi,\eta) =< A_\xi, A_\eta >$. A well known lemma of Linear Algebra (cf. [6] page 61) states that

$$\int_{S^{2n-3}} \Phi(\xi,\xi)\ d\xi = \frac{2\pi^{n-1}}{(2n-2)\Gamma(n-1)}\sum_{s=2}^{n}\left(\Phi(N_s,N_s) + \Phi(JN_s,JN_s)\right),$$

then

$$\int_{S^{2n-3}} |A_\xi|^2\ d\xi = \frac{2\pi^{n-1}}{(n-1)\Gamma(n-1)}\sum_{s=2}^{n}|A_{N_s}|^2 \tag{27}$$

Since $|A_\xi|^2 = 2\ k(\xi)^2$, putting together (13), (25) and (27) we obtain

$$\mu_1(P_\rho) \leq \mu_1(\mathbb{C}P^1(\lambda)_\rho) + \frac{\dfrac{2\ \pi^{n-1}}{(n-1)\Gamma(n-1)} \displaystyle\int_0^\rho (f_\lambda^2)' \frac{1}{2}\ \mathrm{s}_\lambda^{2n-2} \int_P \sum_{s=2}^n |A_{N_s}|^2\ dp\ dr}{\dfrac{2\pi^{n-1}}{\Gamma(n-1)} \displaystyle\int_0^\rho f_\lambda^2\ \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda \left(\mathrm{c}_\lambda^2 \operatorname{volume}(P) - \mathrm{s}_\lambda^2 \frac{1}{2(n-1)} \int_P \sum_{s=2}^n |A_{N_s}|^2\ dp\right) dr}$$

$$= \mu_1(\mathbb{C}P^1(\lambda)_\rho) + \frac{2\lambda \left(\dfrac{1}{n-1} \displaystyle\sum_{s=2}^n a_s - 1\right) \displaystyle\int_0^\rho (f_\lambda^2)'\ \mathrm{s}_\lambda^{2n-2}\ dr}{2 \displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 \left(\mathrm{c}_\lambda^2 - \mathrm{s}_\lambda^2 \lambda \left(\dfrac{1}{n-1} \displaystyle\sum_{s=2}^n a_s - 1\right)\right) dr} \tag{28}$$

Notice that, since $\rho <$ cut distance from P, we have $1 > \mathrm{c}_\lambda^2 - k^2 \mathrm{s}_\lambda^2 > 0$, then the denominator in (28) is positive.

Integration by parts gives

$$\int_0^\rho (f_\lambda^2)'\ \mathrm{s}_\lambda^{2n-2}\ \ dr = \left(\mathrm{s}_\lambda^{2n-2} f_\lambda^2 |_0^\rho - \int_0^\rho (2n-2) \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 dr\right)$$
$$= -(2n-2) \int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2\ dr \tag{29}$$

because $\mathrm{s}_\lambda(0) = 0$ and $f_\lambda(\rho) = 0$. Using that $\mathrm{c}_\lambda^2 = 1 - \lambda \mathrm{s}_\lambda^2$, by substitution of (29) in (28) we obtain

$$\mu_1(P_\rho) \leq \mu_1(\mathbb{C}P^1(\lambda)_\rho) - \frac{\lambda \left(\dfrac{1}{n-1} \displaystyle\sum_{s=2}^n a_s - 1\right) 2(n-1) \displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3}\ \mathrm{c}_\lambda\ (f_\lambda^2)\ \ dr}{\displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2\ dr - \lambda \int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 \mathrm{s}_\lambda^2 \left(\dfrac{1}{n-1} \displaystyle\sum_{s=2}^n a_s\right) dr}$$

$$= \mu_1(\mathbb{C}P^1(\lambda)_\rho) - \frac{2\ \lambda \left(\displaystyle\sum_{s=2}^n a_s - (n-1)\right)}{1 - \lambda \left(\dfrac{1}{n-1} \displaystyle\sum_{s=2}^n a_s\right) \dfrac{\displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 \mathrm{s}_\lambda^2 dr}{\displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 dr}},$$

which is inequality (1) with $C(\rho) = \dfrac{\displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 \mathrm{s}_\lambda^2 dr}{\displaystyle\int_0^\rho \mathrm{s}_\lambda^{2n-3} \mathrm{c}_\lambda\ f_\lambda^2 dr}$, and the inequality satisfied from $C(\rho)$ comes from the positivity of the denominator of (28).

On the other hand, when $P = \mathbb{C}P^1(\lambda)$ we have equality in (1). If $P \neq \mathbb{C}P^1(\lambda)$ the equality can not be achieved because this will imply that f_λ is a solution of (5), which only is true when $k = 0$. Then the equality in (1) holds if and only if P is $\mathbb{C}P^1(\lambda)$, which finishes the proof of the theorem.

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