Bounding the first Dirichlet eigenvalue of a tube around a complex submanifold of $\mathbb{C}P^n(\lambda)$ in terms of the degrees of the polynomials defining it

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(in this version a few small corrections are pending)

Abstract

We obtain upper bounds for the first Dirichlet eigenvalue of a tube around a complex submanifold P of $\mathbb{C}P^n(\lambda)$ which depends only on the radius of the tube, the degrees of the polynomials defining P and the first eigenvalue of the tube around some model centers. The bounds are sharp on these models. Moreover, when the models used are $\mathbb{C}P^q(\lambda)$ or $Q^{n-1}(\lambda)$ these bounds also provide gap phenomena and comparison results.

1 Introduction

This paper deals with classical Dirichlet eigenvalue problem

$$\Delta f = \mu f \quad \text{on} \quad M \qquad \text{and} \qquad f = 0 \quad \text{on} \quad \partial M \tag{1}$$

on a connected compact Riemannian manifold M with boundary ∂M. An active area of research in this problem is to determine "interesting" upper or lower bounds for the first eigenvalue $\mu_1(M)$ in equation (1). Here "interesting" means sharp or related with some special properties of the space M. As examples of the work in this direction there are the results by S. Y. Cheng ([3]), R. Reilly ([17]), M.Gage ([6]), A. Kasue ([11]), J. M. Lee([13]), F. Giménez, A. Lluch, V. Palmer and the second author ([7, 14, 15, 16]), G. P. Bessa and J. F. Montenegro ([2]). In these papers, M is a tube around some compact submanifold of some Riemannian manifold or a manifold with boundary with special bounds on the curvatures of M and ∂M. The bounds obtained for $\mu_1(M)$ are usually the values of this first eigenvalue for some special model tube and, often, the equality characterizes this model tube.

Very close to the problem of obtaining bounds for $\mu_1(M)$ is that of getting estimates for the volume of M. The deep relation between these problems is well known and some of the papers quoted above show or explore it. As for the volume, A. Gray (see [8, 9, 10]) has shown that the volume of a tube around a complex submanifold of $\mathbb{C}P^n(\lambda)$ can be expressed in terms of the degrees of the polynomials defining the center of the tube. It is natural, then, to look for some relation between the first Dirichlet eigenvalue of the tube and the degrees of the polynomials defining its center. Another hope for getting such kind of relation are the recent results by Colbois, Dryden and El Soufi ([4]) where they obtain bounds of the first eigenvalue for the closed problem on algebraic submanifolds in the euclidean space in terms of the degrees of the polynomials defining them. The problem was addressed in [5], where A. Lluch and the authors obtained a bound of the first Dirichlet eigenvalue of a tube around a complex curve in the complex projective space in terms of the degrees of the polynomial defining the complex curve. Our aim in this paper is to complete the work started in [5] by studying tubes around complex submanifolds of any dimension as well as giving bounds of a different flavor, namely by comparing with the eigenvalues of tubes around all the possible homogeneous complex submanifolds for which the eigenfunctions corresponding to $\mu_1(M)$ are radial.

We now state our results with more precision. Let $\mathbb{C}P^n(\lambda)$ be the complex projective space of holomorphic sectional curvature 4λ, and let $\wp : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}P^n(\lambda)$ be the canonical projection. Chow's Theorem states that every complete complex submanifold P^q of $\mathbb{C}P^n(\lambda)$ (of complex dimension q) is the image by $\wp$ of the set of zeros of $n-q$ homogeneous polynomials of degrees $a_{q+1}, ..., a_n$. As a consequence P is compact, because the set $\mathcal{Z}$ of zeros of homogeneous polynomials is a closed subset of $\mathbb{C}^{n+1}$ consisting of lines, then its intersection with the unit sphere S^{2n+1} is compact and the projection $P = \wp(\mathcal{Z} \cap S^{2n+1}) = \wp(\mathcal{Z})$ in $\mathbb{C}P^n(\lambda)$ is compact. For such a submanifold, we shall denote by P_ρ the tube of radius $\rho > 0$ around P and by ∂P^q_ρ its boundary. We shall always consider ρ lower than the cut distance $\mathrm{cut}(P)$ from P^q. In [5] it is studied the Dirichlet eigenvalue problem (1) for $M = P^1_\rho$, where P is a complex curve of $\mathbb{C}P^n(\lambda)$ and an upper bound of $\mu_1(P_\rho)$ is obtained of the form

$$\mu_1(P^1_\rho) \leq \mu_1(\mathbb{C}P^1(\lambda)_\rho) - M^1(\rho, n, a_2, \ldots, a_q), \tag{2}$$

where $\mathbb{C}P^1(\lambda)$ is embedded as a totally geodesic complex submanifold of $\mathbb{C}P^n(\lambda)$, $M^1(\rho, n, 1, \ldots, 1) = 0$ and $M^1(\rho, n, a_2, \ldots, a_q) > 0$ for $(a_2, \ldots, a_n) \neq (1, \ldots, 1)$. Moreover, the equality is attained if and only if $P^1 = \mathbb{C}P^1(\lambda)$ (that is, $(a_2, \ldots, a_q) = (1, \ldots, 1)$).

This bound shows how $\mu_1(P_r)$ is related to the degrees of the polynomials defining P. Moreover it is also a comparison theorem with $\mu_1(\mathbb{C}P^1(\lambda)_r)$ and shows a gap phenomenon for $\mu_1(P^1_\rho)$ between the case $P^1 = \mathbb{C}P^1(\lambda)$ (which corresponds to $a_2 = \cdots = a_n = 1$) and the other complex submanifolds P, and states that the gap is measured by the degrees a_s (a similar gap phenomenon occurs in the study of the closed eigenvalue problem for complex submanifolds by J. P. Bourguignon, P. Li and S. T. Yau ([1])).

In this paper we address the problem of getting bounds of the same nature that (2) for higher dimensions of P^q ($q \geq 1$) and comparing with tubes around some model complex submanifolds. These models, denoted by $\mathfrak{P}$, are the complex submanifolds of $\mathbb{C}P^n(\lambda)$ having constant normal curvatures (constant means here that they do not depend on the point nor the direction). They were classified by Kimura ([12]) and are listed at the end of section 2. As in [5], we use the deep ideas in the work of A. Gray ([8], [9], [10]) to get theorems. But here we use them more from the root, which gives simpler computations and also more general results. Of course, another ingredient is the work of Kimura ([12]). We shall prove:

Theorem 1.1 *For $1 \leq q \leq n-1$, $q = \dim_{\mathbb{C}} P = \dim_{\mathbb{C}} \mathfrak{P}$, the first eigenvalue $\mu_1(P^q_\rho)$ of the Dirichlet eigenvalue problem* (1) *satisfies the inequality*

$$\mu_1(P^q_\rho) \leq \mu_1(\mathfrak{P}_\rho) + M_{\mathfrak{P}}(\rho, n, q, a_{q+1}, ..., a_n), \tag{3}$$

where $M_{\mathfrak{P}}(\rho, n, q, a_{q+1}, ..., a_n)$ is a well defined constant which depends only on $\mathfrak{P}, \rho, n, q$, $a_{q+1}, ..., a_n$. Moreover:

1. *$M_{\mathbb{C}P^q(\lambda)} \leq 0$ and the equality holds if and only if $P = \mathbb{C}P^q(\lambda)$, which is equivalent to $\sum_{i=q+1}^n a_i = n - q$. As a consequence,* (3) *gives a gap between $\mu_1(\mathbb{C}P^n(\lambda)_\rho)$ and the corresponding eigenvalues of tubes with the same radius around complex submanifolds defined by polynomials of higher degree.*

2. *When $q = n-1$ and $\mathfrak{P} = Q^{n-1}(\lambda)$ (the complex hyperquadric), there is a ρ_0, $0 < \rho_0 \leq \mathrm{cut}(P)$, depending on the degree a_n of the polynomial defining P, such that, for every $\rho < \rho_0$, $M_{Q^{n-1}(\lambda)} \leq 0$ if $P \neq \mathbb{C}P^n(\lambda)$ (which is equivalent to $a_n > 1$) and, in this case, equality holds if and only if the polynomial defining P has degree 2. As a consequence, for every $\rho < \rho_0$:*

   (a) $Q^{n-1}(\lambda)$ gives the biggest first eigenvalue of problem (1) *among all the tubes around a complex hypersurface defined by a homogeneous polynomial of degree 2, and*

   (b) For all the complex hypersurfaces defined by polynomials of degree ≥ 3, there is a gap between $\mu_1({Q^{n-1}(\lambda)}_\rho)$ and $\mu_1(P_\rho)$.

Remark 1.2 *We think that $\rho_0 = \mathrm{cut}(P)$, but we have not enough precise bounds for ρ_0 to assure it. For the other models $\mathfrak{P}$ different from $\mathbb{C}P^n(\lambda)$ and $Q^{n-1}(\lambda)$ we have no control on the term $M_{\mathfrak{P}}$. This is what makes us unable to obtain any kind of comparison theorem from* (3) *for these other models.*

If $\mathfrak{P}$ is not any of the models we have considered, the μ_1-eigenfunction is not radial, and our method cannot give an upper bound of $\mu_1(P_\rho) - \mu_1(\mathfrak{P}_\rho)$ depending only on $\mathfrak{P}$, n, q, ρ and the degrees of the polynomials defining P.

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2 Preliminaries on complex submanifolds and the tubes around them

Given a complex submanifold P of $\mathbb{C}P^n(\lambda)$ of real dimension $2q$, we shall denote by r the distance to P in $\mathbb{C}P^n(\lambda)$. Let us denote by $\mathcal{N}P$ the normal bundle of P, by A_ξ the Weingarten map of P in the direction of $\xi \in \mathcal{N}P$, $|\xi| = 1$. Besides, we consider the functions s_λ and c_λ defined by

$$\mathrm{s}_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}, \qquad \mathrm{c}_\lambda(t) = \cos(\sqrt{\lambda}t), \qquad \mathrm{ta}_\lambda(t) = \frac{\mathrm{s}_\lambda(t)}{\mathrm{c}_\lambda(t)}$$

which satisfy the computation rules $\mathrm{s}_\lambda' = \mathrm{c}_\lambda$ and $\mathrm{c}_\lambda^2 + \lambda \mathrm{s}_\lambda^2 = 1$.

Since P is a complex submanifold, given $\xi \in \mathcal{N}P$, $|\xi| = 1$, the Weingarten map A_ξ, has eigenvalues $k_1(\xi), -k_1(\xi), ..., k_q(\xi), -k_q(\xi)$. The trace of the Weingarten map $S(t)$ of ∂P_t is (cf. [10], page 125, formula (7.25)):

$$\operatorname{tr} S = 2\mathrm{s}_\lambda(\rho)\mathrm{c}_\lambda(\rho)\ h_P(\rho) - (2n-2q-1)\frac{\mathrm{c}_\lambda(\rho)}{\mathrm{s}_\lambda(\rho)} + \lambda \frac{\mathrm{s}_\lambda(\rho)}{c_\lambda(\rho)}, \tag{4}$$

$$\text{where } h_P(\rho) = \sum_{i=1}^{q} \frac{(\lambda + k_i^2)}{\mathrm{c}_\lambda^2(\rho) - k_i^2 \mathrm{s}_\lambda^2(\rho)} \tag{5}$$

On the other hand, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞ function, one has (cf. [15] for instance):

$$\Delta(f \circ r) = -f'' \circ r + \operatorname{tr} S\ f' \circ r \tag{6}$$

From now on we shall omit "$\circ\ r$" when it can be understood from the context.

The volume element ω of a tube P_ρ in Fermi coordinates around P can be written (cf. [10], page 125 formula (7.26)) as

$$\omega = \theta(p,\xi,r)\ d\xi\ dp\ dr, \text{ with } \theta(p,\xi,r) = \mathrm{s}_\lambda^{2n-2q-1}\ \mathrm{c}_\lambda\ v(p,\xi,r), \tag{7}$$

$$\text{where } v(p,\xi,r) = \prod_{j=1}^{q}(\mathrm{c}_\lambda^2 - \mathrm{s}_\lambda^2\ k_j(\xi)^2) = \mathrm{c}_\lambda^{2q} \prod_{j=1}^{q}(1 - \mathrm{ta}_\lambda^2\ k_j(\xi)^2),$$

where dp and $d\xi$ denote the volume elements of P and $S^{2n-2q-1}$, respectively. We remark that, for p and ξ fixed, the first positive value of r where $v(p,\xi,r)$ (and hence $\theta(p,\xi,r)$) vanishes is lower than $\operatorname{cut}(P)$.

Expanding the product in the above formula, one obtains

$$\prod_{i=1}^{q}(1 - \mathrm{ta}_\lambda^2\ k_i(\xi)^2) = \sum_{c=0}^{q} \Psi_{2c}(\xi,\ldots,\xi)\ \mathrm{ta}_\lambda^{2c}, \text{ where} \tag{8}$$

$$\Psi_{2c}(\xi,\ldots,\xi) = (-1)^c \sum_{\substack{i_1,\ldots,i_c = 1 \\ i_1 < \cdots < i_c}}^{q} k_{i_1}^2(\xi)\ldots k_{i_c}^2(\xi), \tag{9}$$

and satisfies (cf. [10], pages 65 and 125):

$$\int_{S^{2n-2q-1}} \Psi_{2c}(\xi,\ldots,\xi)\ d\xi =: I_{2c}(\Psi_{2c}) = a(c)\ \mathcal{C}^{2c}((R^P - R^{\mathbb{C}P^n(\lambda)})^c), \tag{10}$$

with

$$a(c) = \frac{2\pi^{n-q}}{c!\ (2c)!\ 2^c\ (n-q+c-1)!} \tag{11}$$

and $\mathcal{C}^{2c}((R^P - R^{\mathbb{C}P^n(\lambda)})^c)$ is a contraction of the curvature operator $(R^P - R^{\mathbb{C}P^n(\lambda)})^c$ which is related to the Chern form $\overline{\gamma}_c = \gamma_c(R^P - R^{\mathbb{C}P^n(\lambda)})$ of the curvature operator $(R^P - R^{\mathbb{C}P^n(\lambda)})$ by

$$b(c) := \int_P \mathcal{C}^{2c}((R^P - R^{\mathbb{C}P^n(\lambda)})^c)\; dp = \frac{i!\;(2c)!\;(2\pi)^c}{(q-c)!} \int_P \overline{\gamma}_c \wedge F^{q-c}\; dp. \tag{12}$$

One can look at [10] page 56 for a precise definition of $\mathcal{C}^{2c}$, and page 88 for the definition of γ_c. On the other hand, $\overline{\gamma}_c$ is related with the degrees a_j of the polynomials defining P (cf. [10] page 141) by

$$[\overline{\gamma}] = \left[\gamma(R^P - R^{\mathbb{C}P^n(\lambda)})\right] = \left[\frac{1}{\displaystyle\prod_{j=q+1}^{n}\left(1 + \frac{(a_j-1)\lambda}{\pi}F\right)}\right], \tag{13}$$

where $[\cdot]$ denotes the cohomology class of the corresponding differential form and the $\overline{\gamma}_c$ are defined from $\overline{\gamma}$ by

$$\overline{\gamma} = 1 + \overline{\gamma}_1 + \cdots + \overline{\gamma}_q + \ldots\;.$$

It follows that

$$1 = \left(1 + \overline{\gamma}_1 + \cdots + \overline{\gamma}_q + \ldots\right) \prod_{j=q+1}^{n}\left(1 + (a_j - 1)\frac{\lambda}{\pi}F\right),$$

and, for $c = 1, \ldots, q$,

$$\overline{\gamma}_c = (-1)^c \beta_c \left(\frac{\lambda}{\pi}F\right)^c \quad \text{where} \quad \beta_c = \sum_{\substack{j_1, \ldots, j_c = q+1 \\ j_1 \le \cdots \le j_c}}^{n} (a_{j_1} - 1)\ldots(a_{j_c} - 1) \tag{14}$$

By substitution of (14) in (12) one obtains

$$b(c) = (-1)^c \frac{c!\;(2c)!\;(2\pi)^c\;\lambda^c\;\beta_c}{(q-c)!\;\pi^c} \int_P F \wedge \overset{q}{\cdots} \wedge F\; dp = (-1)^c \frac{c!\;(2c)!\;2^c\;\;q!}{(q-c)!} \lambda^c\;\beta_c\; vol(P). \tag{15}$$

In [12], Kimura classified all the complex submanifolds of $\mathbb{C}P^n(\lambda)$ whose principal curvatures are constant in the sense that they depend neither on the point of the submanifold nor on the normal vector. They are:

- Totally geodesic $\mathbb{C}P^q(\lambda)$. It has $k_i = 0$ (16)
- The complex hyperquadric $Q^{n-1}(\lambda)$, where $k_i(\xi) = \sqrt{\lambda}$ for $i = 1, ..., n-1$. (17)
- $\mathbb{C}P^1(\lambda) \times \mathbb{C}P^{m-1}(\lambda) \subset \mathbb{C}P^{2m-1}(\lambda)$ for $m \geq 3$ (then $n = 2m-1 \geq 5$ (18)

   and $q = m$), where $k_1 = k_2 = \sqrt{\lambda}$, $k_3 = ... = k_m = 0$
- $SU(5)/S(U(3) \times U(2)) \subset \mathbb{C}P^9(\lambda)$ (then $n = 9$, $q = 6$), (19)

   where $k_1 = k_2 = k_3 = k_4 = \sqrt{\lambda}$, $k_5 = k_6 = 0$.
- $SO(10)/U(5) \subset \mathbb{C}P^{15}(\lambda)$ (then $n = 15, q = 10$), (20)

   where $k_1 = ... = k_6 = \sqrt{\lambda}$, $k_7 = ... = k_{10} = 0$

3 Proof of Theorem 1.1

We shall denote by $S_{\mathfrak{P}}$ the Weingarten map of a tubular hypersurface centered at a model comparison $\mathfrak{P}$ and by $f_{\mathfrak{P}}$ an eigenfunction corresponding to $\mu_1(\mathfrak{P}_\rho)$. According to (6), $f_{\mathfrak{P}}$ is the solution of the equation

$$-f''_{\mathfrak{P}} + \operatorname{tr} S_{\mathfrak{P}}\ f'_{\mathfrak{P}} = \mu_1(\mathfrak{P}_\rho) f_{\mathfrak{P}}, \qquad f_{\mathfrak{P}}(\rho) = 0. \qquad f'_{\mathfrak{P}}(0) = 0. \tag{21}$$

This function satisfies the inequalities (cf. [14] or [15] for instance)

$$f_{\mathfrak{P}} > 0 \text{ on } [0, \rho[\quad \text{and} \quad f'_{\mathfrak{P}} < 0 \text{ on }]0, \rho]. \tag{22}$$

By applying Raileigh's theorem using $f_{\mathfrak{P}} \circ r$ as a test function, we have

$$\mu_1(P_\rho) \leq \frac{\int_{P_\rho} f_{\mathfrak{P}}(\Delta f_{\mathfrak{P}})}{\int_{P_\rho} f_{\mathfrak{P}}^2}. \tag{23}$$

Let us compute the right hand side of the above inequality. From (4), (6) and (21), we get

$$\begin{aligned}\Delta(f_{\mathfrak{P}} \circ r) &= -f''_{\mathfrak{P}} \circ r + \operatorname{tr} S_{\mathfrak{P}}\ f'_{\mathfrak{P}} \circ r + (\operatorname{tr} S\ - \operatorname{tr} S_{\mathfrak{P}})\ f'_{\mathfrak{P}} \circ r \\ &= \mu_1(\mathfrak{P}_\rho) f_{\mathfrak{P}} + 2 \mathrm{s}_\lambda \circ r\ \mathrm{c}_\lambda \circ r\, (h_P - h_{\mathfrak{P}})\ f'_{\mathfrak{P}}.\end{aligned} \tag{24}$$

From (23), (24), (16) and (17) one gets

$$\begin{aligned}\mu_1(P_\rho) &\leq \mu_1(\mathfrak{P}_\rho) + \frac{\displaystyle\int_{P_\rho} f_{\mathfrak{P}}\ 2\ \mathrm{s}_\lambda\ \mathrm{c}_\lambda\, (h_P - h_{\mathfrak{P}})\ f'_{\mathfrak{P}}\ \omega}{\displaystyle\int_{P_\rho} f_{\mathfrak{P}}^2\ \omega} \\ &= \mu_1(\mathfrak{P}_\rho) + \frac{\displaystyle\int_0^\rho \int_P \int_{S^{2n-2q-1}} 2 f_{\mathfrak{P}} f'_{\mathfrak{P}}\ \mathrm{s}_\lambda^{2n-2q}\ \mathrm{c}_\lambda^2\ \ (h_P - h_{\mathfrak{P}})\ \ v(p, \xi, r)\ d\xi\ dp\ dr}{\displaystyle\int_0^\rho \int_P \int_{S^{2n-2q-1}} f_{\mathfrak{P}}^2\ \theta(p, \xi, r)\ d\xi\ dp\ dr},\end{aligned} \tag{25}$$

where we have used the expression (7) of the volume element ω of P^q_ρ in Fermi coordinates.

Let us work with the integrand of the numerator in (25). Using (7), (8) and (9) we get

$$
\begin{aligned}
h_P\ v(p,\xi,r) &= \left(\sum_{i=1}^{q} \frac{k_i(\xi)^2+\lambda}{\mathrm{c}_\lambda^2 - k_i(\xi)^2\mathrm{s}_\lambda^2}\right)\prod_{j=1}^{q}(\mathrm{c}_\lambda^2 - k_j(\xi)^2\mathrm{s}_\lambda^2) \\
&= \sum_{i=1}^{q}(k_i(\xi)^2+\lambda)\ \mathrm{c}_\lambda^{2q-2} \prod_{\substack{j=1\\ j\neq i}}^{q} (1-k_j(\xi)^2\mathrm{ta}_\lambda^2)
\end{aligned}
\tag{26}
$$

$$
h_{\mathfrak{P}}\ v(p,\xi,r) = h_{\mathfrak{P}} \prod_{j=1}^{q}(\mathrm{c}_\lambda^2 - k_j(\xi)^2\mathrm{s}_\lambda^2) = h_{\mathfrak{P}}\ \mathrm{c}_\lambda^{2q}\sum_{c=0}^{q}(-1)^c\ \mathrm{ta}_\lambda^{2c} \sum_{\substack{i_1,\dots,i_c=1\\ i_1<\cdots<i_c}}^{q} k_{i_1}^2\dots k_{i_c}^2
\tag{27}
$$

But a direct computation gives

$$
\sum_{i=1}^{q} k_i(\xi)^2 \prod_{\substack{j=1\\ j\neq i}}^{q} (1-k_j(\xi)^2\mathrm{ta}_\lambda^2) = \sum_{c=1}^{q}(-1)^{c-1}\ c\ \mathrm{ta}_\lambda^{2c-2} \sum_{\substack{i_1,\dots,i_c=1\\ i_1<\cdots<i_c}}^{q} k_{i_1}^2\dots k_{i_c}^2
\tag{28}
$$

and

$$
\sum_{i=1}^{q} \lambda \prod_{\substack{j=1\\ j\neq i}}^{q} (1-k_j(\xi)^2\mathrm{ta}_\lambda^2) = \lambda\sum_{c=1}^{q}(-1)^{c-1}\ (q-c+1)\ \mathrm{ta}_\lambda^{2c-2} \sum_{\substack{i_2,\dots,i_c=1\\ i_2<\cdots<i_c}}^{q} k_{i_2}^2\dots k_{i_c}^2
\tag{29}
$$

From (26), (28), (29), (9), (10) and (12) it follows

$$
\begin{aligned}
&\int_P\int_{S^{2n-2q-1}} h_P\ v(p,\xi,r) \\
&\quad = \mathrm{c}_\lambda^{2q-2}\left(-\sum_{i=1}^{q} i\ \mathrm{ta}_\lambda^{2i-2}\ a(i)\ b(i) + \lambda\sum_{i=1}^{q}(q-i+1)\ \mathrm{ta}_\lambda^{2i-2}\ a(i-1)\ b(i-1)\right)
\end{aligned}
\tag{30}
$$

and

$$
\int_P\int_{S^{2n-2q-1}} h_{\mathfrak{P}}\ v(p,\xi,r) = h_{\mathfrak{P}}\ \mathrm{c}_\lambda^{2q}\sum_{i=0}^{q}\mathrm{ta}_\lambda^{2i}\ a(i)\ b(i)
\tag{31}
$$

Now, after the substitution of (30) and (31) in the numerator of (25), having into account that $2f_{\mathfrak{P}}f'_{\mathfrak{P}} = (f^2_{\mathfrak{P}})'$, we compute for that numerator:

$$\int_0^\rho \int_P \int_{S^{2n-2q-1}} 2 f_{\mathfrak{P}} f'_{\mathfrak{P}}\ \mathrm{s}_\lambda^{2n-2q}\ \mathrm{c}_\lambda^2\ \left(h_P - h_{\mathfrak{P}}\right)\quad v(p,\xi,r)\ d\xi\ dp\ dr$$

$$= \int_0^\rho (f_{\mathfrak{P}}^2)' \mathrm{s}_\lambda^{2n-2q}\ \mathrm{c}_\lambda^{2q} \left(- \sum_{i=1}^{q} i\ \mathrm{ta}_\lambda^{2i-2}\ a(i)\ b(i) + \lambda \sum_{i=1}^{q} (q-i+1)\ \mathrm{ta}_\lambda^{2i-2}\ a(i-1)\ b(i-1) \right.$$
$$\left. - h_{\mathfrak{P}}\ \mathrm{c}_\lambda^2 \sum_{i=0}^{q} \mathrm{ta}_\lambda^{2i}\ a(i)\ b(i) \right) dr$$

$$= \int_0^\rho (f_{\mathfrak{P}}^2)'\ \mathrm{s}_\lambda^{2n-2q}\ \mathrm{c}_\lambda^{2q} \sum_{i=0}^{q} \left(-i\ \mathrm{ta}_\lambda^{2i-2} + \lambda\ (q-i)\ \mathrm{ta}_\lambda^{2i} - h_{\mathfrak{P}}\ \mathrm{c}_\lambda^2 \mathrm{ta}_\lambda^{2i}\right) a(i)\ b(i)\ dr =: \mathcal{N} \tag{32}$$

But in all our comparison models $\mathfrak{P}$ the possible values for k_i are $\pm\sqrt{\lambda}$ or 0. Let $\mathfrak{z}$ be the number of normal curvatures with value $\sqrt{\lambda}$. With this conventon, it follows from (5) that the expression for $h_{\mathfrak{P}}\ \mathrm{c}_\lambda^2$ is

$$h_{\mathfrak{P}}\ \mathrm{c}_\lambda^2 = \mathfrak{z}\ \lambda \frac{1}{\mathrm{c}_{4\lambda}} + q\lambda \tag{33}$$

We continue with the above computation (32)

$$\mathcal{N} = \int_0^\rho (f_{\mathfrak{P}}^2)' \mathrm{s}_\lambda^{2n-2q}\ \mathrm{c}_\lambda^{2q} \sum_{i=0}^{q} \left(-i\ \mathrm{ta}_\lambda^{2i-2} \mathrm{c}_\lambda^{-2} + \lambda\ q\ \mathrm{ta}_\lambda^{2i} - \left(\mathfrak{z}\ \lambda \frac{1}{\mathrm{c}_{4\lambda}} + q\lambda\right) \mathrm{ta}_\lambda^{2i}\right) a(i)\ b(i)\ dr$$

$$= \sum_{i=0}^{q} \int_0^\rho (f_{\mathfrak{P}}^2)' \left(-i\ - \mathfrak{z}\lambda \frac{\mathrm{s}_\lambda^2}{\mathrm{c}_{4\lambda}}\right) \mathrm{s}_\lambda^{2(n-q+i-1)} \mathrm{c}_\lambda^{2(q-i)}\ a(i)\ b(i)\ dr$$

$$= \sum_{i=0}^{q} \int_0^\rho (f_{\mathfrak{P}}^2)' \left(\frac{-i\ \mathrm{c}_\lambda^2 + (-\mathfrak{z} + i)\lambda \mathrm{s}_\lambda^2}{\mathrm{c}_{4\lambda}}\right) \mathrm{s}_\lambda^{2(n-q+i-1)} \mathrm{c}_\lambda^{2(q-i)}\ a(i)\ b(i)\ dr$$

$$= \int_0^\rho (f_{\mathfrak{P}}^2)' \frac{1}{\mathrm{c}_{4\lambda}} \mathrm{s}_\lambda^{2(n-q-1)} \mathrm{c}_\lambda^{2q} \sum_{i=0}^{q} \left(-i\ \mathrm{c}_\lambda^2 + (i-\mathfrak{z})\lambda \mathrm{s}_\lambda^2\right) \mathrm{ta}_\lambda^{2i}\ a(i)\ b(i)\ dr \tag{34}$$

On the other hand, for the denominator of (25), from (7), the remark after (7), (8), (10) and (12) it follows:

$$0 < \int_0^\rho \int_P \int_{S^{2n-2q-1}} f_{\mathfrak{P}}^2 \theta(p,\xi,r)\ d\xi\ dpdr = \int_0^\rho f_{\mathfrak{P}}^2\ \mathrm{s}_\lambda^{2n-2q-1} \mathrm{c}_\lambda^{2q+1} \sum_{i=0}^{q} \mathrm{ta}_\lambda^{2i}\ a(i)\ b(i)\ dr$$
$$= \sum_{i=0}^{q} a(i)\ b(i) \int_0^\rho f_{\mathfrak{P}}^2 \mathrm{s}_\lambda^{2n-2q+2i-1} \mathrm{c}_\lambda^{2q-2i+1}\ dr \tag{35}$$

Now we shall use the notation

$$\mu_{\mathfrak{P}} = f_{\mathfrak{P}}^2\ \mathrm{s}_\lambda^{2n-2q-1}\ \mathrm{c}_\lambda^{2q+1} \quad \text{and} \quad \nu_{\mathfrak{P}} = (f_{\mathfrak{P}}^2)' \frac{\mathrm{s}_\lambda^{2n-2q-1}\ \mathrm{c}_\lambda^{2q}}{\mathrm{c}_{4\lambda}}. \tag{36}$$

Substitution of (34) and (35) in (25) gives

$$\mu_1(P_\rho) \le \mu_1(\mathfrak{P}_\rho) + \frac{\displaystyle\int_0^\rho \sum_{i=0}^{q} \left(-i\ \mathrm{c}_\lambda^2 + (i-\mathfrak{z})\lambda \mathrm{s}_\lambda^2\right) \mathrm{ta}_\lambda^{2i}\ a(i)\ b(i) \nu_{\mathfrak{P}}\ dr}{\displaystyle\sum_{i=0}^{q} \int_0^\rho \mathrm{s}_\lambda^{2i} \mathrm{c}_\lambda^{-2i}\ a(i)\ b(i)\ \mu_{\mathfrak{P}}\ dr}, \tag{37}$$

From the formulae (11) and (15) for $a(i)$ and $b(i)$ it follows that

$$\frac{a(i)\ b(i)}{a(0)\ b(0)} = \frac{(-1)^i\ (n-q-1)!\ q!\ \lambda^i\ \beta_i}{(q-i)!\ (n-q+i-1)!} = (-1)^i \frac{\binom{n-1}{q-i}}{\binom{n-1}{q}} \lambda^i\ \beta_i. \tag{38}$$

Then

$$\begin{aligned}
&\frac{\binom{n-1}{q}}{a(0)b(0)} \sum_{i=0}^{q} \left(-i\ \mathrm{c}_\lambda^2 + (i-\mathfrak{z})\lambda \mathrm{s}_\lambda^2\right) \mathrm{ta}_\lambda^{2i}\ a(i)\ b(i) \\
&= \sum_{i=0}^{q} \left(-i\ \mathrm{c}_\lambda^2 + (i-\mathfrak{z})\lambda \mathrm{s}_\lambda^2\right) \mathrm{ta}_\lambda^{2i}\ (-1)^i \binom{n-1}{q-i} \lambda^i\ \beta_i \\
&= -\ \lambda\ \mathfrak{z}\ \mathrm{s}_\lambda^2 \binom{n-1}{q} \\
&\quad - \left(-\mathrm{s}_\lambda^2 + (1-\mathfrak{z})\lambda \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^2\right) \lambda \binom{n-1}{q-1}\ \beta_1 \\
&\quad + \left(-2\mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^2 + (2-\mathfrak{z})\lambda \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^4\right) \lambda^2 \binom{n-1}{q-2}\ \beta_2 \\
&\quad - \left(-3\mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^4 + (3-\mathfrak{z})\lambda \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^6\right) \lambda^3 \binom{n-1}{q-3}\ \beta_3 \\
&\quad \cdots\cdots\cdots \\
&\quad + (-1)^q \left(-q \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^{2q-2} + (q-\mathfrak{z})\lambda \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^{2q}\right) \lambda^q\ \beta_q \\
&= \sum_{i=0}^{q-1} (-1)^i \lambda^{i+1} \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^{2i} \left((i-\mathfrak{z})\ \binom{n-1}{q-i} \beta_i + (i+1) \binom{n-1}{q-i-1} \beta_{i+1}\right) \\
&\qquad + (-1)^q (q-\mathfrak{z})\lambda \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^{2q} \lambda^q\ \beta_q
\end{aligned} \tag{39}$$

Multiplying numerator and denominator of (37) by $\dfrac{\binom{n-1}{q}}{a(0)\ b(0)}$, having into account (11) and (15), we obtain

$$\mu_1(P_\rho) \leq \mu_1(\mathfrak{P}_\rho) + \frac{\sum_{i=0}^{q-1} (-1)^i \lambda^{i+1} \left((i-\mathfrak{z}) \binom{n-1}{q-i} \beta_i + (i+1) \binom{n-1}{q-i-1} \beta_{i+1} \right) B_i(\rho) + (-1)^q (q-\mathfrak{z}) \lambda \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^{2q} \lambda^q \ \beta_q B_q(\rho)}{\sum_{i=0}^{q} (-1)^i \binom{n-1}{q-i} \lambda^i \beta_i C_i(\rho)} \tag{40}$$

where $B_i(\rho) = \int_0^\rho \mathrm{s}_\lambda^2 \mathrm{ta}_\lambda^{2i} \ \nu_{\mathfrak{P}} \ dr$ and $C_i(\rho) = \int_0^\rho \mathrm{s}_\lambda^{2i} \mathrm{c}_\lambda^{-2i} \mu_{\mathfrak{P}} \ dr$.

When $\mathfrak{P} = \mathbb{C}P^q(\lambda)$, $\mathfrak{z} = 0$, and in order to study the sign of $M_{\mathbb{C}P^q(\lambda)}$ we consider its expression given by the second summand in (25). In this case, from the definition (5) of h_P, one has $h_P - h_{\mathbb{C}P^q(\lambda)} = \sum_{i=1}^q \frac{k_i^2}{\mathrm{c}_\lambda^2(\mathrm{c}_\lambda^2 - k_i^2 \mathrm{s}_\lambda^2)} \geq 0$ for $r < \mathrm{cut}(P)$. Since also $\theta \geq 0$ for $r < \mathrm{cut}(P)$ and $f'_{\mathfrak{P}} < 0$ on $]0, \rho]$, one has that $M_{\mathbb{C}P^q(\lambda)} \leq 0$. On the other hand, the equality in (3) implies the equality in (23), which implies that $f_{\mathbb{C}P^q(\lambda)}$ is an eigenfunction with eigenvalue $\mu_1(\mathbb{C}P^q(\lambda))$. From this and (24), it follows that $h_P - h_{\mathbb{C}P^q(\lambda)} = 0$, and, from the above expression this can happen only if $k_i = 0$, that is, if $P = \mathbb{C}P^q(\lambda)$.

When $\mathfrak{P} = Q^{n-1}(\lambda)$, one has $\mathfrak{z} = q = n-1$ and there is only one polynomial defining P, with degree a_n, then $\beta_i = (a_n - 1)^i$ and the numerator of $M_{Q^{n-1}(\lambda)}$ becomes

$$\begin{aligned} &\sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \left((i-(n-1)) \binom{n-1}{n-1-i} (a_n-1)^i + (i+1) \binom{n-1}{n-1-i-1} (a_n-1)^{i+1} \right) B_i(\rho) \\ &\quad = \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \left((i-(n-1)) \binom{n-1}{n-1-i} + (i+1) \binom{n-1}{n-1-i-1} (a_n-1) \right) (a_n-1)^i B_i(\rho) \end{aligned} \tag{41}$$

But

$$\begin{aligned} (i-(n-1)) \binom{n-1}{n-1-i} &= -(n-1-i) \binom{n-1}{n-1-i} \\ &= -\frac{(n-1)\cdots(n-i-1)}{i!} = -(i+1) \binom{n-1}{n-1-i-1}, \end{aligned} \tag{42}$$

which, substituted in (41) gives

$$\begin{aligned} &\sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \frac{(n-1)\cdots(n-i-1)}{i!} (-1 + a_n - 1)(a_n-1)^i B_i(\rho) \\ &= (a_n - 2) \sum_{i=0}^{n-2} (-1)^i \lambda^{i+1} \frac{(n-1)\cdots(n-i-1)}{i!} (a_n-1)^i B_i(\rho) \end{aligned} \tag{43}$$

Then $M_{Q^{n-1}(\lambda)}$ vanishes when $a_n = 2$. As a consequence, among all the complex hypersurfaces P defined by a polynomial of degree 2, the complex hyperquadric gives the maximum value of $\mu_1(P)$.

Now, let us study the sign of $M_{Q^{n-1}(\lambda)}$. First, recall that its denominator is positive for $\rho < \mathrm{cut}(P)$ (as we noticed in (35)). To check the sign of the numerator, the observation that when $2[(n-2)/2]+1 > n-2$, $n-2[(n-2)/2]-2 = 0$, allows us to write the sum in (43) in the following way:

$$\sum_{i=0}^{n-2}(-1)^i\lambda^{i+1}\frac{(n-1)\cdots(n-i-1)}{i!}(a_n-1)^i B_i(\rho)$$
$$= \sum_{j=0}^{[(n-2)/2]}(-1)^{2j}\lambda^{2j+1}\frac{(n-1)\cdots(n-2j-1)}{(2j)!}(a_n-1)^{2j}\left(B_{2j}(\rho)-\lambda\frac{(n-2j-2)}{(2j+1)}(a_n-1)B_{2j+1}(\rho)\right)$$
$$= \sum_{j=0}^{[(n-2)/2]}\lambda^{2j+1}\frac{(n-1)\cdots(n-2j-1)}{(2j)!}(a_n-1)^{2j}\int_0^\rho \mathrm{s}_\lambda^2\mathrm{ta}_\lambda^{4j}\ \nu_{\mathfrak{P}}\left(1-\lambda\frac{(n-2j-2)}{(2j+1)}(a_n-1)\mathrm{ta}_\lambda^2\right)\,dr$$

which is negaitive for $\rho \le \rho_1 := \min_{0\le j\le[(n-2)/2]} \mathrm{ta}_\lambda^{-1}\left(\sqrt{\frac{2j+1}{\lambda(n-2j-2)}}\right)$, where ta_λ^{-1} means the inverse function of ta_λ with image in $[0,\pi/2\sqrt{\lambda}[$. Then, taking $\rho_0 = \min\{\mathrm{cut}(P),\rho_1\}$, we have that $M_{Q^{n-1}(\lambda)} < 0$ for $a_n \ge 3$. This gives a gap between $\mu_1(Q^{n-1}(\lambda)_\rho)$ and $\mu_1(P_\rho)$ for all complex hypersurfaces defined by polynomials of degree ≥ 3.

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