COEFFICIENT MULTIPLIERS ON BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. Motivated by an old paper of Wells [J. London Math. Soc. 2 (1970), 549–556] we define the space $X \otimes Y$, where X and Y are "homogeneous" Banach spaces of analytic functions on the unit disk $\mathbb{D}$, by the requirement that f can be represented as $f = \sum_{j=0}^{\infty} g_n * h_n$, with $g_n \in X$, $h_n \in Y$ and $\sum_{n=1}^{\infty} \|g_n\|_X \|h_n\|_Y < \infty$. We show that this construction is closely related to coefficient multipliers. For example, we prove the formula $((X \otimes Y), Z) = (X, (Y, Z))$, where (U, V) denotes the space of multipliers from U to V, and as a special case $(X \otimes Y)^* = (X, Y^*)$, where $U^* = (U, H^\infty)$. We determine $H^1 \otimes X$ for a class of spaces that contains H^p and ℓ^p $(1 \le p \le 2)$, and use this together with the above formulas to give quick proofs of some important results on multipliers due to Hardy and Littlewood, Zygmund and Stein, and others.

1. Introduction

Let $\mathcal{S}$ denote the space of all (formal) power series $f = \sum_{j=0}^{\infty} \hat{f}(j) z^j = \{\hat{f}(j)\}_{j=0}^{\infty}$ with complex-valued coefficients. We introduce the locally convex vector topology on X by means of the seminorms $p_j(f) = \hat{f}(j)$, $j \ge 0$. Thus $f_n \to f$ $(n \to \infty)$ in $\mathcal{S}$ if and only if $\hat{f}_n(j) \to \hat{f}(j)$ for each j. Then $\mathcal{S}$ is metrizable and complete and therefore it is an F-space. The Hadamard product of f and g is defined as

$$f * g = \sum_{j=0}^{\infty} \hat{f}(j) \hat{g}(j) z^j.$$

A Banach space X will be called *$\mathcal{S}$-admissible* if $\mathcal{P}$, the set off all polynomials, is contained in X, and $X \subset \mathcal{S}$ with continuous inclusion.

Let $X_{\mathcal{P}}$ denote the closure of $\mathcal{P}$ in X, $e_j(z) = z^j$ and $\gamma_j(f) = \hat{f}(j)$ for $j \ge 0$. Of course if X is $\mathcal{S}$-admissible so it is $X_{\mathcal{P}}$. On the other hand for an $\mathcal{S}$-admissible Banach space X one has that $e_j \in X$ and $\gamma_j \in X'$, where X' stands for the topological dual space. Hence $(X_{\mathcal{P}})'$ is also an $\mathcal{S}$-admissible Banach space, identifying $\phi \in X'$ with the power series $\phi(z) = \sum_j \phi(e_j) z^j$.

Note that ℓ^p, $1 \le p \le \infty$, the space of all complex sequences $a = \{\hat{a}(j)\}_0^{\infty}$ such that $\|a\|_{\ell^p} := \left(\sum_{j=0}^{\infty} |\hat{a}(j)|^p \right)^{1/p} < \infty$, can be regarded as a subspace of $\mathcal{S}$, denoted $A(\mathbb{T})$ for $p = 1$, by putting $a = \sum_{j=0}^{\infty} \hat{a}(j) z^j$. Further examples of

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$\mathcal{S}$-admissible spaces are $c_0 = (\ell^\infty)_{\mathcal{P}}$, H^∞, i.e. the space of bounded analytic functions, $A(\mathbb{D}) = (H^\infty)_{\mathcal{P}}$, and $\mathcal{A}$, the space of Abel summable series (i.e. there exists $\lim_{r\to 1}\sum_{n=0}^\infty \hat{f}(n)r^n$ with the norm given by $\|f\|_{\mathcal{A}} = \sup_{n\geq 0}|\sum_{j=0}^n \hat{f}(j)| < \infty$).

Given two $\mathcal{S}$-admissible Banach spaces X, Y we denote

$$(X, Y) = \{\lambda \in \mathcal{S} : \lambda * f \in Y \text{ for all } f \in X\}.$$

Then (X, Y) becomes an $\mathcal{S}$-admissible Banach space with its natural norm (see Theorem 2.1).

We keep the notation X' for the topological dual and denote $X^K = (X, A(\mathbb{T}))$ (the Köthe dual), $X^* = (X, H^\infty)$, $X^\# = (X, A(\mathbb{D}))$ and $X^a = (X, \mathcal{A})$ (the Abel dual).

Since H^∞, $A(\mathbb{T})$, $A(\mathbb{D})$ and $\mathcal{A}$ are $\mathcal{S}$-admissible Banach spaces then $X^K, X^*, X^\#$ and X^a are also $\mathcal{S}$-admissible Banach spaces.

Following Wells [34] (see also [11] and [33, Sections V.4, VI.3]), given X and Y $\mathcal{S}$-admissible Banach spaces we define $X \otimes Y$ as the space of series $h \in \mathcal{S}$ such that $h = \sum_{n=0}^\infty f_n * g_n$, where the series converges in $\mathcal{S}$, $f_n \in X$, $g_n \in Y$ and $\sum_{n=0}^\infty \|f_n\|_X \|g_n\|_Y < \infty$. It is not difficult to see that $X \otimes Y$, normed in a natural way, is also $\mathcal{S}$-admissible (see Theorem 2.2).

We shall show in the paper a quite useful formula connecting multipliers and tensors of $\mathcal{S}$-admissible Banach spaces (see Theorem 2.3)

$$(X \otimes Y, Z) = (X, (Y, Z)). \tag{1.1}$$

We are mainly interested in the case where X and Y are Banach spaces of analytic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$, i.e., $f = \sum \hat{f}(j)z^j$ with $\limsup_j \sqrt[j]{|\hat{f}(j)|} \leq 1$. Let $\mathbb{D}_R \subset \mathbb{C}$ denote the open disk of radius R centered at zero (we put $\mathbb{D}_1 = \mathbb{D}$) and let E be a complex Banach space. We write $\mathcal{H}(\mathbb{D}_R)$ (respect. $\mathcal{H}(\mathbb{D}_R, E)$) for the vector space of all functions analytic in $\mathbb{D}_R$ (respect. with values in E), which endowed with "$\mathcal{H}$-topology", i.e., the topology of uniform convergence on compact subsets of $\mathbb{D}_R$, becomes a locally convex F-space. This topology can be described by the family of the norms $N_\rho(f) = \sup_{|z|<\rho} \|f(z)\|_E$, $0 < \rho < R$. Since $\mathcal{H}(\mathbb{D}_R) \subset \mathcal{S}$, we see that, formally, there are two topologies on $\mathcal{H}(\mathbb{D}_R)$: $\mathcal{H}$-topology and $\mathcal{S}$-topology. However, it is well known and easy to see that they coincide on $\mathcal{H}(\mathbb{D}_R)$.

Several authors have formulated some natural conditions (which hold in most of classical spaces such as Hardy, Bergman, Besov, etc.) to develop a general theory of spaces of analytic functions. Two basic ones first appeared in the work by A.E. Taylor (see [29]) are the following:

(P1) There exists $A_1 > 0$ such that $|\hat{f}(j)| \leq A_1\|f\|$, $j \in \{0, 1, \ldots\}$.

(P2) There exists $A_2 > 0$ such that $\|e_j\| \leq A_2$, $j \in \{0, 1, \ldots\}$.

This perfectly fitted with Hardy spaces (see [30]) but, unfortunately these conditions are too restrictive to include many of the interesting spaces appearing in the literature. We shall propose in this paper some weaker ones.

A Banach space $X \subset \mathcal{S}$ will be called *$\mathcal{H}$-admissible* if $X \subset \mathcal{H}(\mathbb{D})$ with continuous inclusion, $\mathcal{H}(\mathbb{D}_R) \subset X$ for all $R > 1$, and the map $f \mapsto f|_{\mathbb{D}}$ is continuous from $\mathcal{H}(\mathbb{D}_R)$ to X.

Clearly $\mathcal{H}$-admissible spaces are also $\mathcal{S}$-admissible. Denote, as usual, $C(z) = \frac{1}{1-z}$ the Cauchy kernel and $f_w(z) = f(wz)$ for $w \in \bar{\mathbb{D}}$. In particular $f_r = C_r * f$.

We shall show that in the setting of $\mathcal{H}$-admissible Banach spaces, the map $w \to f_w$ defines an X-valued analytic function, i.e. $F(w) = f_w \in \mathcal{H}(\mathbb{D}, X)$. In particular

$$M_X(r, f) = \sup_{|w|=r} \|f_w\|_X$$

becomes an increasing function (where, as usual, we denote $M_p(r, f)$ for the Hardy spaces $X = H^p$). We shall pay special attention to the subspace of functions such that $F \in H^\infty(\mathbb{D}, X)$ and denote

$$\tilde{X} = \{f \in \mathcal{H}(\mathbb{D}) : \sup_{0<r<1} M_X(r, f) < \infty\}.$$

Of course if X and Y are $\mathcal{H}$-admissible then (X, Y) and $X \otimes Y$ are also $\mathcal{H}$-admissible (see Theorem 3.1).

Inspired by the Besov-type spaces we denote, for $1 \le q \le \infty$, by $\mathfrak{B}^{X,q}$ the space of functions in $\mathcal{H}(\mathbb{D})$ such that $(1 - r^2)M_X(r, Df) \in L^q((0, 1), \frac{rdr}{1-r^2})$ where $Df(z) = \sum_{n=0}^{\infty}(n + 1)\hat{f}(n)z^n$.

It is clear that $\tilde{X}$ and $\mathfrak{B}^{X,q}$ are also $\mathcal{H}$-admissible Banach spaces. It fact they automatically have better properties.

In the original paper A.E. Taylor also considered some particular properties (see [29]):

(P3) If $f \in X$ then $f_{e^{i\theta}} \in X$ and $\|f_{e^{i\theta}}\|_X = \|f\|_X$, $\theta \in [0, 2\pi]$.

(P4) If $f \in X$ then $f_r \in X$ with $\|f_r\|_X \le A_4\|f\|_X$, $0 \le r < 1$, for some $A_4 > 0$.

In this paper we propose a general class of $\mathcal{H}$-admissible Banach spaces of analytic functions, which cover many of the classical function spaces, and is well-adapted to the study of multipliers.

We shall say that an $\mathcal{H}$-admissible Banach space X is *homogeneous* if (P3) and (P4) holds, that is, it satisfies $\|f_\xi\|_X = \|f\|_X$ for all $|\xi| = 1$ and $f \in X$, and $M_X(r, f) \le K\|f\|_X$ for all $0 \le r < 1$ and $f \in X$.

That is to say, for homogeneous spaces, $w \to f_w$ defines a function in $H^\infty(\mathbb{D}, X)$. In particular $X \subset \tilde{X}$.

Note that the spaces $\tilde{X}$ and $\mathfrak{B}^{X,q}$ become automatically homogeneous for any $\mathcal{H}$-admissible Banach space X. Of course if X and Y are homogeneous so are (X, Y) and $X \otimes Y$.

We shall also show in this setting that (see Theorem 7.1)

$$\mathfrak{B}^{X,1} \subset H^1 \otimes X \subset X_{\mathcal{P}} \tag{1.2}$$

or that (see Theorem 4.1)

$$(\mathfrak{B}^{X,1}, Y) = \mathfrak{B}^{(X,Y),\infty}. \tag{1.3}$$

Many more properties are relevant according to the problem in study. For instance, the class of spaces invariant under Moebious transformations or G-invariant spaces, i.e. $X \subset \mathcal{H}(\mathbb{D})$ such that there exists $K > 0$ such that $\|f \circ \phi\|_X \le K\|f\|_X$ whenever $f \in X$ and ϕ belongs to the group of Moebious transformation of $\mathbb{D}$, have been considered by several authors (see [3, 12, 31]). Among the G-invariant spaces there are maximal and minimal spaces in the scale, namely the Bloch space and the Besov class (see [6, 26, 32]). Similarly, in our setting of homogeneous Banach spaces of analytic functions one has (see Proposition 4.3) that

$$\mathfrak{B}^{X,1} \subset X_{\mathcal{P}} \subset \tilde{X} \subset \mathfrak{B}^{X,\infty}.$$

Let us finally recall some extra properties also considered by Taylor:

(P5) If $f \in X$ then $f_r \in X$ and $\|f\|_X = \lim_{r\to 1} \|f_r\|_X$.

(P6) If $f \in X$ then $f_r \in X$ and $\lim_{r\to 1} \|f_r - f\|_X = 0$.

Of course these two conditions are connected to the density of polynomial in the space X. In fact if X is $\mathcal{H}$-admissible then $X_{\mathcal{P}}$ satisfies (P6) (and therefore (P5)).

Another one which appears naturally is the following:

(P7) If $f \in \mathcal{H}(\mathbb{D})$ satisfies that $f_r \in X$ and $\sup_{r\to 1} \|f_r\|_X < \infty$ then $f \in X$ and $\|f\|_X = \lim_{r\to 1} \|f_r\|_X$.

This is satisfied by $\tilde{X}$ and $\mathfrak{B}^{X,q}$. Clearly c_0 or $A(\mathbb{D})$ fail this property. We shall consider a variation of (P7) useful for our purposes. An homogeneous space X is said to have (F)-*property* (Fatou property) if there exists $A > 0$ such that for any sequence $(f_n) \in X$ with $\sup_n \|f_n\|_X \le 1$ and $f_n \to f$ in $\mathcal{H}(\mathbb{D})$ one has that $f \in X$ and $\|f\|_X \le A$. (F)-property will be shown to be equivalent to the fact that $X = \tilde{X}$ or $X = X^{**}$ with equivalent norms (see Proposition 5.1).

One of our main goals is to characterize $H^1 \otimes X$. In order to do that we shall consider a new property, namely, we say that X has the (HLP)-property if $X \subset \mathfrak{B}^{X,2}$. For instance ℓ^q fails to have (HLP) for $q > 2$, because $\mathfrak{B}^{\ell^q,2} = \ell(q,2)$ (see Proposition 3.6), and H^p has (HLP) for $1 \le p \le 2$ due to the Hardy and Littlewood result (see [10, 15]) states that, for $1 \le p \le 2$,

$$\int_0^1 (1-r^2) M_p^2(r, f') r dr \le C \|f\|_p^2, \quad f \in H^p.$$

The vector-valued version of the Hardy-Littlewood theorem was considered in [5]. A Banach space E was said to have the (HL)-property if

$$\int_0^1 (1-r^2) M_1^2(r, F') r dr \le C \|F\|^2_{H^1(\mathbb{D},X)}, \quad F \in H^1(\mathbb{D}, E).$$

Since $F(w) = f_w \in H^\infty(\mathbb{D}, X)$ and $\|F\|_{H^1(\mathbb{D},X)} = \|f\|_X$ for any $f \in X$ and any homogeneous space X, one concludes that any homogeneous Banach space X having the (HL)-property satisfies (HLP). The reader is referred to [5] for examples of such spaces and connections with other properties in Banach space theory. In particular it was shown ([5, Prop. 4.4]) that $L^p(\mu)$ has (HL) if and only if $1 \le p \le 2$. Therefore, besides Hardy spaces, also Bergman spaces $X = A^p$ or $X = \ell^p$ for $1 \le p \le 2$ and many other obtained via interpolation satisfy (HLP).

We shall show that if X has (HLP) property then (see Theorem 7.2)

$$H^1 \otimes X = \mathfrak{B}^{X,1}. \tag{1.4}$$

A combination of our main results (1.4), (1.1) and (1.3) allow us to recover a number of know results about multipliers. Namely, for spaces with (HLP) one has

$$(H^1, X^*) = (X, BMOA) = \mathfrak{B}^{X^*,\infty}.$$

From this one can recapture many known results on multipliers and to obtain new ones selecting other spaces with (HLP).

The paper is organized as follows: Sections 2 is devoted to introduce and prove the basic properties about the $\mathcal{S}$-admissibility showing there the basic formula (1.1). Section 3 deals with the notion of $\mathcal{H}$-admissibility . We also introduce in that section the spaces $\tilde{X}$ and $\mathfrak{B}^{X,q}$. We deal with the notion of homogeneous Banach spaces in Section 4, showing there the basic result of multipliers (1.3). The Fatou property is studied in Section 5. In Section 6 we present some new facts on "solid" spaces (introduced and studied by Anderson and Shields [2]). We use Section 7 to study the space $H^1 \otimes X$ and to show (1.2) and (1.4). Finally Section 8 is devoted to applications.

2. $\mathcal{S}$-admissible Banach spaces: Multipliers and tensors

Definition 2.1. *A Banach space X will be called $\mathcal{S}$-*admissible *if $\mathcal{P} \subset X$ and $X \subset \mathcal{S}$ with continuous inclusion, i.e. for each $j \geq 0$ there exists C_j such that $|\hat{f}(j)| \leq C_j \|f\|_X$.*

Definition 2.2. *Let X and Y be $\mathcal{S}$-admissible Banach spaces. A series $\lambda \in \mathcal{S}$ is said to be a (coefficient) multiplier from X to Y if $\lambda * f \in Y$ for each $f \in X$.*

We denote the set of all multipliers from X to Y by (X, Y) and define

$$\|\lambda\|_{(X,Y)} = \sup\{ \|\lambda * f\|_Y : \|f\|_X \leq 1 \}.$$

Theorem 2.1. *If X and Y are $\mathcal{S}$-admissible then (X, Y) is an $\mathcal{S}$-admissible Banach space.*

Proof. An application of the closed graph theorem shows that the functional $\|\cdot\|_{(X,Y)}$ is finite. That $\|\lambda\|_{(X,Y)} = 0$ implies $\lambda = 0$ follows the condition $\mathcal{P} \subset X$. The other properties of the norm are immediate consequences of the definition. Also, it is clear that $\mathcal{P} \subset (X, Y)$. That the inclusion $(X, Y) \subset \mathcal{S}$ is continuous follows from the inequality

$$|\hat{\lambda}(j)| = |(\widehat{\lambda * e_j})(j)| \leq C_j \|\lambda * e_j\|_Y \leq C_j \|e_j\|_X \|\lambda\|_{(X,Y)}.$$

Finally, to prove that (X, Y) is complete, assume that

$$\|\lambda_m - \lambda_n\|_{(X,Y)} \to 0 \quad \text{as } m, n \to \infty. \tag{+}$$

This implies that there is a bounded linear operator $T : X \mapsto Y$ such that $\|T - T_n\| \to 0$ as $n \to \infty$, where the linear operator T_n is defined by $T_n f = \lambda_n * f$. Hence $\|Tf - \lambda_n * f\|_Y \to 0$ as $n \to \infty$, for each $f \in X$. Since the inclusion $Y \subset \mathcal{S}$ is continuous, we see that

$$\lambda_n * f \to Tf \text{ in } \mathcal{S}. \tag{$*$}$$

On the other hand, from (+) and the continuity of the inclusion $(X, Y) \subset \mathcal{S}$ it follows that $\lambda_m - \lambda_n \to 0$ $(m, n \to \infty)$ in $\mathcal{S}$, which implies that there is a $\lambda \in \mathcal{S}$ such that $\lambda_n * f \to \lambda * f$ in $\mathcal{S}$. This and $(*)$ show that $Tf = \lambda * f$, which completes the proof. □

We have another procedure of generate $\mathcal{S}$-admissible Banach spaces.

Definition 2.3. *We define the space $X \otimes Y$, to be the set of all $h \in \mathcal{S}$ that can be represented in the form $h = \sum_{n=0}^{\infty} f_n * g_n$, $\quad f_n \in X$, $g_n \in Y$ so that the series converges in $\mathcal{S}$ and*

$$\sum_{n=0}^{\infty} \|f_n\|_X \|g_n\|_Y < \infty \tag{2.1}$$

The norm in $X \otimes Y$ is given by

$$\|h\|_{X \otimes Y} = \inf \sum_{n=0}^{\infty} \|f_n\|_X \|g_n\|_Y,$$

where the infimum is taken over all the above representations.

It follows from the definition that if (2.1) holds, then $\sum_{n=0}^{\infty} f_n * g_n \in X \otimes Y$, and

$$\Big\| \sum_{n=0}^{\infty} f_n * g_n \Big\|_{X\otimes Y} \leq \sum_{n=0}^{\infty} \|f_n\|_X \, \|g_n\|_Y \, .$$

The norm in $X \otimes Y$ is based on Schatten's definition of greatest crossnorm.

Theorem 2.2. *If X and Y are $\mathcal{S}$-admissible space then $X \otimes Y$ is an $\mathcal{S}$-admissible Banach space.*

Proof. Let us first show that the functional $\|\cdot\|_{X\otimes Y}$ is actually a norm.

Only the implication $\|h\| = 0 \implies h = 0$ requires a proof. Let $\|h\|_{X\otimes Y} = 0$. Let $\varepsilon > 0$. Then $h = \sum_{n=0}^{\infty} f_n * g_n$, where $\sum_{n=0}^{\infty} \|f_n\|_X \, \|g_n\|_Y < \varepsilon$. Since X and Y are continuously embedded in $\mathcal{S}$, we have $|\hat{f}_n(j)| \leq C_j \|f_n\|_X$ and $|\hat{g}_n(j)| \leq D_j \|g_n\|_Y$, where C_j and D_j are constant depending only on j. Hence

$$|\hat{h}(j)| = \Big| \sum_{n=0}^{\infty} \hat{f}_n(j)\hat{g}_n(j) \Big| \leq \sum_{n=0}^{\infty} C_j D_j \|f_n\|_X \, \|g_n\|_Y \leq C_j D_j \varepsilon.$$

Thus $\hat{h}(j) = 0$ because ε was arbitrary.

Incidentally, this shows also that $X \otimes Y \subset \mathcal{S}$ with continuity. The fact that $\mathcal{P} \subset X \otimes Y$ is immediate. It remains to show that the space $X \otimes Y$ is complete.

Let $h_n \in X \otimes Y$ $(n \geq 0)$ be such that $\sum_{n=0}^{\infty} \|h_n\|_{X\otimes Y} < \infty$. We have $h_n = \sum_{k=0}^{\infty} f_{k,n} * g_{k,n}$, where $\sum_{k=0}^{\infty} \|f_{k,n}\|_X \, \|g_{k,n}\|_Y \leq 2 \|h_n\|$. It is easily verified that $h := \sum_{n=0}^{\infty} h_n$ converges in $\mathcal{S}$ and therefore $h \in X \otimes Y$. It remains to prove that

$$\Big\| \sum_{n=m}^{\infty} h_n \Big\|_{X\otimes Y} \to 0, \quad m \to \infty.$$

But this follows from

$$\Big\| \sum_{n=m}^{\infty} h_n \Big\|_{X\otimes Y} \leq \sum_{k=0}^{\infty} \sum_{n=m}^{\infty} \|f_{k,n}\|_X \, \|g_{k,n}\|_Y \leq \sum_{n=m}^{\infty} 2 \|h_n\| \, ,$$

concluding the proof. □

Proposition 2.1. *If $\mathcal{P}$ is dense in X or Y, then $\mathcal{P}$ is a dense subset of $X \otimes Y$. In particular $(X_{\mathcal{P}} \otimes Y)_{\mathcal{P}} = X_{\mathcal{P}} \otimes Y$.*

Proof. By symmetry of the definition, let assume that $\mathcal{P}$ is dense in X. Let $h \in X \otimes Y$, and $\varepsilon > 0$. Then, by the definition, there are a positive integer n and $f_k \in X$, $g_k \in Y$ $(0 \leq k \leq n)$ such that

$$\Big\| h - \sum_{k=0}^{n} f_k * g_k \Big\|_{X\otimes Y} < \varepsilon/2.$$

Choose polynomials P_k so that $\|f_k - P_k\|_X < \varepsilon \|g_k\|_Y / 2n$. Then we have

$$\begin{aligned} \Big\| h - \sum_{k=0}^{n} P_k * g_k \Big\|_{X\otimes Y} &\leq \Big\| h - \sum_{k=0}^{n} f_k * g_k \Big\|_{X\otimes Y} + \Big\| \sum_{k=0}^{n} (f_k - P_k) * g_k \Big\|_{X\otimes Y} \\ &\leq \varepsilon/2 + \sum_{k=0}^{n} \|f_k - P_k\|_X \, \|g_k\|_Y \leq \varepsilon \end{aligned}$$

This concludes the proof because $\sum_{k=0}^{n} P_k * g_k$ is a polynomial. □

The following fact can help in determining $X \otimes Y$ in simple situations. Recall that a quasinorm on a (complex) vector space A is a functional $\|\cdot\|$ on A satisfying the following conditions:

(i) $\|f\| \geq 0$; $\|f\| = 0$ iff $f = 0$.
(ii) $\|tf\| = |t|\,\|f\|$, for all $t \in \mathbb{C}$, $f \in A$.
(iii) $\|f+g\| \leq K(\|f\| + \|g\|)$ for all f, $g \in A$, where $K \geq 1$ is a constant.

The couple $(A, \|\cdot\|)$ is called a quasi-normed space. A *complete* quasinormed space is called a quasi-Banach space. "Complete" means that if $\{f_k\} \subset A$ is a sequence such that $\lim_{m,k} \|f_m - f_k\| = 0$, then there is $f \in A$ such that $\lim_k \|f_k - f\| = 0$. If A', the space of all bounded linear functionals on A, separates points in A, then there is the smallest Banach space, $[A]$, such that $A' = [A]'$. More precisely, let

$$\|f\|_1 = \sup\{|\Lambda f| : \Lambda \in A',\ \|\Lambda\| \leq 1\}.$$

Then $\|\cdot\|_1$ is a norm on A, and we define $[A]$ to be the completion of $(A, \|\cdot\|_1)$.

If $A \subset \mathcal{S}$ with continuous inclusion, then the dual A' separates points in A because $f \mapsto \hat{f}(j)$, for each j, is in A'. Then we can realize $[A]$ as the subset of $\mathcal{S}$ consisting of those f that can be represented in the form

$$f = \sum_{n=1}^{\infty} f_n \quad \text{with} \sum_{n=1}^{\infty} \|f_n\|_A < \infty. \tag{‡}$$

Moreover we have

$$\|f\|_{[A]} = \inf \sum_{n=1}^{\infty} \|f_n\|_A,$$

where the infimum is taken over all representations of the form (‡). It follows from the condition $\sum_n \|f_n\|_A < \infty$ that the series $\sum_n f_n$ converges in $\mathcal{S}$.

Proposition 2.2. *Let X and Y be $\mathcal{S}$-admissible Banach spaces.*

(i) *If there exists a Banach space Z such that*

$$X * Y = \{f * g \colon f \in X,\ g \in Y\} \subset Z,$$

then $X \otimes Y \subset Z$.

(ii) *If $X * Y = A$ is a quasi-Banach space then $X \otimes Y = [A]$.*

Proof. (i) An application of the closed graph theorem to the operators $f \mapsto f * g$ shows that

$$\sup_{\|f\|_X \leq 1} \|f * g\|_Z < \infty.$$

Hence, by the Banach-Steinhauss theorem,

$$\sup_{\|f\|_X \leq 1,\, \|g\|_Y \leq 1} \|f * g\|_Z < \infty. \tag{†}$$

Now, assuming that $X * Y \subset Z$, let

$$\sum_{j=1}^{\infty} \|f_n\|_X \|g_n\|_Y \ < \infty,$$

where $f_n \in X$, $g_n \in Y$. From this and (†) we obtain

$$\sum_{n=1}^{\infty} \|f_n * g_n\|_Z \ < \infty,$$

whence $\sum_n f_n * g_n$ converges in Z. The result follows.

(ii) Let $X * Y = A$. Since $A \subset [A]$, we have $X \otimes Y \subset [A]$, by (i).

In the other direction, let $f \in [A]$. Choose $\{f_n\}_1^\infty \subset X * Y = A$ so that

$$f = \sum_{n=0}^{\infty} f_n \quad \text{and } \|f\|_{[A]} \leq 2 \sum_{n=1}^{\infty} \|f_n\|_A.$$

Choose $g_n \in X$ and $h_n \in Y$ so that $f_n = g_n * h_n$. Then, as above, $\|g_n * h_n\|_A \leq C\|g_n\|_X \|h_n\|_Y$, where C is independent of n. The result now follows. □

Corollary 2.1. *Let $1 \leq p, q \leq \infty$ and $p * q = \max\left\{\frac{pq}{p+q}, 1\right\}$ where $pq/(p+q) = \infty$ if $p = \infty$ or $q = \infty$. Then $\ell^p \otimes \ell^q = \ell^{p*q}$.*

Proof. It is easily seen that, for p, $q > 0$,

$$\ell^p * \ell^q = \ell^s, \quad \text{where } \frac{1}{s} = \frac{1}{p} + \frac{1}{q}. \tag{2.2}$$

The result now follows from Proposition 2.2. □

Here there is a basic formula connecting tensors and multipliers.

Theorem 2.3. *Let X, Y, Z be $\mathcal{S}$-admissible Banach spaces. Then*

$$(X \otimes Y, Z) = (X, (Y, Z)).$$

Proof. Let $\lambda \in (X \otimes Y, Z)$. We have to prove that $\lambda * f \in (Y, Z)$, for all $f \in X$, i.e., that $\lambda * f * g \in Z$, for all $f \in X$, $g \in Y$. But, since $f * g \in X \otimes Y$, the hypothesis $\lambda \in (X \otimes Y, Z)$ implies $\lambda * (f * g) \in Z$. Hence we have proved that $(X \otimes Y, Z) \subset (X, (Y, Z))$.

In the other direction, assume that $\lambda \in (X, (Y, Z))$, and let $h \in X \otimes Y$. Then

$$h = \sum_{n=1}^{\infty} f_n * g_n, \quad f_n \in X,\ g_n \in Y,$$

and

$$\sum_{n=1}^{\infty} \|f_n\|_X \|g_n\|_Y \leq 2\|h\|_{X \otimes Y}.$$

Hence $\lambda * h = \sum_{n=1}^{\infty} \lambda * f_n * g_n$ (convergence in $\mathcal{S}$). Since $\lambda * f_n \in (Y, Z)$, we have $\lambda * f_n * g_n \in Z$, whence

$$\Big\| \sum_{n=1}^{\infty} \lambda * f_n * g_n \Big\|_Z \leq \|\lambda * f_n\|_{(Y,Z)} \|g_n\|_Y \leq \|\lambda\|_{(X,(Y,Z))} \|f_n\|_X \|g_n\|_Y < \infty.$$

Since Z is complete we have that

$$\lambda * \sum_{n=1}^{\infty} f_n * g_n = \sum_{n=1}^{\infty} \lambda * f_n * g_n \in Z,$$

i.e., $\lambda \in (X \otimes Y, Z)$. This completes the proof of the theorem. □

Corollary 2.2. *Let X and Y be $\mathcal{S}$-admissible Banach spaces. Then*

$$(X \otimes Y)^K = (X, Y^K),\ (X \otimes Y)^* = (X, Y^*),\ (X \otimes Y)^a = (X, Y^a).$$

3. $\mathcal{H}$-admissible Banach spaces

Definition 3.1. *A Banach space $X \subset \mathcal{S}$ is said to be $\mathcal{H}$-admissible if*

(i) *$X \subset \mathcal{H}(\mathbb{D})$ with continuous inclusion, and*
(ii) *$\mathcal{H}(\mathbb{D}_R) \subset X$ for each $R > 1$ and $f \mapsto f|_{\mathbb{D}}$ is continuous from $\mathcal{H}(\mathbb{D}_R)$ to X.*

Proposition 3.1. *Let X be $\mathcal{H}$-admissible. Then*

(i) *$C_X(z) = \sum_{n=0}^{\infty} e_n z^n \in \mathcal{H}(\mathbb{D}, X)$.*
(ii) *$C_{X'}(z) = \sum_{n=0}^{\infty} \gamma_n z^n \in \mathcal{H}(\mathbb{D}, X')$.*
(iii) *The mapping $f \to F$ where $F(w) = f_w$ defines a continuous inclusion $X \subset \mathcal{H}(\mathbb{D}, X_{\mathcal{P}})$.*

Proof. (i) Observe first if X is $\mathcal{H}$-admissible then for any $0 < r < 1$ there is a constant $A_r < \infty$, depending only on r, such that

$$M_\infty(r, f) \le A_r \|f\|_X, \qquad f \in X.$$

In particular, $r^n \le A_r \|e_n\|$ for all $n \in \mathbb{N}$. On the other hand, for each $R > 1$ and $f \in \mathcal{H}(\mathbb{D}_R)$ then $f \in X$ and there exists $C_R > 0$ such that

$$\|f\|_X \le C_R \sup_{|z|<R} |f(z)|,$$

equivalently if $f \in \mathcal{H}(\mathbb{D})$ then $f_r \in X$, for every $r \in (0,1)$, and there holds the inequality

$$\|f_r\|_X \le B_r \|f\|_\infty \quad (0 < r < 1).$$

In particular, $r^{-n}\|e_n\|_X \le B_r$ for all $n \in \mathbb{N}$.

From these estimates one easily deduces that

$$\lim_{n\to\infty} \sqrt[n]{\|e_n\|_X} = 1,$$

Therefore (i) follows.

(ii) On the other hand

$$\|\gamma_n\|_{X'} = \sup_{\|f\|_X \le 1} |\hat{f}(n)| \le r^{-n} A_r$$

and $1 \le \|\gamma_n\|_{X'} \|e_n\|_X$. This gives

$$\lim_{n\to\infty} \sqrt[n]{\|\gamma_n\|_{X'}} = 1,$$

which implies (ii).

(iii) It follows from (i) that if $f \in X$ then

$$f_w = \sum_{n=0}^{\infty} \gamma_n(f) e_n w^n$$

is absolutely convergent in X. Hence $f_w \in X_{\mathcal{P}}$ for any $w \in \mathbb{D}$ and $w \to f_w$ is an $X_{\mathcal{P}}$-valued analytic function on the unit disk $\mathbb{D}$. □

Proposition 3.2. *Let X is $\mathcal{H}$-admissible and, for $0 < r < 1$, write*

$$M_X(r, f) = \sup_{|w|=r} \|f_w\|_X.$$

Then

(i) *$M_X(r, f)$ is increasing.*
(ii) *$M_\infty(r, f) \le A_X(r)\|f\|_X$, $f \in X$, where $A_X(r) = \|(C_{X'})_r\|_{C(\mathbb{T}, X')}$.*
(iii) *$M_X(r, f) \le B_X(r)\|f\|_\infty$, $f \in A(\mathbb{D})$, where $B_X(r) = \|(C_X)_r\|_{L^1(\mathbb{T}, X)}$.*

Proof. (i) Since $F(w) = f_w \in \mathcal{H}(\mathbb{D}, X)$ then $w \to \|F(w)\|_X$ is subharmonic. Therefore $M_X(r, f) = \sup_{|w|=r} \|f_w\|_X$ is increasing in r.

(ii) Note that $C_{X'}(z) = \sum_{n=0}^\infty \gamma_n z^n \in \mathcal{H}(\mathbb{D}, X')$ and, for each $0 < r < 1$, the series $(C_{X'})_r(z) = \sum_{n=0}^\infty \gamma_n z^n r^n$ is absolutely convergent in $C(\mathbb{T}, X')$. Hence

$$f_r(z) = \sum_{n=0}^\infty \gamma_n(f) r^n e_n = (C_{X'})_r(z)(f),$$

which implies that $M_\infty(r, f) \le A_X(r)\|f\|_X$.

(iii) We write, for $f \in A(\mathbb{D})$,

$$f_w = \int_0^{2\pi} f(e^{-i\theta}) C_{we^{i\theta}} \frac{d\theta}{2\pi}.$$

Now, for $|w| = r$, applying Minkowski's inequality

$$\|f_w\|_X \le \int_0^{2\pi} |f(e^{-i\theta})| \|C_{we^{i\theta}}\|_X \frac{d\theta}{2\pi} \le \|f\|_\infty \int_0^{2\pi} \|(C_X)_r(e^{i\theta})\|_X \frac{d\theta}{2\pi}.$$

This gives the result. □

Given $v : \mathbb{D} \to [0, \infty)$ a continuous weight, let H_v^∞ denote the space of $f \in \mathcal{H}(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty$. Hence (ii) in Proposition 3.2 shows the following fact.

Corollary 3.1. *Let X be $\mathcal{H}$-admissible and define $v_1^{-1}(z) = A_X(|z|) = \|(C_{X'})_{|z|}\|_{C(\mathbb{T},X')}$. Then $X \subset H_{v_1}^\infty$ with continuous inclusion.*

Let us now show that also taking multipliers and tensors preserve $\mathcal{H}$-admissiblity.

Theorem 3.1. *Let X and Y be $\mathcal{H}$-admissible. Then (X, Y) and $X \otimes Y$ are $\mathcal{H}$-admissible Banach spaces.*

Proof. Let us take $\lambda \in (X, Y)$ and observe that, using Proposition 3.2,

$$M_\infty(r, \lambda) \le A_Y(r)\|\lambda * C_r\|_Y \le A_Y(r)\|\lambda\|_{(X,Y)}\|C_r\|_X.$$

This gives that $(X, Y) \subset \mathcal{H}(\mathbb{D})$ with continuity.

Also note that if $\lambda \in \mathcal{H}(\mathbb{D})$ then

$$\begin{aligned}
\|\lambda_{r^2}\|_{(X,Y)} &= \sup_{\|f\|_X\le 1} \|(\lambda * f_r)_r\|_Y \\
&\le B_Y(r) \sup_{\|f\|_X\le 1} M_\infty(r, \lambda * f) \\
&\le B_Y(r)\|\lambda\|_\infty \sup_{\|f\|_X\le 1} M_\infty(r, f) \\
&\le B_Y(r)A_X(r)\|\lambda\|_\infty.
\end{aligned}$$

This is equivalent to $\mathcal{H}(\mathbb{D}_R) \subset (X, Y)$ for any $R > 1$.

To show that $X \otimes Y$ is $\mathcal{H}$-admissible Let $h = \sum_{n=0}^\infty f_n * g_n$ where the series converges in $\mathcal{S}$ and $\sum_{n=0}^\infty \|f_n\|_X \|g_n\|_Y < \infty$. Observe that for each $0 < r < 1$

$$h_{r^2} = \sum_{n=0}^\infty (f_n)_r * (g_n)_r.$$

Hence

$$\begin{aligned} M_\infty(r^2, h) &\leq \sum_{n=0}^\infty M_\infty(r, f_n) M_\infty(r, g_n) \\ &\leq A_X(r) A_Y(r) \sum_{n=0}^\infty \|f_n\|_X \|g_n\|_Y. \end{aligned}$$

Hence, taking the infimum over all representations, $M_\infty(r^2, h) \leq A_X(r)A_Y(r)\|h\|_{X\otimes Y}$. This shows that $X \otimes Y \subset \mathcal{H}(\mathbb{D})$ with continuity.

Let us now take $h \in \mathcal{H}(\mathbb{D}_R)$ and fix $1 < S < R$. Hence $\sum_{n=0}^\infty |\hat{h}(n)| S^n < \infty$. Using that $\lim_{n\to\infty} \sqrt[n]{\|e_n\|_X \|e_n\|_Y} = 1$, we can write $h = \sum_{n=0}^\infty \hat{h}(n) e_n * e_n$, with convergence in $\mathcal{H}(\mathbb{D})$ and

$$\sum_{n=0}^\infty \|\hat{h}(n) e_n\|_X \|e_n\|_Y \leq K \sum_{n=0}^\infty S^{-n} \|e_n\|_X \|e_n\|_Y < \infty.$$

□

Definition 3.2. *If X is an $\mathcal{H}$-admissible Banach space we define $\tilde{X}$ as the space of functions in $\mathcal{H}(\mathbb{D})$ such that $w \to f_w \in H^\infty(\mathbb{D}, X)$. We write*

$$\|f\|_{\tilde{X}} = \sup_{0<r<1} M_X(r, f).$$

For instance $\widetilde{H^p} = H^p$ or $\widetilde{A(\mathbb{D})} = H^\infty$.

Let us collect some properties of $\tilde{X}$ in the next proposition.

Proposition 3.3. *Let $X \subset \mathcal{H}(\mathbb{D})$ be $\mathcal{H}$-admissible. Then*

(i) *$\tilde{X}$ is $\mathcal{H}$-admissible.*

(ii) *$\tilde{X}_{\mathcal{P}} \subset X_{\mathcal{P}}$ and $\tilde{X} = \widetilde{(X_{\mathcal{P}})} = \tilde{\tilde{X}}$.*

(iii) *$X^\# \subset X^* \subset (X_{\mathcal{P}})^\# \subset (\tilde{X})^*$ with continuous inclusions. In particular $(X_{\mathcal{P}})^* = (X_{\mathcal{P}})^\#$.*

Proof. (i) The fact that $\|\cdot\|_{\tilde{X}}$ is a norm and complete is standard. Due to (i) in Proposition 3.2 one has that for $0 < r < 1$

$$M_{\tilde{X}}(r, f) = \|f_r\|_{\tilde{X}} = M_X(r, f).$$

From this one easily shows that $\tilde{X}$ is also $\mathcal{H}$-admissible.

(ii) Note that

$$\|f_r\|_{\tilde{X}} = M_{X_{\mathcal{P}}}(r, f) = M_{\tilde{X}}(r, f),$$

which gives that $\tilde{X} = \widetilde{X_{\mathcal{P}}}$. On the other hand if $f \in \mathcal{P}$ then

$$\|f\|_X = \lim_{r\to 1} \|f_r\|_X \leq \sup_{0<r<1} M_X(r, f) = \|f\|_{\tilde{X}}.$$

(iii) The first inclusion is immediate. For the second one note that $(H^\infty)_{\mathcal{P}} = A(\mathbb{D})$ and that $(X, Y) \subset (X_{\mathcal{P}}, Y_{\mathcal{P}})$.

Let $g \in (X_{\mathcal{P}})^\#$. Since $f_r \in X_{\mathcal{P}}$ one has

$$\|(g * f)_r\|_{A(\mathbb{D})} \leq C\|f_r\|_X \leq C\|f\|_{\tilde{X}}.$$

This shows that $g \in (\tilde{X})^*$. □

Let us now present some useful lemmas to be used in the sequel.

Lemma 3.1. *Let $X \subset \mathcal{H}(\mathbb{D})$ be an $\mathcal{H}$-admissible Banach space. If $f, g \in \mathcal{H}(\mathbb{D})$ then*

$$M_X(rs, f * g) \leq M_1(r, f) M_X(s, g),$$

Proof. Let $0 \leq r, s < 1$, $|w| = r$ and $|w'| = s$

$$(f * g)_{ww'} = \sum_{n=0}^{\infty} \gamma_n(f_w)\gamma_n(g_w)e_n$$

where the series is absolutely convergent in X. Hence one concludes

$$(f * g)_{ww'} = \int_0^{2\pi} f(we^{-i\theta}) g_{w'e^{i\theta}} \frac{d\theta}{2\pi},$$

where the integral is understood in the vector valued sense. Using Minkowski's inequality

$$\|(f * g)_{ww'}\|_X \leq \int_0^{2\pi} |f(we^{-i\theta})| \|g_{w'e^{i\theta}}\|_X \frac{d\theta}{2\pi} \leq M_1(r, f) M_X(s, g).$$

This implies the result. □

Lemma 3.2. *Let $X \subset \mathcal{H}(\mathbb{D})$ be an $\mathcal{H}$-admissible Banach space and $f \in \mathcal{H}(\mathbb{D})$. Then*

$$M_X(rs, Df) \leq \frac{1}{1 - r^2} M_X(s, f), \tag{3.1}$$

$$M_X(r, f)dr \leq \int_0^1 M_X(rs, Df) ds, \tag{3.2}$$

where $Df(z) = \sum_{n=1}^{\infty} (n+1)\hat{f}(n) z^n$.

Proof. Recall that $De_n = (n+1)e_n$ and $Df = K * f$ where $K(z) = \frac{1}{(1-z)^2}$. Use Lemma 3.1 to obtain (3.1).

To see (3.2) simply use that, for each $0 \leq r < 1$ and $|\xi| = 1$, one has

$$r f_{r\xi} = \int_0^r (Df)_{s\xi} ds$$

as X-valued function. Hence, by Minkowski's inequality,

$$r M_X(r, f) dr \leq \int_0^r M_X(s, Df) ds = r \int_0^1 M_X(rs, Df) ds.$$

□

Definition 3.3. *If X is an $\mathcal{H}$-admissible Banach space and $1 \leq q < \infty$ we write $\mathfrak{B}^{X,q}$ for the spaces of holomorphic functions such that*

$$\|f\|_{\mathfrak{B}^{X,q}} = (\int_0^1 (1 - r^2)^{q-1} M_X^q(r, Df) r dr)^{1/q} < \infty.$$

The case $q = \infty$ corresponds to

$$\|f\|_{\mathfrak{B}^{X,\infty}} = \sup_{0<r<1} (1 - r^2) M_X(r, Df).$$

Clearly $\mathfrak{B}^{H^p,q}$ coincides with $\mathfrak{B}^{p,q}$, $1 \le p,\, q \le \infty$, consisting of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathfrak{B}^{p,q}} := \left(|f(0)|^q + \int_0^1 M_p^q(r,f')(1-r)^{q-1} r\,dr\right)^{1/q} < \infty.$$

(These spaces are called in [14] Hardy-Bloch spaces.) In the case $q = \infty$, this should be interpreted as

$$|f(0)| + \sup_{0<r<1} M_p(r,f')(1-r) < \infty.$$

Clearly $\mathfrak{B}^{\infty,\infty}$ coincides with the Bloch space $\mathfrak{B}$.

It is easy to see that also $\mathfrak{B}^{\ell^q,q} = \ell^q$.

Definition 3.4. *Let $0 < p,\, q \le \infty$. The space $\ell(p,q)$ introduced by Kellogg* [18], *consists of complex sequences $\{\hat{a}(k)\}_0^\infty$ such that*

$$\left\{\Big(\sum_{j\in I_k} |\hat{a}(j)|^p\Big)^{1/p}\right\}_{k=0}^\infty \in \ell^q,$$

where $I_k = \{j : 2^{k-1} \le j < 2^k\}$, for $k \ge 1$, and $I_0 = \{0\}$. The quasinorm in $\ell(p,q)$ is given by

$$\|\{\hat{a}(j)\}\|_{\ell(p,q)} = \left\|\left\{\Big(\sum_{j\in I_k} |\hat{a}(j)|^p\Big)^{1/p}\right\}_{k=0}^\infty\right\|_{\ell^q}.$$

It follows that $\ell(p,p)$ is identical with ℓ^p. It is not difficult to show that, for $q < \infty$, the dual of $\ell(p,q)$ is (isometrically) isomorphic to $\ell(p',q')$, with the duality pairing given by

$$(a,b) \mapsto \sum_{j=0}^\infty \hat{a}(j)\hat{b}(j)$$

(the series being absolutely convergent and $p' = \infty$ for $p \le 1$. Hence, the norm in $\ell(p,q)$, where $1 \le p \le \infty$ and $1 \le q < \infty$, by means of the formula

$$\|a\|_{\ell(p,q)} = \sup\left\{\Big|\sum_{j=0}^\infty \hat{a}(j)\hat{b}(j)\Big| : \|b\|_{\ell(p',q')} \le 1\right\}.$$

This can be used to derive the following formula for the Banach envelope of $\ell(p,q)$:

$$[\ell(p,q)] = \begin{cases} \ell^1, & \text{if } p,q \le 1, \\ \ell(p,1), & \text{if } 1 < p \le \infty,\ q < 1, \\ \ell(1,q), & \text{if } p < 1,\ 1 < q \le \infty. \end{cases} \tag{3.3}$$

Given $0 < u, v < \infty$ let us denote

$$u \ominus v = \begin{cases} \dfrac{uv}{u-v}, & \text{if } v < u < \infty, \\ v, & \text{if } u = \infty, \\ \infty & \text{if } u \le v. \end{cases}$$

(The notation $u \ominus v$ was introduced in [7].) Kellogg proved the following extension of Hölder duality result.

Proposition 3.4. *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$. Then*

$$(\ell(p_1,q_1), \ell(p_2,q_2)) = \ell(p_1 \ominus p_2, q_1 \ominus q_2)$$

with equal norms.

It is not hard to generalize the formula (2.2) to the setting of the Kellogg spaces.

$$\ell(p_1,q_1) * \ell(p_2,q_2) = \ell(s_1,s_2),$$

where

$$\frac{1}{s_j} = \frac{1}{p_j} + \frac{1}{q_j}.$$

Then, using Proposition 2.2 and formula (3.3), one proves the following result.

Proposition 3.5. *Let* $1 \le p_j, q_j \le \infty$. *Then*

$$\ell(p_1,q_1) \otimes \ell(p_2,q_2) = \ell(p_1 * p_2, q_1 * q_2).$$

Proposition 3.6. *Let* $1 \le p, q \le \infty$. *Then* $\mathfrak{B}^{\ell^p,q} = \ell(p,q)$.

Proof. The case $q = \infty$ follows from the observation that $f \in \ell(p,\infty)$ can be rewritten by the condition

$$\sum_{n=0}^{\infty} |(n+1)\hat{f}(n)|^p r^{np} \le \frac{C}{(1-r)^p}.$$

The case $q < \infty$ follows from the inequalities, for $p, \alpha > 0$ and $a_k \ge 0$, (see [20] or also [4, Lemma 2.1])

$$\begin{aligned} A_{p,\alpha} \sum_{n=0}^{\infty} 2^{-n\alpha} (\sum_{k \in I_n} a_k)^p &\le \int_0^1 (1-r)^{p\alpha-1} (\sum_{k=0}^{\infty} a_k r^k)^p dr \\ &\le B_{p,\alpha} \sum_{n=0}^{\infty} 2^{-n\alpha} (\sum_{k \in I_n} a_k)^p. \end{aligned}$$

□

4. Homogeneous spaces of analytic functions

Definition 4.1. *Let X be an $\mathcal{H}$-admissible Banach space. It is said to be* homogeneous *if it satisfies:*

(i) *If $f \in X$ and $|\xi| = 1$, then $f_\xi \in X$ and $\|f_\xi\|_X = \|f\|_X$.*
(ii) *If $f \in X$ and $0 < r < 1$ then $M_X(r,f) \le K\|f\|_X$, where K is a constant independent of f and r.*

Observe that for homogeneous spaces $C_\xi \in (X,X)$ with $\|C_\xi\|_{(X,X)} = 1$ if $|\xi| = 1$ and $C_r \in (X,X)$ with $\sup_{0<r<1} \|C_r\|_{(X,X)} \le K$. Note also that in this case $\|f_w\|_X = \|f_{|w|}\|_X$ and $\|f_r\| = M_X(r,f)$ and $X \subset \tilde{X}$ with continuity.

We denote by $H^\infty(\mathbb{D},X)$ the space of X-valued bounded analytic functions and $A(\mathbb{D},X)$ those with continuous extension to the boundary, i.e. the closure of X-valued polynomials.

Proposition 4.1. *Let X be homogeneous Banach space.*

(i) *If $f \in X$ then $w \to f_w \in H^\infty(\mathbb{D}, X_{\mathcal{P}})$.*
(ii) *If $f \in X_{\mathcal{P}}$ then $w \to f_w \in A(\mathbb{D}, X_{\mathcal{P}})$.*

Proof. (i) Note that the $\mathcal{H}$-admissibility guarantees that $F(w) = f_w \in \mathcal{H}(\mathbb{D}, X_{\mathcal{P}})$. For homogeneous spaces

$$M_X(r,f) = \sup_{|\xi|=1} \|f_{r\xi}\|_X = \|F_r\|_{H^\infty(\mathbb{D},X)}.$$

Hence $F \in H^\infty(\mathbb{D}, X)$.

(ii) It is clear that if $f \in X_{\mathcal{P}}$ then $\lim_{r\to 1} \|f_r - f\| = 0$. Now use that $\|F - F_r\|_{H^\infty(\mathbb{D},X)} = \|f - f_r\|$ to conclude the result, because $F_r \in A(\mathbb{D}, X)$ for each $0 < r < 1$. □

Proposition 4.2. *Let X and Y be $\mathcal{H}$-admissible Banach spaces. Then*

(i) *$\tilde{X}$ is homogeneous.*
(ii) *If Y is homogeneous then (X, Y) is homogeneous.*
(iii) *If X and Y are homogeneous then $X \otimes Y$ is homogeneous.*

Proof. The $\mathcal{H}$-admissibility of (X, Y), $X \otimes Y$ and $\tilde{X}$ was proved in Theorems 3.1 and 3.3 respectively.

(i) To show that $\tilde{X}$ is homogeneous use that $M_X(r, f)$ is increasing and the facts, for $|\xi| = 1$ and $0 < r, s < 1$,

$$M_X(r, f_\xi) = M_X(r, f) \text{ and } \qquad M_X(s, f_r) = M_X(sr, f).$$

(ii) Given $\lambda \in (X, Y)$ and $f \in X$ one has that

$$\lambda_w * f = (\lambda * f)_w$$

what trivially gives the result using the properties of Y.

(iii) Now given $h \in X \otimes Y$ with $h = \sum_{n=0}^\infty f_n * g_n$ with $\sum_{n=0}^\infty \|f_n\|\|g_n\| < \infty$ one has

$$M_{X\otimes Y}(r^2, h) \le \sum_{n=1}^\infty M_X(r, f_n) M_Y(r, g_n) \le K^2 \sum_{n=1}^\infty \|f_n\|_X \|g_n\|_Y.$$

Therefore $M_{X\otimes Y}(r^2, h) \le \|h\|_{X\otimes Y}$ for all $0 < r < 1$.

Taking into account that

$$h_\xi = \sum_{n=0}^\infty (f_n)_\xi * g_n, \qquad |\xi| = 1$$

one concludes that $\|h_\xi\|_{X\otimes Y} \le \|h\|_{X\otimes Y}$ for $|\xi| = 1$. Therefore $\|h_\xi\|_{X\otimes Y} = \|h\|_{X\otimes Y}$. □

Proposition 4.3. *Let X be $\mathcal{H}$-admissible and $1 \le q \le \infty$. Then*

(i) *$\mathfrak{B}^{X,q}$ is homogeneous.*
(ii) *$(\mathfrak{B}^{X,q})_{\mathcal{P}} = \mathfrak{B}^{X,q}$ for $1 \le q < \infty$.*
(iii) *$(\mathfrak{B}^{X,\infty})_{\mathcal{P}} = \{f \in \mathcal{H}(\mathbb{D}) : \lim_{r\to 1}(1 - r^2) M_X(r, Df) = 0\}$.*
(iv) *$\mathfrak{B}^{X,1} \subset X_{\mathcal{P}}$ and $\tilde{X} \subset \mathfrak{B}^{X,\infty}$.*

Proof. (i) The facts that $\|\cdot\|_{\mathfrak{B}^{X,q}}$ is a norm and the completeness follow from standard arguments which are left to the reader. The $\mathcal{H}$-admissibility and homogeneity follow from the facts $\|f_s\|_{\mathfrak{B}^{X,q}} = M_{\mathfrak{B}^{X,q}}(s, f)$ and Lemmas 3.1 and 3.2.

(ii) Note that $\lim_{s\to 1} M_X(s, f_r - f) = 0$ for each $0 < r < 1$. Hence, using the Lebesgue dominated convergence theorem, one sees that, for $q < \infty$, if $f \in \mathfrak{B}^{X,q}$ then $\|f_r - f\|_{\mathfrak{B}^{X,q}} \to 0$ as $r \to 1$. Since $f_r \in (\mathfrak{B}^{X,q})_{\mathcal{P}}$ the result follows.

(iii) Since any polynomial $d \in \mathcal{P}$ satisfies that $\lim_{r\to 1}(1 - r^2) M_X(r, Df) = 0$ then $(\mathfrak{B}^{X,\infty})_{\mathcal{P}} \subset \{f \in \mathcal{H}(\mathbb{D}) : \lim_{r\to 1}(1 - r^2) M_X(r, Df) = 0\}$. Let $f \in \mathcal{H}(\mathbb{D})$ such that $\lim_{r\to 1}(1 - r^2) M_X(r, Df) = 0$. For each $\varepsilon > 0$ one there exists $r_0 < 1$ such that

$$(1 - s^2) \sup_{r>s} M_X(r, Df) < \varepsilon, r_0 \le s < 1.$$

Now observe that

$$\|f-f_s\|_{\mathfrak{B}^{X,\infty}} \leq M_X(r_0, D(f_s-f))+2(1-s^2)\sup_{r>r_0} M_X(r, Df) \leq M_X(r_0, D(f_s-f))+\varepsilon.$$

Therefore $f_s \in (\mathfrak{B}^{X,\infty})_{\mathcal{P}}$ approaches f.

(iv) It follows from Lemma 3.2 and (ii). □

Proposition 4.4. *Let X and Y be homogeneous Banach spaces. Then*

$$(\mathfrak{B}^{X,1}, Y) = \mathfrak{B}^{(X,Y),\infty}.$$

Proof. Let f be a polynomial and $g \in \mathfrak{B}^{(X,Y),\infty}$. Observe that

$$\begin{aligned} f * g(z) &= \frac{1}{2}\int_0^1 (1-r^2) r^{2n+1} \sum_{n=0}^{\infty} (n+1) n \hat{f}(n)\hat{g}(n) z^n \\ &= \frac{1}{2}\int_0^1 (1-r^2)(Df)_r * ((Dg)_r - g_r)(z) r dr. \end{aligned}$$

Using that $M_{(X,Y)}(r, g) \leq M_{(X,Y)}(r, Dg)$ (see (3.2)) one concludes that

$$\begin{aligned} \|f * g\|_Y &\leq \int_0^1 (1-r^2)\|(Df)_r * ((Dg)_r - g_r)\|_Y r dr \\ &\leq \int_0^1 (1-r^2) M_{(X,Y)}(r, (Dg) - g) M_X(r, Df) r dr \\ &\leq 2\int_0^1 M_X(r, (Df))(1-r^2) M_{(X,Y)}(r, Dg) r dr \\ &\leq 2\|f\|_{\mathfrak{B}^{X,1}} \|g\|_{\mathfrak{B}^{(X,Y),\infty}}. \end{aligned}$$

Using that polynomials are dense in $\mathfrak{B}^{X,1}$ one easily concludes that $\mathfrak{B}^{(X,Y),\infty} \subset (\mathfrak{B}^{X,1}, Y)$.

Let $f \in (\mathfrak{B}^{X,1}, Y)$. Then

$$\begin{aligned} M_{(X,Y)}(r, Df) &= \sup\{\|Df * g_r\|_Y : \|g\|_X \leq 1\} \\ &= \sup\{\|f * Dg_r\|_Y : \|g\|_X \leq 1\} \\ &\leq \|f\|_{(\mathfrak{B}^{X,1},Y)} \sup\{\|Dg_r\|_{\mathfrak{B}^{X,1}} : \|g\|_X \leq 1\} \\ &\leq \|f\|_{(\mathfrak{B}^{X,1},Y)} \sup\{\int_0^1 M_X(s, D^2 g_r) ds : \|g\|_X \leq 1\}. \end{aligned}$$

Observe now that

$$\begin{aligned} \int_0^1 M_X(s, D^2 g_r) s ds &= \int_0^1 M_X(sr, D^2 g) s ds \\ &\leq \int_0^1 \frac{M_X(\sqrt{sr}, Dg)}{1-sr} ds \\ &\leq A\int_0^1 \frac{\|g\|_X}{(1-sr)^2} ds \\ &\leq A'' \frac{\|g\|_X}{(1-r^2)}. \end{aligned}$$

This estimate concludes the proof. □

Corollary 4.1. *If X is homogeneous then*

$$(\mathfrak{B}^{X,1})^{\#} = (\mathfrak{B}^{X,1})^{*} = (\mathfrak{B}^{X,1})' = \mathfrak{B}^{X^*,\infty} \text{ and } (\mathfrak{B}^{X,1})^{K} = \mathfrak{B}^{X^K,\infty}.$$

Let us give some information on the dual of homogeneous Banach spaces.

Note that if X is $\mathcal{H}$-admissible then X^a is continuously embedded into X' by means of the map $\lambda \in X^a \to \phi_\lambda \in X'$ defined by

$$\phi_\lambda(f) = \lim_{r\to 1} \sum_{n=0}^{\infty} \hat{\lambda}(n)\hat{f}(n)r^n.$$

Recall that we use the notation $A^{\#} = (X, A(\mathbb{D}))$. Hence, in particular $X^{\#} \subset X'$ by means of $f \to \lambda * f(1)$ for $\lambda \in X^{\#}$.

Therefore we have the following chain of continuous inclusions between $\mathcal{H}$-admissible Banach spaces:

$$X^K \subseteq X^{\#} \subseteq X^a \subseteq X'.$$

Proposition 4.5. *Let X be an homogeneous Banach space. Then $X^{\#} \subset (X_{\mathcal{P}})' \subset (X_{\mathcal{P}})^{\#}$ with continuity.*

Proof. Let $f \in X^{\#}$ and define $\gamma(g) = f * g(1)$. One has that $\gamma \in (X_{\mathcal{P}})'$ and $\|\gamma\| \le \|f\|_{X^{\#}}$ what shows $X^{\#} \subset (X_{\mathcal{P}})'$.

Given $\gamma \in (X_{\mathcal{P}})'$ define $\lambda(z) = \sum_{n=0}^{\infty} \gamma(e_n) z^n$. Let $f \in X_{\mathcal{P}}$ and observe that from Proposition 4.1 (ii) the function $w \to f_w$ belongs to $A(\mathbb{D}, X)$. Hence

$$\lambda * f(w) = \sum_{n=0}^{\infty} \gamma(e_n)\hat{f}(n)w^n = \gamma(f_w).$$

The continuity of γ implies that $\lambda * f \in A(\mathbb{D})$. Moreover

$$\|\lambda * f\|_{A(\mathbb{D})} = \sup_{|w|<1} |\lambda * f(w)| \le K\|\gamma\|\|f\|.$$

This shows that $\lambda \in (X_{\mathcal{P}})^{\#}$ and $\|\lambda\|_{(X_{\mathcal{P}})^{\#}} \le K\|\gamma\|$. □

Corollary 4.2. *If X is an homogeneous Banach space then $X^* = (X_{\mathcal{P}})^* = (X_{\mathcal{P}})^{\#} = (X_{\mathcal{P}})^a = (X_{\mathcal{P}})'$ with equivalent norms.*

Proof. Since $X \subset \tilde{X}$ it follows from Proposition 3.3 that

$$\tilde{X}_{\mathcal{P}} = X_{\mathcal{P}} \text{ and } X^* = (X_{\mathcal{P}})^{\#}.$$

For the other equalities use the previous proposition. □

Proposition 4.6. *Let X be homogeneous. Then $X_{\mathcal{P}} \subset X^{**}$ and there exists $A > 0$ that*

$$\|f\|_{X^{**}} \le \|f\|_X \le K\|f\|_{X^{**}}, \quad f \in X_{\mathcal{P}}.$$

In particular, $X_{\mathcal{P}} = (X^{**})_{\mathcal{P}}$.

Proof. The inclusion and the first inequality are straightforward.

Let now $f \in X_{\mathcal{P}}$. From Corollary 4.2 and Hanh-Banach theorem,

$$\begin{aligned}
\|f\|_X &= \sup\{|\gamma(f)| : \gamma \in (X_{\mathcal{P}})', \|\gamma\| \le 1\} \\
&\le A \sup\{|g * f(1)| : g \in (X_{\mathcal{P}})^{\#}, \|g\|_{(X_{\mathcal{P}})^{\#}} \le 1\} \\
&\le A \sup\{\|g * f\|_\infty : g \in (X_{\mathcal{P}})^{\#}, \|g\|_{(X_{\mathcal{P}})^{\#}} \le 1\} \\
&= A \sup\{\|g * f\|_\infty : g \in X^*, \|g\|_{X^*} \le 1\} \\
&\le A\|f\|_{X^{**}}.
\end{aligned}$$

□

5. The Fatou property

In this section we shall now consider a property closely related to (P7).

Definition 5.1. *Let $X \subset \mathcal{H}(\mathbb{D})$ be an homogeneous Banach space. X is said to satisfy F-property, to be denoted (FP), if there exists $A > 0$ such that for any sequence $(f_n) \in X$ with $\sup_n \|f_n\|_X \leq 1$ and $f_n \to f$ in $\mathcal{H}(\mathbb{D})$ one has that $f \in X$ and $\|f\|_X \leq A$.*

Proposition 5.1. *Let X and Y be $\mathcal{H}$-admissible Banach spaces. Then*

(i) *$\tilde{X}$ and $\mathfrak{B}^{X,q}$, $1 \leq q \leq \infty$, have (FP).*
(ii) *If Y is homogeneous with (FP) then (X, Y) has (FP).*

Proof. (i) Let $(f_n) \in \tilde{X}$ such that $\|f_n\|_{\tilde{X}} \leq 1$ and $f_n \to f$ in $\mathcal{H}(\mathbb{D})$. Using that $\lim_{n\to\infty} M_X(r, f_n) = M_X(r, f)$ one concludes that $f \in \tilde{X}$. Similar argument works for $\mathfrak{B}^{X,q}$.

(ii) Let $(f_n) \in (X, Y)$ such that $\|f_n\|_{(X,Y)} \leq 1$ and $f_n \to f$ in $\mathcal{H}(\mathbb{D})$. Hence for a given $g \in X$ with $\|g\|_X = 1$ we have $(f_n * g) \in Y$ such that $\|f_n * g\|_{(X,Y)} \leq 1$ and $f_n * g \to f * g$ in $\mathcal{H}(\mathbb{D})$. Since Y has (FP), one has that $f * g \in Y$ and $\|f * g\|_Y \leq A$. Therefore $f \in (X, Y)$ with $\|f\|_{(X,Y)} \leq A$. □

Let us formulate some equivalent conditions of this property.

Theorem 5.1. *Let X be homogeneous. The following are equivalent:*

(i) *X has (FP).*
(ii) *If $f \in \mathcal{H}(\mathbb{D})$ and $\sup_{w\in\mathbb{D}} \|f_w\|_X < \infty$ then $f \in X$.*
(iii) *$X = \tilde{X}$ with equivalent norms.*
(iv) *$X = X^{**}$.*

Proof. (i) $\Longrightarrow$ (ii) Take $f \in \mathcal{H}(\mathbb{D})$ with $0 < \sup_{0\leq r<1} M_X(r, f) = A < \infty$. Select a sequence r_n converging to 1 and put $f_n = A_n f_{r_n}$ where $A_n^{-1} = M_X(r_n, f)$. Of course $f_n \to A^{-1} f$ in $\mathcal{H}(\mathbb{D})$ and $\|f_n\|_X \leq 1$. Applying the assumption one gets that $f \in X$.

(ii) $\Longrightarrow$ (iii) Note that if X is homogeneous one has $X \subset \tilde{X}$ and $\|f\|_{\tilde{X}} \leq K\|f\|_X$. The assumption means that $\tilde{X} \subset X$. The continuity follows from the open map theorem.

(iii) $\Longrightarrow$ (iv) Take $f \in X^{**}$. Then $f_r \in (X^{**})_{\mathcal{P}}$ which, according to Proposition 4.6, coincides with $X_{\mathcal{P}}$. Hence we have

$$M_X(r, f) \leq K M_{(X_{\mathcal{P}})^{**}}(r, f) \leq K'\|f\|_{(X_{\mathcal{P}})^{**}}.$$

This gives $f \in \tilde{X} = X$.

(iv) $\Longrightarrow$ (i) If $X = X^{**}$ then X has (FP) because (X^*, H^∞) has (FP) according to Proposition 5.1. □

This characterization allows us to give examples failing to have (FP), for instance $X = c_0$ or $X = A(\mathbb{D})$.

To see that it suffices to consider the Cauchy kernel $C = (\hat{f}(j))_j$ where $\hat{f}(j) = 1$ for all j. Hence $C \in \ell^\infty \setminus c_0$, but, however, $C_w * f = C_w \in c_0$ for any $|w| < 1$ and $\sup_{w\in X} \|C_w * f\|_{c_0} = 1$. Thus c_0 fails (FP). Select $f \in H^\infty \setminus A(\mathbb{D})$ and observe that $\sup_{w\in X} \|C_w * f\|_{A(\mathbb{D})} = \|f\|_\infty$. Thus $A(\mathbb{D})$ fails (FP).

In fact both examples are particular cases of the following corollary.

Corollary 5.1. *If $X_{\mathcal{P}}$ has (FP), then $X = X_{\mathcal{P}}$.*

Remark 5.1. There exists a notion closely related to (FP) in Banach space theory. Recall that a complex Banach space E is said to have the $ARNP$ if any bounded E-valued function has boundary limits a.e, i.e if $F : \mathbb{D} \to E$ is holomorphic and bounded then $\lim_{r\to 1} F(re^{i\theta})$ exists a.e. in E (see [8, 9]).

Since $F(w) = f_w \in H^\infty(\mathbb{D}, X)$, one sees that any homogeneous Banach space X with the $ARNP$ satisfies (FP) (note that $f_{e^{i\theta}} \in X$ for almost all θ implies that $f \in X$.)

Since H^∞ fails $ARNP$ but has (FP) they are not equivalent properties.

Although the space $X \otimes Y$ needs not to have (FP) if only one of the spaces has (FP) (take $X = \ell^\infty$ and $Y = c_0$ and note that $X \otimes Y = c_0$) the following result says that the result holds true if both spaces have (FP).

Theorem 5.2. *Let X and Y be homogeneous with (FP). Then $X \otimes Y$ has (FP).*

Proof. Let $(h_n) \in X \otimes Y$ such that $\|h_n\|_{X\otimes Y} \le 1$ for all n such that $h_n \to h$ in $\mathcal{H}(\mathbb{D})$. Let us take a decomposition such that $h_n = \sum_{j=1}^\infty f_{n,j} * g_{n,j}$ where $\|f_{n,j}\|_X = \|g_{n,j}\|_Y$ and

$$\|h_n\|_{X\otimes Y} \le \sum_{j=0}^\infty \|f_{n,j}\|_X \|g_{n,j}\|_Y \le \|h_n\|_{X\otimes Y} + 1/n \le 2.$$

Therefore for any sequence $(a_j)_j \in \ell^2$ with $\|(a_j)\|_2 = 1$ one has that

$$\max\{\|\sum_j a_j f_{n,j}\|_X, \|\sum_j a_j g_{n,j}\|_Y\} \le 2.$$

Denoting $\phi_n = \sum_j a_j f_{n,j}$ and $\psi_n = \sum_j a_j g_{n,j}$, one has that $\sup_n \|\phi_n\|_X \le 2$ and $\sup_n \|\psi_n\|_X \le 2$. Since $X \subset (X^\#)'$ and $Y \subset (Y^\#)'$, the Banach-Alaoglu theorem implies that there exists a subsequence $k(n)$ such that $\phi_{k(n)}$ converges in the *weak**-topology to ϕ and $\psi_{k(n)}$ converges in the *weak**-topology to ψ. In particular $\phi_{k(n)} \to \phi$ in $\mathcal{H}(\mathbb{D})$ and $\psi_{k(n)} \to \psi$ in $\mathcal{H}(\mathbb{D})$. Using the (FP) in both spaces X and Y one obtains that $\phi \in X$ and $\psi \in Y$ with $\|\phi\|_X \le 2$ and $\|\psi\|_Y \le 2$.

Let us now select $(a_j)_j$ the canonical basis of ℓ^2 and write f_j and g_j the functions ϕ and ψ corresponding to such cases. In particular, using a diagonal process there exists a subsequence $k'(n)$ such that $f_{k'(n),j} \to f_j$ and $g_{k'(n),j} \to g_j$ in $\mathcal{H}(\mathbb{D})$ for all $j \in \mathbb{N}$. Taking limits one gets $f = \sum_{j=1}^\infty f_j * g_j$ in $\mathcal{S}$. To show that $\sum_j \|f_j\|_X \|g_j\|_Y < \infty$ we shall see that $\sum_j \|f_j\|_X^2 < \infty$ and $\sum_j \|g_j\|_Y^2 < \infty$. This follows using that $\phi = \phi((a_j))$ and $\psi = \psi((a_j))$ coincide with $\phi = \sum_j a_j f_j$ and $\psi = \sum_j a_j g_j$ and the facts $\|\sum_j a_j f_j\|_X \le 2$ and $\|\sum_j a_j g_j\|_Y \le 2$. □

Theorem 5.3. *Let X and Y be homogeneous spaces.*

(i) *If Y has (FP), then $(X, Y) = (X \otimes Y^*)^*$.*
(ii) *If X and Y have (FP), then $X \otimes Y = (X, Y^*)^*$.*

Proof. (i) Use that $Y^{**} = Y$ and Corollary 2.2 to get $(X \otimes Y^*)^* = (X, Y)$.

(ii) We have $(X\otimes Y)^{**} = X\otimes Y$ by Theorems 5.2 and 5.1. Again use $(X\otimes Y)^* = (X, Y^*)$ to conclude the proof. □

6. $\ell^\infty \otimes Y$ and solid Banach spaces

Definition 6.1. *(see* [2]*) A set $A \subset \mathcal{S}$ is said to be* solid *if for any $f \in A$ and $g \in \mathcal{S}$ with $|\hat{g}(j)| \leq |\hat{f}(j)|$, $j \geq 0$, implies that $g \in A$.*

Remark 6.1. Let X be an $\mathcal{S}$-admissible Banach space. X is solid iff $\ell^\infty \subset (X, X)$.

Let us mention the following elementary facts.

Proposition 6.1. *If X or Y are solid $\mathcal{S}$-admissible Banach spaces, then so are (X, Y) and $X \otimes Y$.*

Proof. Let $(\hat{f}(j))_j \in \ell^\infty$ and $\lambda \in (X, Y)$. To show that $f * \lambda \in (X, Y)$ take $g \in X$ and observe that $(f * \lambda) * g = \lambda * (f * g) = f * (\lambda * g)$. This shows that $(f * \lambda) * g \in Y$ whenever X or Y are solid.

The case $X \otimes Y$ follows from Remark 6.1 together with the trivial inclusion $X \subset (Y, X \otimes Y)$ and Theorem 2.3. If X is solid then

$$\ell^\infty \subset (X, X) \subset (X, (Y, X \otimes Y) = (X \otimes Y, X \otimes Y).$$

□

Proposition 6.2. *(see* [2]*) If $X \subset \mathcal{S}$ is an $\mathcal{S}$-admissible Banach space, then there is a largest solid $\mathcal{S}$-admissible Banach space $s(X) \subset X$. Furthermore $s(X)$ is the largest solid subset of X and we have*

$$s(X) = (\ell^\infty, X).$$

Proof. Denote $s(X) = (\ell^\infty, X)$. It is an $\mathcal{S}$-admissible Banach space, by Theorem 2.1. From Proposition 6.1 one has that $s(X)$ is a solid subspace of X. Now let $Y \subset X$ be any other solid subset. If $f \in Y$ and $g \in \ell^\infty$, then $g * f \in Y \subset X$. Hence $f \in (\ell^\infty, X)$ and so $Y \subset (\ell^\infty, X)$. □

Proposition 6.3. [2, 7] *If $X \subset \mathcal{S}$, then there is a smallest solid superset $S(X) \supset X$. Furthermore,*

$$S(X) = \ell^\infty * X, \quad \textit{and}$$

$$S(X) = \{g \in \mathcal{S} \colon \exists f \in A \textit{ such that } |\hat{f}(j)| \geq |\hat{g}(j)| \textit{ for all } j\}. \tag{†}$$

Proof. Clearly, $S(X)$ is the intersection of all solid sets containing X. Since the set $\ell^\infty * X$ is solid, we have $S(X) \subset \ell^\infty * X$. On the other hand,

$$\ell^\infty * X \subset \ell^\infty * S(X) \quad (\text{because } X \subset S(X))$$

and $\ell^\infty * S(X) = S(X)$, whence $\ell^\infty * X \subset S(X)$, and so $\ell^\infty * X = S(X)$.

For (†), let

$$B = \{g \in \mathcal{S} \colon \exists f \in X \text{ such that } |\hat{f}(j)| \geq |\hat{g}(j)| \text{ for all } j\}.$$

It is trivial to check that B is a solid superset of X. Let D be any solid superspace of A, and let $g \in B$. Then there is $f \in X$ such that $|\hat{f}(j)| \geq |\hat{g}(j)|$ for all j. Then $f \in D$, and since D is solid we have $g \in D$. Thus $B \subset D$, whence $B = S(X)$. □

Denote $S_b(X) = \ell^\infty \otimes X$. Of course $S(X) \subset S_b(X)$.

Theorem 6.1. *Let X be an $\mathcal{S}$-admissible Banach space. Then $S_b(X)$ is the smallest solid Banach space containing X. More precisely, if Y is a solid Banach space containing X, then $S_b(X) \subset Y$ with continuity.*

Proof. Let $h \in \ell^\infty \otimes X$. Then

$$h = \sum_{n=1}^{\infty} b_n * f_n, \quad \text{where} \quad b_n \in \ell^\infty, \quad f_n \in X, \quad \text{and}$$

$$\|b_n\|_{\ell^\infty} = 1, \quad \sum_{n=1}^{\infty} \|f_n\|_X < \infty.$$

The series $\sum_{n=1}^{\infty} b_n * f_n$ converges in Y because

$$\sum_{n=1}^{\infty} \|b_n * f_n\|_Y \le \sum_{n=1}^{\infty} C\|f_n\|_Y \le \sum_{n=1}^{\infty} CC_1\|f_n\|_X < \infty.$$

The sum in Y of this series is equal to h because X and Y are continuously embedded in $\mathcal{S}$. Thus $h \in Y$, which was to be proved. □

Corollary 6.1. *If $S(X)$ is an $\mathcal{S}$-admissible Banach space then $S(X) = \ell^\infty \otimes X$, with equivalent norms.*

Proof. Since $S(X) \subset \ell^\infty \otimes X$, by definition, and $\ell^\infty \otimes X \subset S(X)$, by Theorem 6.1, we see that $S(X)$ and $\ell^\infty \otimes X$ are equal as sets. The norms are equivalent because these spaces are complete and $\ell^\infty \otimes X \subset S(X)$, by Theorem 6.1. □

Theorem 6.2. *If X and Y are $\mathcal{S}$-admissible Banach spaces, then*

$$(S_b(X), Y) = (X, s(Y)) = s\big((X, Y)\big).$$

Proof. We have, by Theorem 2.3,

$$(\ell^\infty \otimes X, Y) = (\ell^\infty, (X, Y)) = s\big((X, Y)\big),$$

and

$$(X \otimes \ell^\infty, Y) = (X, (\ell^\infty, Y)) = (X, s(Y)).$$

□

It is not hard to see that if X is solid, then $X^a = X^K$, but, in the general case, we always have $X^K \subset X^a$.

Theorem 6.3. *If X is an $\mathcal{S}$-admissible Banach space, then*

$$(S_b(X))^a = (S_b(X))^K = s(X^a) = X^K.$$

Proof. By Theorem 2.3, we have

$$(\ell^\infty \otimes X, \mathcal{A}) = (\ell^\infty, X^a) = s(X^a),$$

and

$$(X \otimes \ell^\infty, \mathcal{A}) = (X, (\ell^\infty, \mathcal{A})) = X^K,$$

where we have used the easily verified relation $(\ell^\infty)^a = \ell^1$. □

7. Computing $H^1 \otimes X$ in some cases.

The aim of this section is to identify $H^1 \otimes X$ for some homogeneous Banach spaces X. According to Theorem 5.3 one can state the following general result.

Proposition 7.1. *If X has (FP) then we have that*

$$H^1 \otimes X = (H^1, X^*)^* = (X, BMOA)^*.$$

However this is not a direct description of the space, but relies upon the knowledge of the multiplier space. The following lemma is relevant for our purposes.

Lemma 7.1. *Let X be a homogeneous Banach space. Then there exist $A_1, A_2 > 0$ such that*

$$A_1 r^m \|f\|_X \leq M_X(r,f) \leq A_2 r^k \|f\|_X, \quad 0<r<1$$

whenever $f(z) = \sum_{j=k}^m a_j z^j$ where $0 \leq k < m$.

Proof. It is well known (see Lemma 3.1 [21]) that

$$r^m \|f\|_\infty \leq M_\infty(r,f) \leq r^k \|f\|_\infty, \quad 0<r<1.$$

Using Proposition 4.6 one has

$$\begin{aligned} r^m\|f\|_X &\approx r^m \|f\|_{X^{**}} \\ &\approx \sup\{r^m \|f * g\|_\infty : \|g\|_{X^*} = 1\} \\ &\leq C \sup\{M_\infty(r, f*g) : \|g\|_{X^*} = 1\} \\ &\approx \|f_r\|_{X^{**}} \approx M_X(r,f) \\ &\leq C r^n \sup\{\|f*g\|_\infty : \|g\|_{X^*} = 1\} \\ &\leq C r^n \|f\|_{X^{**}} \leq A r^n \|f\|_X. \end{aligned}$$

□

Lemma 7.2. *Let $X \subset \mathcal{H}(\mathbb{D})$ be homogeneous and $P(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \hat{P}(k) e_k$. Then there exist constants B_1 and B_2 such that*

$$B_1 2^n \|P*f\|_X \leq \|P * Df\|_X \leq B_2 2^n \|P*f\|_X, f \in X \tag{7.1}$$

Proof. We apply Lemma 7.1 to obtain

$$A_1 r^{2^{n+1}} \|P\|_X \leq M_X(r,P) \leq A_2 r^{2^{n-1}} \|P\|_X. \tag{7.2}$$

To show (7.1) apply (7.2) for $r_n = 1 - 2^{-n}$ and (3.1) to get first

$$\begin{aligned} \|P * Df\|_X &= \|D(P*f)\|_X \\ &\leq A M_X(r_n, D(P*f)) \\ &\leq A 2^n M_X(r_n, P*f) ds \\ &\leq A 2^n \|P*f\|_X. \end{aligned}$$

Also applying (3.2) one gets

$$\begin{aligned} \|P*f\|_X &\approx M_X(r_n, P*f) \\ &\leq A \int_0^{r_n} M_X(s, P*Df) ds \\ &\leq A \int_0^{r_n} s^{2^n} \|P*Df\|_X ds \\ &\leq A 2^{-n} \|P*Df\|_X. \end{aligned}$$

□

Theorem 7.1. *Let X be an homogeneous Banach space. Then*

$$\mathfrak{B}^{X,1} \subset H^1 \otimes X \subset X_{\mathcal{P}}.$$

Proof. From Proposition 2.2 it suffices to show that if $f \in H^1$ and $g \in X$ then $f * g \in X_{\mathcal{P}}$. From Lemma 3.1

$$M_X(r^2, f * g) \leq M_1(r, f) M_X(r, g) \leq K\|f\|_1 \|g\|_X.$$

Using Proposition 2.1 the polynomials are dense in $H^1 \otimes X$ and $H^1 \otimes X \subset X_p$ is shown.

Let us now show that $\mathfrak{B}^{X,1} \subset H^1 \otimes X$.

Let $\{W_n\}_0^\infty$ be a sequence of polynomials such that

$$\operatorname{supp}(\hat{W}_n) \subset [2^{n-1}, 2^{n+1}] \quad (n \geq 1), \quad \operatorname{supp}(\hat{W}_0) \subset [0, 1], \quad \sup_n \|W_n\|_1 < \infty$$

$$f = \sum_{n=0}^{\infty} W_n * f, \qquad f \in \mathcal{H}(\mathbb{D}).$$

Such a sequence exists (see, e.g., [4, 23, 17, 25] for possible constructions). Note that

$$\|(W_n * f)_r\|_X \leq K\|W_n\|_1 \|f_r\|_X \leq C\|f\|_X,$$

Hence, since $W_n * f$ is a polynomial, $\|W_n * f\|_X \leq C\|f\|_X$.

Denoting $Q_n = W_{n-1} + W_n + W_{n+1}$ we can write

$$f = \sum_{n=0}^{\infty} Q_n * W_n * f,$$

for all $f \in \mathcal{H}(\mathbb{D})$.

Note now that Lemma 7.2 allow us to conclude

$$\begin{aligned}
\sum_{n=0}^{\infty} \|Q_n\|_1 \|W_n * f\|_X &\leq K \sum_{n=0}^{\infty} \|W_n * f\|_X \\
&\leq K \sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+1)}} 2^n r^{2^n} \|W_n * f\|_X dr \\
&\leq K \sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+1)}} r^{2^n} \|W_n * Df\|_X dr \\
&\leq K \sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+1)}} M_X(r, W_n * Df) dr \\
&\leq K \sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+1)}} M_X(r, Df) dr \\
&= K \int_0^1 M_X(r, Df) dr \\
&= K\|f\|_{\mathfrak{B}^{X,1}}.
\end{aligned}$$

□

A property that turns out to be crucial for our purposes is the following one already mentioned in the introduction.

Definition 7.1. *Let $X \subset \mathcal{H}(\mathbb{D})$ be an homogeneous Banach space. We say that X satisfies (HLP) if $X \subset \mathfrak{B}^{X,2}$, i.e. there exits a constant $A > 0$ such that*

$$\int_0^1 (1-r)M_X^2(r, Df)dr \leq A\|f\|_X \tag{7.3}$$

Theorem 7.2. *Let X be an homogeneous Banach space satisfying (HLP). Then $H^1 \otimes X = \mathfrak{B}^{X,1}$.*

Proof. Due to Theorem 7.1 we only need to show that $H^1 \otimes X \subset \mathfrak{B}^{X,1}$. It suffices to see that $f * g \in \mathfrak{B}^{X,1}$ for each $f \in H^1$ and $g \in X$. Now using Lemma 3.1 we have,

$$\begin{aligned}
\int_0^1 M_X(r, D(f*g))rdr &\leq A\int_0^1 (\int_0^r M_X(s, D^2(f*g))ds)rdr \\
&\leq A\int_0^1 (1-s)M_X(s, D^2(f*g))ds \\
&\leq 2A(\int_0^1 (1-r^2)M_1(r, Df)M_X(r, Dg)rdr
\end{aligned}$$

Now from Cauchy-Schwarz (7.3) for $\mathbb{C}$-valued functions and (HLP) one obtains

$$\begin{aligned}
\int_0^1 (1-r^2)M_1(r, Df)M_X(r, Df)rdr &\leq (\int_0^1 (1-r^2)M_1^2(r, Df)rdr)^{1/2} \\
&. \ (\int_0^1 (1-r^2)M_X^2(r, Dg)rdr)^{1/2} \\
&\leq K\|f\|_1\|g\|_X
\end{aligned}$$

□

8. Applications

Our techniques allow us to describe $X \otimes Y$ in several cases. We only exhibit some applications, although many others can be achieved in a similar fashion.

As a consequence of Theorem 7.2 and Proposition 3.6 one obtains the following result.

Corollary 8.1. *Let $1 \leq p \leq 2$. Then*
 (i) $H^1 \otimes H^p = \mathfrak{B}^{p,1}$.
 (ii) $H^1 \otimes \ell^p = \ell^{p,1}$.

Let $1 \leq p, q \leq \infty$ and let $H^{p,q,\alpha}$ denote the mixed norm spaces of analytic functions in the unit disc given by the condition

$$\|f\|_{H^{p,q,\alpha}} = (\int_0^1 (1-r)^{\alpha q - 1} M_p(r, f)dr)^{1/q} < \infty, \quad q < \infty$$

and

$$\|f\|_{H^{p,\infty,\alpha}} = \sup_{0<r<1} (1-r)^\alpha M_p(r, f) < \infty, \quad q = \infty.$$

Recall that $p \ominus q$ stands for the value ∞ whenever $q \geq p$ and $\frac{1}{p \ominus q} = \frac{1}{q} - \frac{1}{p}$ whenever $q < p$, and that $\frac{1}{p*q} = \min\{1, \frac{1}{p} + \frac{1}{q}\}$.

Corollary 8.2. *Let $1 \leq q, u, v \leq \infty$. Then $\mathfrak{B}^{1,q} \otimes \mathfrak{B}^{u,v} = \mathfrak{B}^{u,q*v}$.*

Proof. This follows from Theorem 5.3, applying that the spaces $\mathfrak{B}^{p,q}$ have (FP) together with the facts that

$$(\mathfrak{B}^{p,q}, H^\infty) = \mathfrak{B}^{p',q'}, \qquad p, q \geq 1,$$

(see [1] for $p = 1$, $1 < q < \infty$; see [13] for the remaining cases) and

$$(\mathfrak{B}^{1,q}, \mathfrak{B}^{u',v'}) = \mathfrak{B}^{u',q\ominus v'}, \qquad q, u, v \geq 1. \tag{8.1}$$

Relation (8.1) is only a reformulation the following result on multipliers (see [17, Theorem 3.5]):

$$(H(1, q, 1), H(u', v', 1)) = \{\lambda \in \mathcal{H}(\mathbb{D}) : D\lambda \in H(u', q \ominus v', 1)\}.$$

□

We can now use our techniques to characterize the space of multipliers from H^1 in some cases.

Theorem 8.1. *Let X be a homogeneous Banach space with (HLP). Then*

$$(H^1, X^*) = \mathfrak{B}^{X^*,\infty}.$$
$$(H^1, X^K) = \mathfrak{B}^{X^K,\infty}.$$

Proof. Apply Theorem 2.3 together with Theorem 7.2 and Proposition 4.4 to obtain

$$(H^1, X^*) = (H^1 \otimes X, H^\infty) = (\mathfrak{B}^{X,1}, H^\infty) = \mathfrak{B}^{X^*,\infty}.$$

The other case is analogous. □

In particular the previous theorem yields the following results on multipliers from H^1 due, among others, to Hardy and Littlewood, Stein and Zygmund, Sledd (the cases H^q), to Mateljević and Pavlović (the case $BMOA$) and to Duren (the case ℓ^q).

Corollary 8.3. *Let $2 \leq q < \infty$. Then*

$$(H^1, H^q) = \mathfrak{B}^{q,\infty} \text{ (see [16],[28], [27])},$$
$$(H^1, BMOA) = \mathfrak{B} \text{ (see [22])},$$
$$(H^1, \ell^q) = \ell(q, \infty).$$

Also we can use our results to obtain spaces of multipliers into $BMOA$ in some cases.

Theorem 8.2. *Let X be a homogeneous Banach space with (HLP). Then*

$$(X, BMOA) = \mathfrak{B}^{X^*,\infty}.$$

Proof. Combining again Theorem 2.3 together with Theorem 7.2 and Proposition 4.4 one gets

$$(X, BMOA) = (X, (H^1, H^\infty)) = (X \otimes H^1, H^\infty) = (\mathfrak{B}^{X,1}, H^\infty) = \mathfrak{B}^{X^*,\infty}.$$

□

Corollary 8.4. *Let $1 \leq p \leq 2$. Then*

$$(H^p, BMOA) = \mathfrak{B}^{p',\infty} \text{ (see [24] and [17])},$$
$$(\ell^p, BMOA) = \ell(p', \infty).$$

The results allow also to recapture some of the multiplier results for Hardy-Lorentz spaces appearing in [19] using similar approaches.

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