Timing of Service, Frequency and Pricing

The following formal and numerical analysis corresponds with the comments towards the end of the paper *"Mixed Oligopoly, Product Differentiation and Competition for Public Transport Services"*. Specifically, it examines demand for each service so as to formally address the point that there is (direct) multiproduct firm competition by considering each service as a different product, no matter it is offered by the train or the bus company.

There are a number of papers dealing with multiproduct firms in address models of product differentiation. Thus, Martínez-Giralt and Neven (1988), Klemperer (1992) and De Fraja (1993) are representative references that consider horizontal product differentiation; Irmen and Thisse (1998) study a location-then-price game in a (horizontal) multi-characteristics space. As for vertical differentiation, the papers by Champsaur and Rochet (1989) and De Fraja (1996) merit to be cited. Two relevant contributions that combine horizontal and vertical differentiation in a setting are those of Neven and Thisse (1990) and Dos Santos Ferreira and Thisse (1996). Their analyses are confined to a setting with single-product firms. To the best of our knowledge, Ireland (1991), Gilbert and Matutes (1993) and Canoy and Peitz (1997) look at multiproduct firm competition under horizontal and vertical differentiation. Ireland (1991) does not model strategic interaction between modes whereas the latter two papers adopt a sequential game in the choice of product variants. Thus, the analysis that follows can be seen as a further step in modelling horizontal and vertical differentiation by letting firms select how many products (services) to offer. We explore the mixed duopoly regime and also the social optimum to complement the analysis in the main paper and check whether policy recommendations might change when e.g. a bus service is scheduled at a time between two train services.

1 The Model

Assume two firms, the bus and the train company. Each company can offer one or two products (services). A product is defined as a pair (x_i, h), denoting its position in the variety and quality spaces. Varietal characteristic x_i lies on a circumference of length 1 denoted by C; the quality level h belongs to the interval $[0, 1]$ and is exogenous. The space of product characteristics is thus a cylinder $C \times [0,1]$. Consumers do not rank the product varieties in the same way so that the first characteristic corresponds to some horizontal differentiation. The second characteristic portrays vertical differentiation since all consumers prefer a high quality to a low quality product.

Each consumer is defined by two parameters, z and y. z is interpreted as his most preferred variety (service) and lies on C, the circumference of length 1. y denotes the income of each consumer, and it lies on the interval $[y_1, y_1+1]$. The space of consumers' characteristics (z, y) is the cylinder $C\times [y_1, y_1+1]$. Assume that y_1 is large enough for all consumers to find a product for which their utility is positive in equilibrium; i.e. all consumers travel one journey. Consumers are uniformly distributed over $C\times$ $[y_1, y_1+1]$ with a total mass equal to one. A consumer of type (z, y) derives the following indirect utility from travelling by train

$$U_t(y) = y - p_{it} - v\,|z - x_i| \tag{1}$$

where y is the consumer's income, p_{it} the price charged by the ith train service, $|z - x_i|$ is the difference in time of the ith service from the consumer's ideal departure time, and v is the associated cost to such inconvenience. If travelling is by bus then a consumer derives indirect utility

$$U_b = h(y - p_{ib} - v\,|z - x_i|) \tag{2}$$

where p_{ib} denotes the price of the ith bus service. The parameter h captures a quality differential between both means of transport. We have arbitrarily chosen the train transport to be of a higher quality than the bus transport. We wish to solve the following two-stage game. In the first stage, both companies decide simultaneously and independently whether to supply one or two services. Services are located equidistantly from one another and the provision of one service entails a fixed cost ε. In the second stage, and knowing the first-stage choices, companies simultaneously and independently choose prices. There are four subgames to be solved.

- Suppose that each company chooses one service. This subgame is devoted by the pair $(1,1)$. Given (1) and (2) we can obtain the set of consumers indifferent between travelling by train and by bus.

$$\bar{y}(z) = \frac{2p_t - 2hp_b + 2vz(1+h) - vh}{2(1-h)} \tag{3}$$

This is a linear and increasing function of z and partitions total demand $C \times [y_1, y_1+1]$ in two groups of consumers (see Figure 1, the grey shaded are identifies the consumers travelling by bus). In order to obtain total demand for the bus company we have to integrate the function $\bar{y}(z)$ over $[y_1, y_1+1]$. There are several possibilities depending on the position and slope of $\bar{y}(z)$.[1] We will focus on the case where $\bar{y}(z)$ crosses the vertical sides of the square in the figure. This means that the quality difference dominates the difference in variety and is referred to as *vertical dominance*. Formally, $\left|\frac{\partial \bar{y}(z)}{\partial z}\right| <$ slope of the diagonal, that is, $\frac{v(1+h)}{(1-h)} < 2$. This results in a linear demand as follows,

$$D_b(1,1) = 2\int_0^{1/2} \bar{y}(z)dz = \frac{4p_t - 4hp_b + v(1-h)}{4(1-h)} \tag{4}$$

which is valid as long as p_b belongs to the interval $[p_b', p_b'']$, where p_b' is the price for which $\bar{y}(z=0) = y_1$ and p_b'' is the price for which $\bar{y}(z=1/2) = y_1+1$; otherwise, the function $\bar{y}(z)$ would not cross the vertical sides of the square. Similarly, demand for the train company is given by,

$$D_t(1,1) = 2(\frac{1}{2} - D_b) = \frac{4hp_b - 4p_t + (4-v)(1-h)}{4(1-h)} \tag{5}$$

which is valid as long as p_t belongs to the interval $[p_t', p_t'']$, where p_t' is the price for which $\bar{y}(z=1/2) = y_1+1$ and p_t'' is the price for which $\bar{y}(z=0) = y_1$. Each company maximizes profits, $\pi_b = p_b D_b$ and $\pi_t = p_t D_t$ resulting in the following equilibrium prices

$$p_b^*(1,1) = \frac{(1-h)(4+v)}{12h} \qquad p_t^*(1,1) = \frac{(1-h)(8-v)}{12} \tag{6}$$

provided that $p_b^* \in [p_b'(p_t^*), p_b''(p_t^*)]$ and $p_t^* \in [p_t'(p_b^*), p_t''(p_b^*)]$. Equilibrium profits are

$$\pi_b^*(1,1) = \frac{(1-h)(4+v)^2}{12h} - \varepsilon \quad \pi_t^*(1,1) = \frac{(1-h)(8-v)^2}{12} - \varepsilon \tag{7}$$

- Suppose that the train company offers one service and the bus company

[1] See Neven and Thisse (1990).

offers two services. This subgame is denoted by the pair $(1, 2)$. See Figure 2. We proceed as above to obtain

$$\breve{y}(z) = \frac{3p_t - 3hp_{b1} - vh + 3vz(1+h)}{3(1-h)} \tag{8}$$

where p_{b1} denotes the price of bus service one; vertical dominance occurs if $\frac{v(1+h)}{(1-h)} < 3$. Note now that, with two dimensions, all consumers in the backgarden of the bus service do not necessarily travel by bus; depending on their income and the degree of vertical differentiation some might travel by train. We need to find the set of consumers indifferent by solving $y - p_t - vz = h(y - p_{b1} - v(z - \frac{1}{3}))$, that is,

$$\tilde{y}(z) = \frac{3p_t - 3hp_{b1} + vh}{3(1-h)} + vz \tag{9}$$

so that total demand for bus service one is

$$D_{b1} = \int_0^{1/3} \breve{y}(z)dz + \int_{1/3}^{1/2} \tilde{y}(z)dz \tag{10}$$

and demand for the train service is $\frac{1}{2} - D_{b1}$. Since we are considering a circumference and bus services are (only) horizontally differentiated the indifferent z between bus service one and bus service two would be given by $z = \frac{p_{b2} - p_{b1} + v}{2v}$. Price discrimination is not allowed but, given the symmetry in the model, the symmetric solution $p_{b1} = p_{b2}$ will be an equilibrium. Hence, total demand for bus services is

$$D_b(1,2) = 2\frac{36p_t - 36hp_b + v(9-5h)}{72(1-h)} \tag{11}$$

which is valid as long as p_b belongs to the interval $[p_b', p_b'']$, where p_b' is the price for which $\bar{y}(z = 0) = y_1$ and p_b'' is the price for which $\tilde{y}(z = 1/2) = y_1 + 1$, and total demand for the train service is given by

$$D_t(1,2) = 2\frac{36hp_b - 36p_t + 36(1-h) - v(9-5h)}{72(1-h)} \tag{12}$$

which is valid as long as p_t belongs to the interval $[p_t', p_t'']$, where p_t' is the price for which $\tilde{y}(z = 1/2) = y_1 + 1$ and p_t'' is the price for which $\bar{y}(z = 0) =$

y_1.Profit maximization by each company results in the following equilibrium prices

$$p_b^*(1,2) = \frac{36(1-h)+v(9-5h)}{108h} \qquad p_t^*(1,2) = \frac{72(1-h)-v(9-5h)}{108} \tag{13}$$

provided that $p_b^* \in [p_b'(p_t^*), p_b''(p_t^*)]$ and $p_t^* \in [p_t'(p_b^*), p_t''(p_b^*)]$. Equilibrium profits are

$$\pi_b^*(1,2) = \frac{[36(1-h)+v(9-5h)]^2}{11664h(1-h)} - 2\varepsilon \quad \pi_t^*(1,2) = \frac{[72(1-h)-v(9-5h)]^2}{11664(1-h)} - \varepsilon \tag{14}$$

• Suppose that the train company offers two services and the bus company offers one service. See Figure 3. We proceed as above to obtain the following equilibrium prices

$$p_b^*(2,1) = \frac{36(1-h)-v(9h-5)}{108h} \qquad p_t^*(2,1) = \frac{72(1-h)+v(9h-5)}{108} \tag{15}$$

and equilibrium profits

$$\pi_b^*(2,1) = \frac{[36(1-h)-v(9h-5)]^2}{11664h(1-h)} - \varepsilon \quad \pi_t^*(2,1) = \frac{[72(1-h)+v(9h-5)]^2}{11664(1-h)} - 2\varepsilon \tag{16}$$

where $(2,1)$ means that the train company offers two services and the bus company offers one service.

• Suppose finally that both the train and the bus company offer two services each. See Figure 4. Then, equilibrium prices are given by

$$p_b^*(2,2) = \frac{(1-h)(8+v)}{24h} \qquad p_t^*(2,2) = \frac{(1-h)(16-v)}{24} \tag{17}$$

and equilibrium profits are

$$\pi_b^*(2,2) = \frac{(1-h)(8+v)^2}{576h} - 2\varepsilon \quad \pi_t^*(2,2) = \frac{(1-h)(16-v)^2}{576} - 2\varepsilon \tag{18}$$

where $(2,2)$ denotes that each company offers two services.

2 Characterization of Equilibrium

We begin by proving that, if the companies had to choose their location then they will not choose the same time. Suppose that in the $(1,1)$ subgame both the train and the bus service are located at point 0. Solving for the set of consumers indifferent between each mode of travel, i.e from equating $y-p_t-vz$ to $h(y-p_b-vz)$, we obtain $\dot{y}(z)=\frac{p_t-hp_b}{1-h}+vz$. Integrating this function between 0 and 1/2, and multiplying by two the demand for the bus service is obtained as $D_b=\frac{4p_t-4hp_b-4v(1-h)}{4(1-h)}$. Similarly, the demand for the train service is given by $D_t=\frac{4hp_b-4p_t+(4-v)(1-h)}{4(1-h)}$. Then, profit maximization results in the same equilibrium prices $p_b^*(1,1)$ and $p_t^*(1,1)$. However, equilibrium demands are different, $D_b^*=\frac{4+v}{12}$ and $D_t^*=\frac{8-v}{12}$ and so are equilibrium profits. These are

$$\pi_b(1,1)=\frac{(1-h)(4+v)^2}{144h}-\varepsilon \quad \pi_t(1,1)=\frac{(1-h)(8-v)^2}{144}-\varepsilon \tag{19}$$

which are obviously smaller than $\pi_b^*(1,1)$ and $\pi_t^*(1,1)$ above. Therefore, *it is the case that transport companies earn higher profits by scheduling their services at the opposite ends of the circumference diameter. Services will not be scheduled at the same time.*

Before computing the Nash equilibrium in the choice of the number of services we must bear in mind some restrictions. Firstly, the restriction for vertical dominance in the four subgames; it must the case that $\frac{v(1+h)}{(1-h)}<2$ or $h<\frac{2-v}{2+v}$. Since $h\in(0,1)$ then $v\in(0,2)$. Secondly, we must check that equilibrium prices are positive; this is indeed the case. Thirdly, there is a bound on the size of ε to ensure non-negative profits. Specifically, ε must be smaller than the $min\{\frac{[72(1-h)+v(9h-5)]^2}{2\times 11664(1-h)},\frac{[36(1-h)+v(9-5h)]^2}{2\times 11664h(1-h)},\frac{(1-h)(16-v)^2}{2\times 576}\}\equiv\bar{\varepsilon}$.[2] Figure 3 displays the areas in (v,h) space where each of these three bounds apply.

The train company will provide two services rather than one, provided that there is one bus service if $\pi_t^*(2,1)-\pi_t^*(1,1)>0$, that is, if

$$\frac{(72-9h(8-v)-7v)v}{1458(1-h)}>\varepsilon \tag{20}$$

[2]These computations are available in a Mathematica notebook.

Denote the l.h.s. by $\Psi_1(v,h)$. It can be checked that this function is positive for all $v \in (0,2)$ and $h \in (0, \frac{2-v}{2+v})$ and that it is smaller than $\bar{\varepsilon}$. Similarly, the train company will provide two services rather than one, provided that there are two bus services if $\pi_t^*(2,2) - \pi_t^*(1,2) > 0$, that is, if

$$\frac{(9-h)v(288-27v-h(288-19v))}{46656(1-h)} > \varepsilon \tag{21}$$

Denote the l.h.s. by $\Psi_2(v,h)$. It can be checked that this function is positive for all $v \in (0,2)$ and $h \in (0, \frac{2-v}{2+v})$ and that it is smaller than $\bar{\varepsilon}$. We now study the best response for the bus company. The bus company will provide two services rather than one, provided that there is one train service if $\pi_b^*(1,2) - \pi_b^*(1,1) > 0$, that is, if

$$\frac{v(36(1-h)+v(9-7h))}{1458(1-h)} > \varepsilon \tag{22}$$

Denote the l.h.s. by $\Psi_3(v,h)$. It can be checked that this function is positive for all $v \in (0,2)$ and $h \in (0, \frac{2-v}{2+v})$ and that it is smaller than $\bar{\varepsilon}$. Finally, The bus company will provide two services rather than one, provided that there are two train services if $\pi_b^*(2,2) - \pi_b^*(2,1) > 0$, that is, if

$$\frac{(1-9h)v(-144-19v+9h(16+3v))}{46656(1-h)h} > \varepsilon \tag{23}$$

Denote the l.h.s. by $\Psi_4(v,h)$. It can be checked that this function is positive for all $v \in (0,2)$ and $h \in (\frac{1}{9}, \frac{2-v}{2+v})$ and that it is smaller than $\bar{\varepsilon}$. These four functions Ψ can be ranked as follows:

$$\Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > \Psi_4(v,h) \tag{24}$$

for $v \in (0, 0.44)$ and for $h \in (0, \frac{288+45v-8\sqrt{2v(288+13v)}}{288-19v})$, and for $v \in (0.44, 2)$ and for $h \in (0, \frac{2-v}{2+v})$; otherwise the ordering is

$$\Psi_1(v,h) > \Psi_2(v,h) > \Psi_3(v,h) > \Psi_4(v,h) \tag{25}$$

Suppose that the ordering is given by (24) and recall that functions Ψ are smaller than $\bar{\varepsilon}$. Besides, $\Psi_4(v,h)$ is negative for $h < \frac{1}{9}$. Then, the next propositions characterize the Nash equilibrium in choice of the number of services.

Proposition 1 *For* $v \in (0, 2)$ *and for* $h \in (0, \frac{1}{9})$ *it occurs that*

$$\bar{\varepsilon} > \Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > 0 > \Psi_4(v,h)$$

a.1. If $\Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > \varepsilon > 0$ *then the unique Nash equilibrium is* $(2,1)$.
a.2. If $\Psi_2(v,h) > \Psi_1(v,h) > \varepsilon > \Psi_3(v,h) > 0$ *then the unique Nash equilibrium is* $(2,1)$.
a.3. If $\Psi_2(v,h) > \varepsilon > \Psi_1(v,h) > \Psi_3(v,h) > 0$ *then the unique Nash equilibrium is* $(1,1)$.
a.4. If $\varepsilon > \Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > 0$ *then the unique Nash equilibrium is* $(1,1)$.

Proposition 2 *For* $v \in (0, 0.44)$ *and for* $h \in (\frac{1}{9}, \frac{288+45v-8\sqrt{2v(288+13v)}}{288-19v})$ *and for* $v \in (0.44, 2)$ *and for* $h \in (\frac{1}{9}, \frac{2-v}{2+v})$ *it occurs that*

$$\bar{\varepsilon} > \Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > 0$$

b.1. If $\Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > \varepsilon > 0$ *then the unique Nash equilibrium is* $(2,2)$.
b.2. If $\Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > \varepsilon > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(2,1)$.
b.3. If $\Psi_2(v,h) > \Psi_1(v,h) > \varepsilon > \Psi_3(v,h) > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(2,1)$.
b.4. If $\Psi_2(v,h) > \varepsilon > \Psi_1(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(1,1)$.
b.5. If $\varepsilon > \Psi_2(v,h) > \Psi_1(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(1,1)$.

Suppose now that the ordering is given by (25). This occurs for $v \in (0, 0.44)$ and $h \in (\frac{288+45v-8\sqrt{2v(288+13v)}}{288-19v}, \frac{2-v}{2+v})$. Since $\Psi_4(v,h)$ is negative for $h \in (0, \frac{1}{9})$ and $\frac{1}{9} < \frac{288+45v-8\sqrt{2v(288+13v)}}{288-19v}$ for $v \in (0,2)$ we can state the following proposition.

Proposition 3 *For* $v \in (0, 0.44)$ *and for* $h \in (\frac{288+45v-8\sqrt{2v(288+13v)}}{288-19v}, \frac{2-v}{2+v})$ *it occurs that*

$$\bar{\varepsilon} > \Psi_1(v,h) > \Psi_2(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > 0$$

c.1. If $\Psi_1(v,h) > \Psi_2(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > \varepsilon > 0$ *then the unique Nash equilibrium is* $(2,2)$.
c.2. If $\Psi_1(v,h) > \Psi_2(v,h) > \Psi_3(v,h) > \varepsilon > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(2,1)$.
c.3. If $\Psi_1(v,h) > \Psi_2(v,h) > \varepsilon > \Psi_3(v,h) > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(2,1)$.
c.4. If $\Psi_1(v,h) > \varepsilon > \Psi_2(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(2,1)$.
c.5. If $\varepsilon > \Psi_1(v,h) > \Psi_2(v,h) > \Psi_3(v,h) > \Psi_4(v,h) > 0$ *then the unique Nash equilibrium is* $(1,1)$.

3 The Mixed Duopoly

We now assume that the train company is a public firm and that, in concordance with the literature on mixed oligopolies, it maximizes total surplus when setting prices of train services. Specifically, its objective function will consist of consumer surplus, profits of the bus company and profits of the train company. We therefore need to compute consumer surplus. Consider the case in which each company offers one service. Consumer surplus for those who travel by bus is given by

$$CS_b(1,1) = 2\int_0^{1/2}\left[\int_{y_1}^{\bar{y}(z)} h(y - p_b - vz)dy\right]dz$$

and for those who travel by train it is given by

$$CS_t(1,1) = 2\int_0^{1/2}\left[\int_{\bar{y}(z)}^{y_1+1} (y - p_t - vz)dy\right]dz$$

Consequently, the bus company maximizes $\pi_b = p_b D_b$ and the train company $TS_t = p_t D_t(1,1) + p_b D_b(1,1) + CS_b(1,1) + CS_t(1,1)$, which yields the following equilibrium prices

$$p_b^{M*}(1,1) = \frac{(1-h)(4y_1 - v)}{4(1-2h)} \qquad p_t^{M*}(1,1) = \frac{(1-h)(8hy_1 - v)}{4(1-2h)}$$

and equilibrium profits are

$$\begin{aligned}\pi_b^{M*}(1,1) &= \frac{(1-h)h(4y_1 - v)^2}{16(1-2h)^2} - \varepsilon \\ \pi_t^{M*}(1,1) &= \frac{(1-h)(8hy_1 - v)(h(4y_1 - v) - 4(1-2h))}{16(1-2h)^2} - \varepsilon\end{aligned}$$

where superscript M stands for mixed duopoly. The remainder subgames are solved analogously.[3]

4 A Numerical Example

Consider the following values: $y_1 = .197$, $v = .1$ and $h = .7$. We have the following ordering $\Psi_2(v,h) \equiv .0050 > \Psi_1(v,h) \equiv .0049 > \Psi_3(v,h) \equiv$

[3] All the computations in this paper are available in Mathematica notebooks.

$.0025 > \Psi_4(v,h) \equiv .0023 > 0$. The bound on ε is given by $\frac{[36(1-h)+v(9-5h)]^2}{2\times 11664h(1-h)} \equiv .026$. These values ensure that equilibrium prices are such that all consumers purchase the product in the four subgames; vertical dominance holds and demand is linear. Next we illustrate the subgame perfect Nash equilibrium in prices and frequencies corresponding to proposition 2 above. Let us begin by taking $\varepsilon = .00495$ (part b.4). Then, the equilibrium prices are the following:

$$\begin{aligned} p_t^*(1,1) &= .1975 \; p_b^*(1,1) = .1464; \quad p_t^*(1,2) = .1949 \; p_b^*(1,2) = .1501 \\ p_t^*(2,1) &= .2012 \; p_b^*(2,1) = .1411; \quad p_t^*(2,2) = .1986 \; p_b^*(2,2) = .1446 \end{aligned}$$

so that the high-quality firm sets a higher price and also has a greater market share than the low-quality firm. The Nash equilibrium of the next payoff matrix is $(1,1)$:

train/bus	1 service	2 services
1 service	.1251, .045	.1216, .043
2 services	.1250, .042	.1218, .039

Consumer surplus and total surplus are the following:

$$\begin{aligned} CS(2,2) &= .489 > CS(2,1) = .486 > CS(1,2) = .479 > CS(1,1) = .478 \\ TS(2,1) &= .652 > TS(2,2) = .649 > TS(1,1) = .648 > TS(1,2) = .643 \end{aligned}$$

Now take $\varepsilon = .003$ (part b.3). The equilibrium prices are obviously the same and so happens for consumer surplus. However, the payoff matrix changes to the following,

train/bus	1 service	2 services
1 service	.127, .047	.123, .046
2 services	.129, .043	.126, .043

where $(2,1)$ is the Nash equilibrium. We have that consumer surplus increases and total surplus is ranked as,

$$TS(2,1) = .658 > TS(2,2) = .657 > TS(1,1) = .652 > TS(1,2) = .649$$

Finally, set $\varepsilon = .0015$ (part b.1) to obtain the following payoff matrix,

train/bus	1 service	2 services
1 service	.128, .048	.125, .049
2 services	.132, .045	.129, .046

where $(2,2)$ is the Nash equilibrium. Total surplus is ranked as,

$$TS(2,2) = .664 > TS(2,1) = .663 > TS(1,1) = .656 > TS(1,2) = .654$$

As for the equilibrium under the mixed duopoly, it must be noted that, as in the paper, we need to weight the profits of the train company to verify the non-negativity constraint on profits and hence do not obtain negative equilibrium prices or demands. In particular, the maximization of TS_t in the corresponding subgame includes the weight 1.4 ahead of π_t. The equilibrium prices are the following:

$$\begin{aligned} p_t^M(1,1) &= .0731\ p_b^M(1,1) = .0576;\ \ p_t^M(1,2) = .0688\ p_b^M(1,2) = .0601 \\ p_t^M(2,1) &= .0754\ p_b^M(2,1) = .0513;\ \ p_t^M(2,2) = .0709\ p_b^M(2,2) = .0533 \end{aligned}$$

Consumer surplus and total surplus are the following (for $\varepsilon = .003$):

$$\begin{aligned} CS^M(2,2) &= .614 > CS^M(1,2) = .602 > CS^M(1,1) = .601 > CS^M(2,1) = .597 \\ TS^M(2,2) &= .669 > TS^M(1,1) = .666 > TS^M(2,1) = .661 > TS^M(1,2) = .660 \end{aligned}$$

It should be mentioned that, for these particular values, subgame $(2,2)$ entails all consumers travelling by train. In fact, it is very likely to obtain this type of corner solutions when computing the social optimum or total surplus is the objective function of one of the firms. Thus, maximization of total surplus when each company offers one service yields prices $p_b = .172$ and $p_t = .113$. This would mean that the consumer at distance $1/2$ in Figure 1 obtains negative utility and hence does not travel by bus. We then compute the price such that this consumer would obtain non negative utility; from $h(y_1 - p_b - v(1/2) \geq 0$ we have $p_b = .147$. Inserting this value in the f.o.c. $\partial TS(1,1)/\partial p_t = 0$ and solving yields $p_t = .0879$. Still the whole market would be served by the train service. Even if the train company lowered the price it would not capture any consumers (for the values in this numerical example). Something similar occurs in the remaining subgames. We next report the equilibrium prices noting that although there would be bus services scheduled at particular times, everybody would travel by train and consequently bus profits would be zero.

$$\begin{aligned} p_t^S(1,1) &= .0879; \; p_t^S(1,2) = .0968 \\ p_t^S(2,1) &= .0956; \; p_t^S(2,2) = .1129 \end{aligned}$$

Consumer surplus and total surplus are the following (for $\varepsilon = .003$):

$$\begin{aligned} CS^S(1,1) &= .584 > CS^S(1,2) = .575 > CS^S(2,2) = .572 > CS^S(2,1) = .559 \\ TS^S(1,1) &= .756 > TS^S(2,2) = .679 > TS^S(1,2) = .669 > CS^S(2,1) = .666 \end{aligned}$$

Thus, the comparison of prices under each regime is $p_b^* > p_b^M > p_b^S$ and $p_t^* > p_t^S > p_t^M$. It is also the case that total surplus TS under mixed duopoly does not attain the social optimum but is always higher than that under private duopoly. The fact that the above example yields corner solutions makes it such that the equilibrium choice of frequencies under private duopoly might be $(1,1)$ or $(2,2)$, which are certainly close to the choices that maximize total welfare. In this particular case, and since prices are lower in the social optimum, it seems that authorities, when faced with travellers in the same multimodal station should definitely care for increasing the number of services supplied by the high-quality firm. Furthermore, and in contrast with previous work on mixed oligopolies, the above example discloses that a mixed duopoly does not recover the socially optimal solution.

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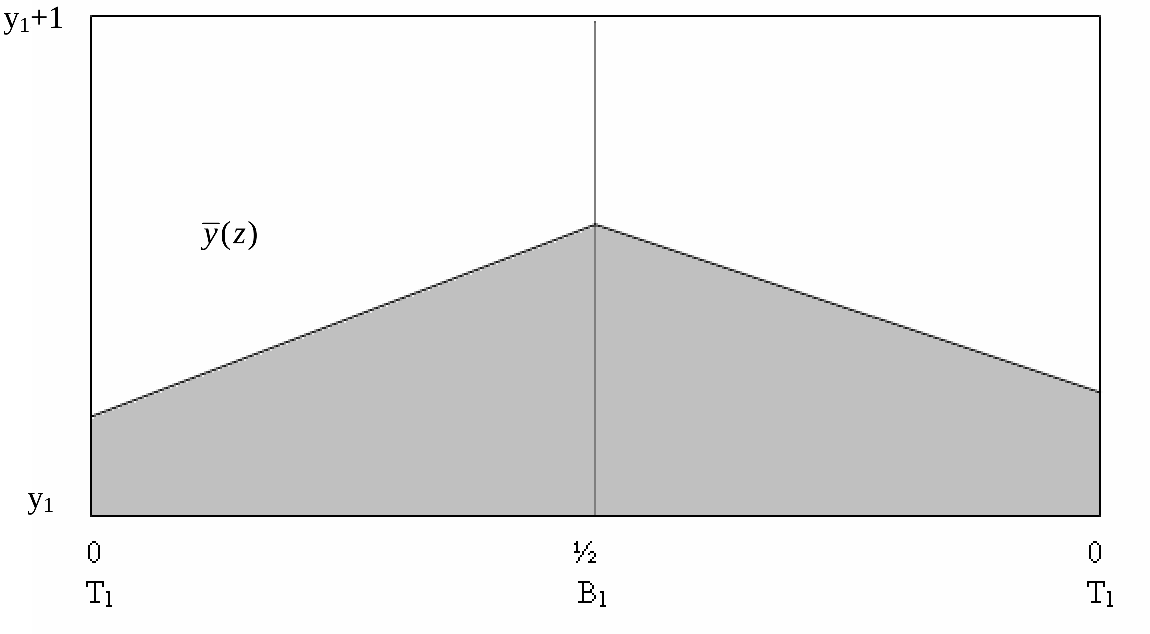
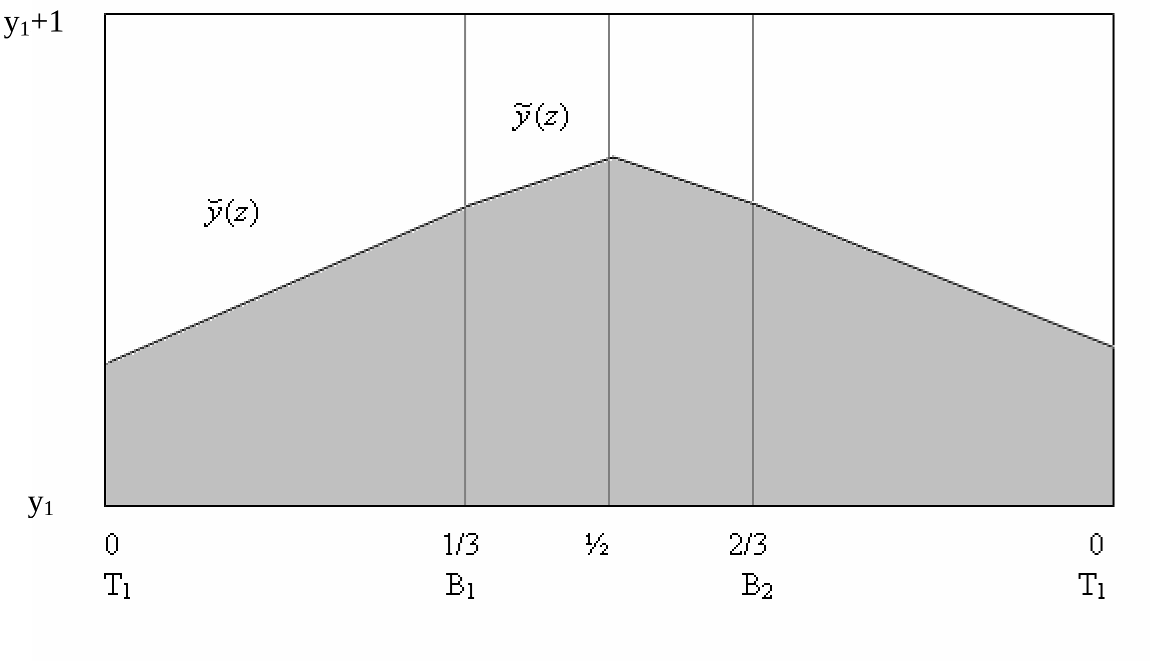
Figure 1



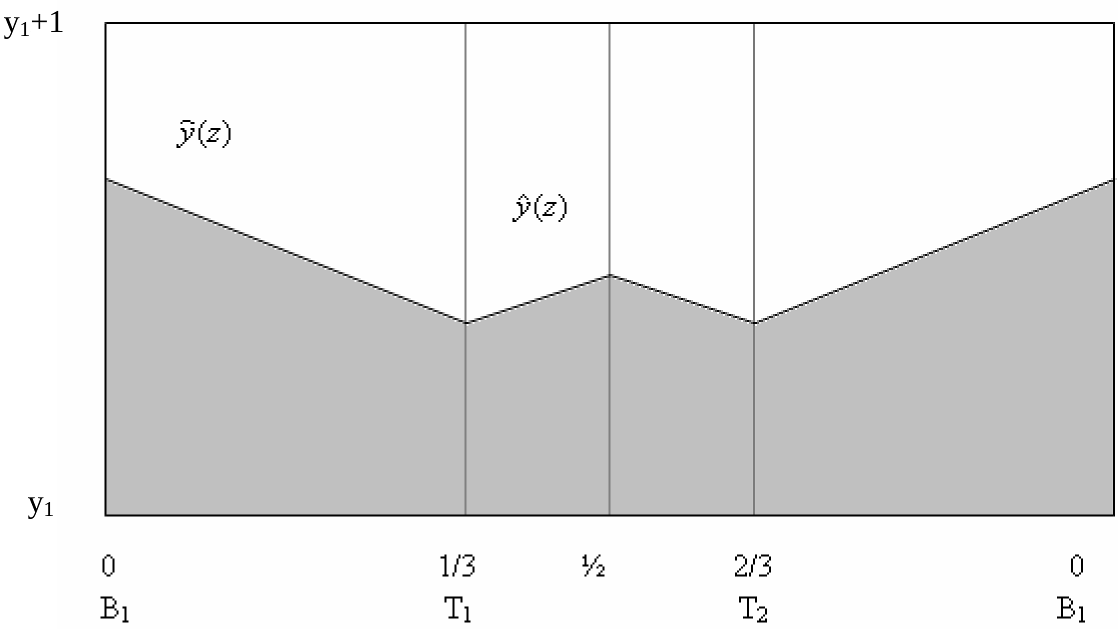
$$\bar{y}(z) = \frac{2p_t - 2hp_b + 2vz(1+h) - vh}{2(1-h)}$$

Figure 2



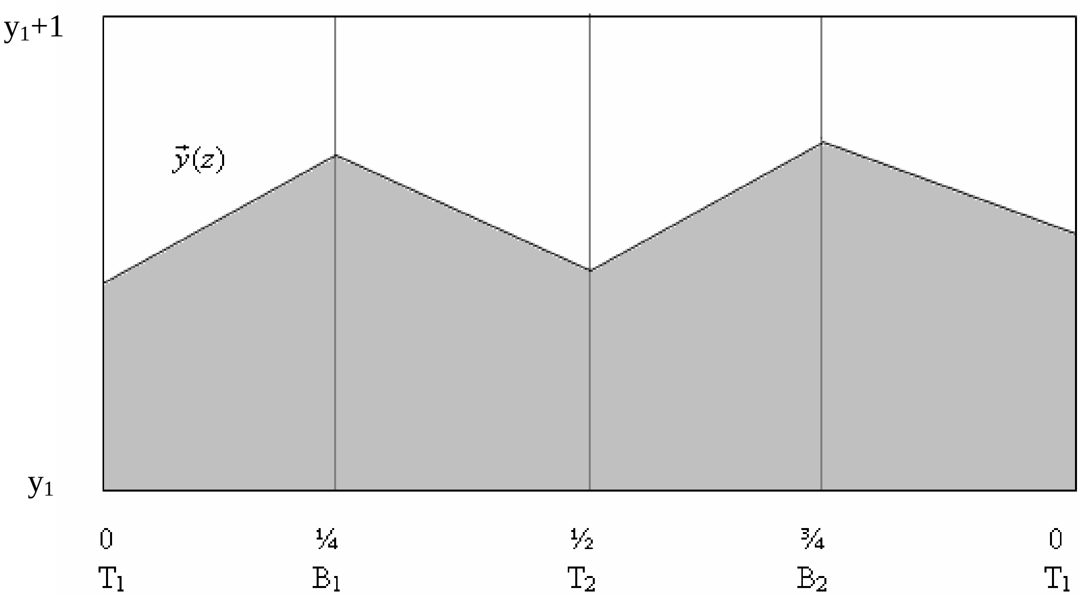
$$\breve{y}(z) = \frac{3p_t - 2hp_{b1} + 3vz(1+h) - vh}{3(1-h)}; \qquad \tilde{y}(z) = \frac{3p_t - 3hp_{b1} + vh}{3(1-h)} + vz$$

Figure 3



$$\breve{y}(z) = \frac{3p_{t1} - 2hp - 3vz(1+h)}{3(1-h)}; \qquad \hat{y}(z) = \frac{3p_{t1} - 3hp_b - v}{3(1-h)} + vz$$

Figure 4



$$\vec{y}(z) = \frac{4p_{t1} - 4hp_{b1} + 4vz(1+h) - vh}{4(1-h)}$$