Undecidability of the Spectral Gap Problem

David Pérez-García

Joint work with Toby Cubitt (Cambridge) and Michael Wolf (Munich)

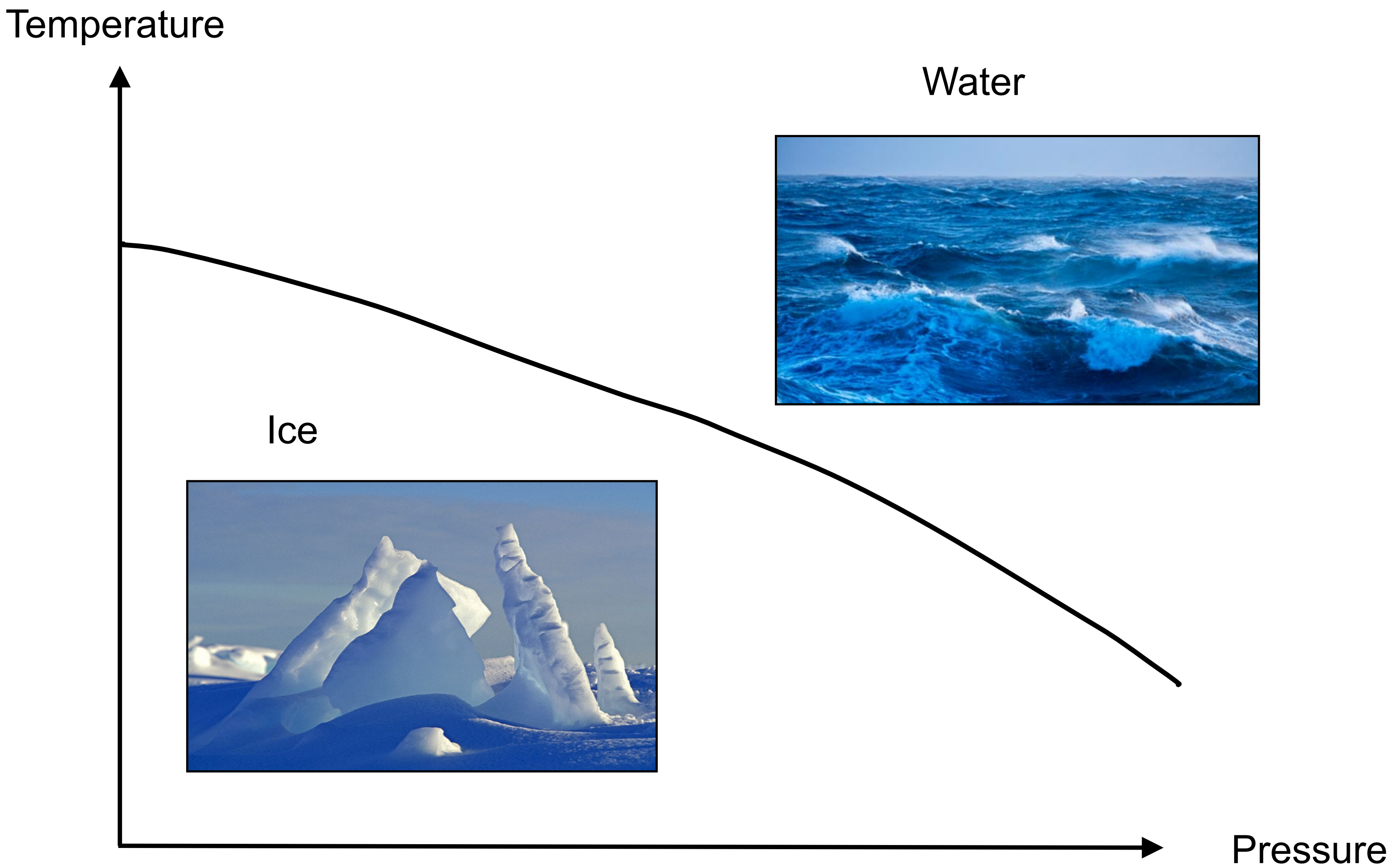
Outlook

- Introduction
- Our result
- Some consequences
- Main ingredients in the proof

INTRODUCTION

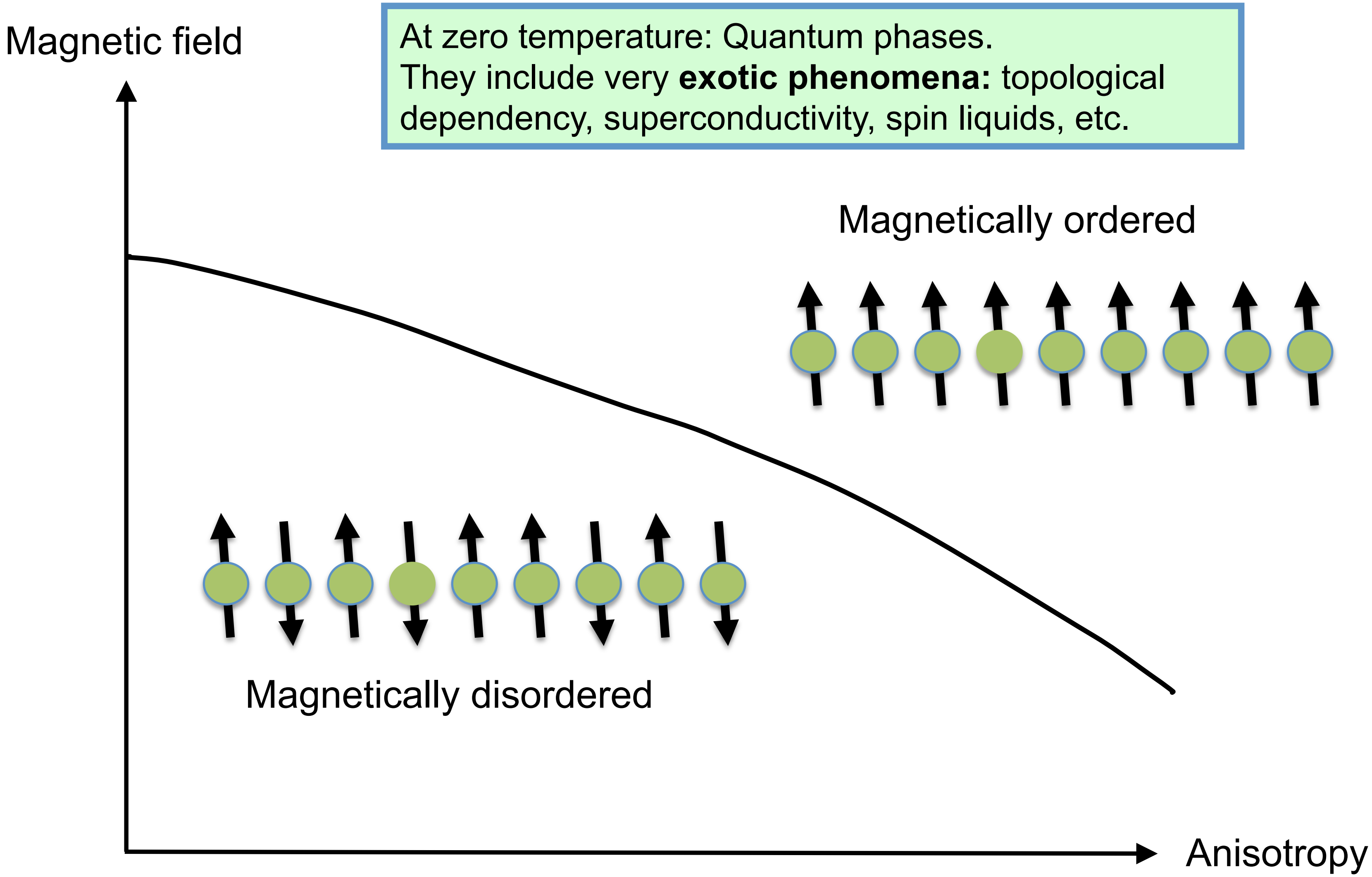
Phases and phase transitions

Phase transitions in 2 slides

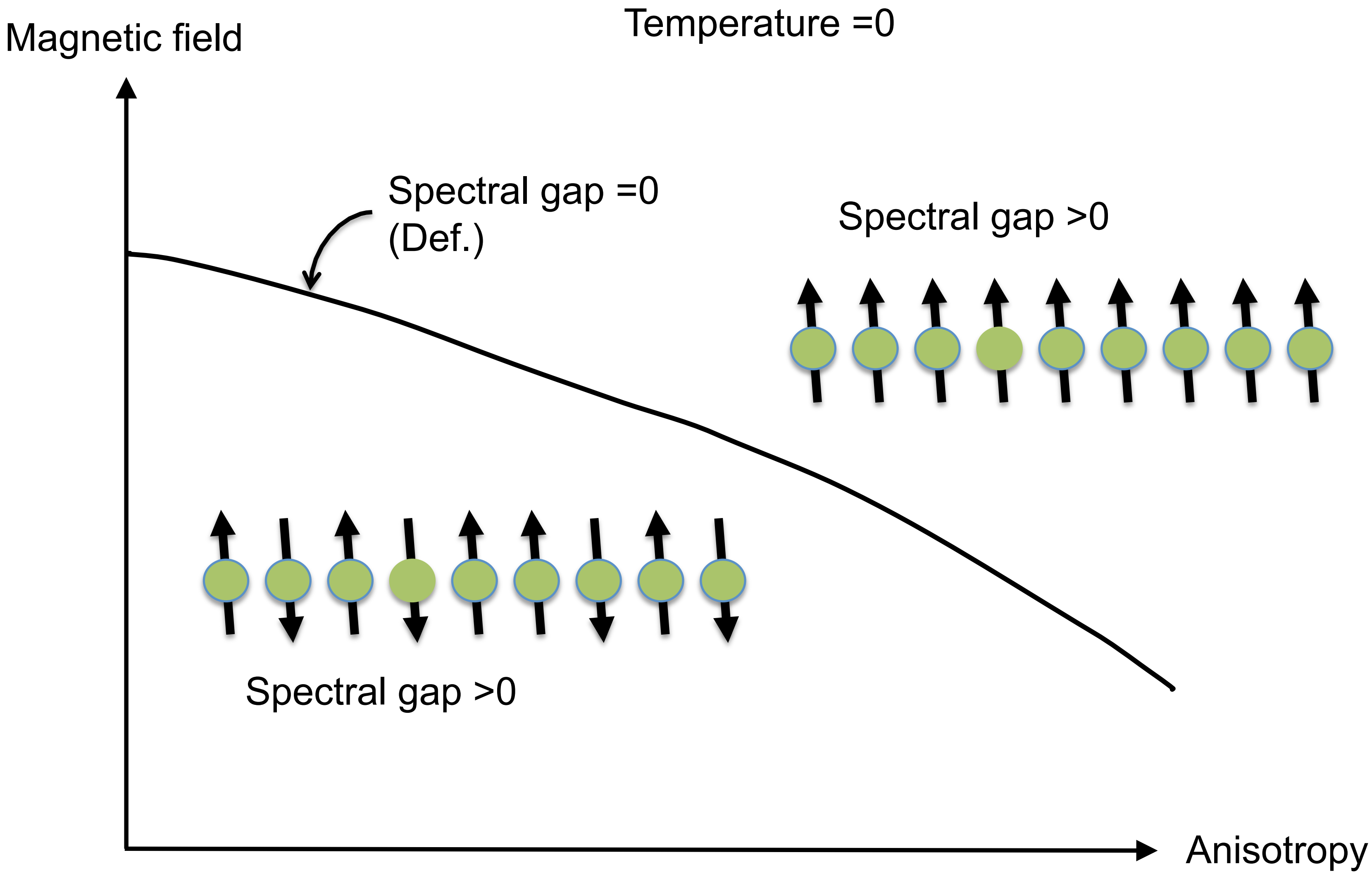


Quantum phase transitions

At zero temperature: Quantum phases.
They include very **exotic phenomena:** topological dependency, superconductivity, spin liquids, etc.

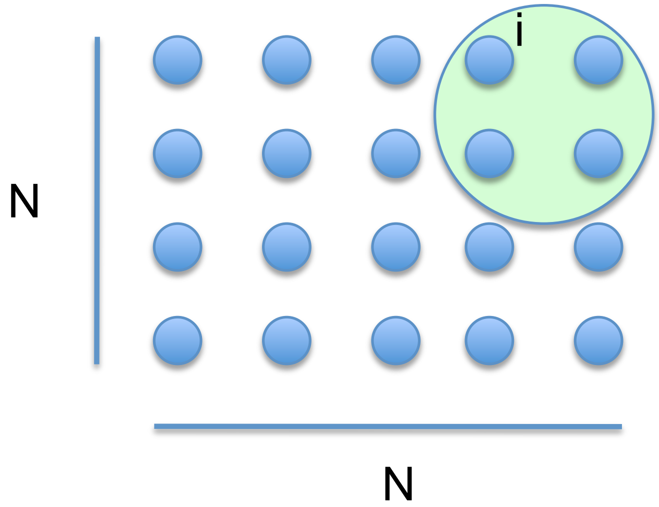


Quantum phase transitions



INTRODUCTION
Spectral gap problem

Quantum mechanics in 2 slides



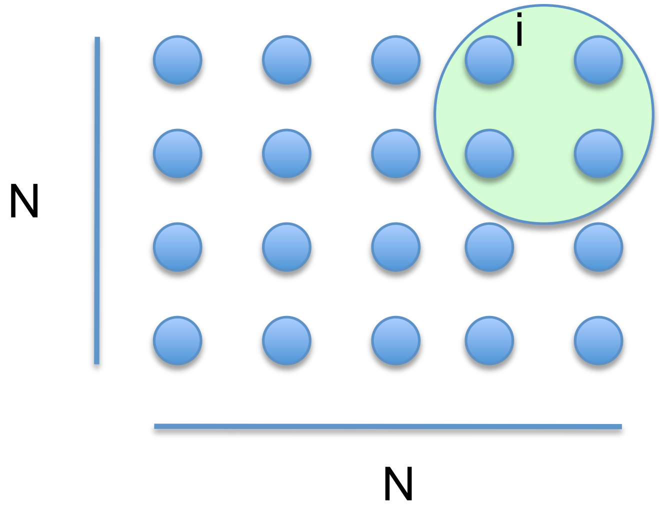
Particles in a lattice

A space $H_i = C^d$ associated to each particle

Space of the joint system = tensor product

$$= \bigotimes_i H_i \cong C^{d^{N^2}}$$

Quantum mechanics in 2 slides



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Huge!!

h_i matrix h located at position i.

Hamiltonian $H = \sum_i h_i \otimes 1_{rest}$ matrix of size $d^{N^2} \times d^{N^2}$

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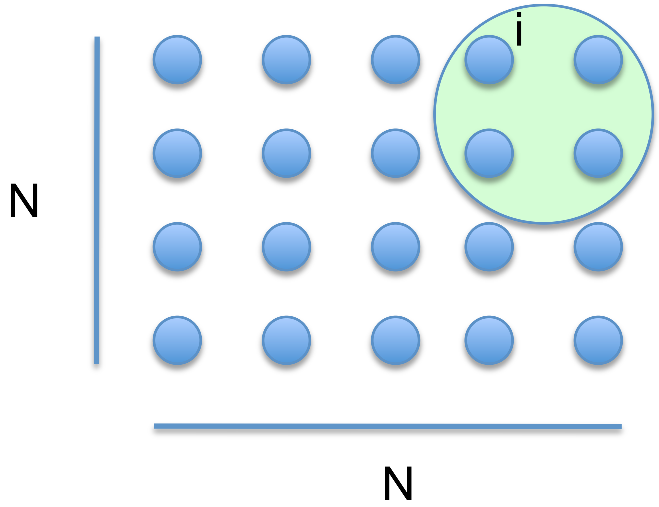

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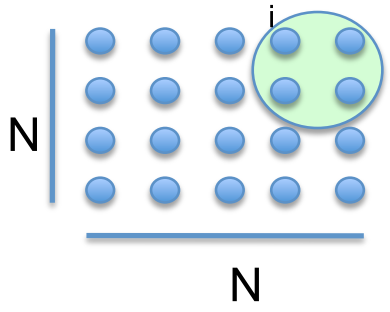
Normalized vectors in $\bigotimes_i H_i$ are the ***states*** of the system (encode all properties)

Hamiltonian H = Energy: A state v has energy = $v^t H v$

Energy levels of the system = eigenvalues of H.

Quantum mechanics in 2 slides



States with minimal energy = eigenvector of $\lambda_0(N)$
Called ***ground states***.

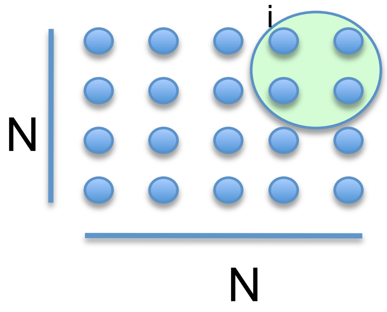
Eigenvectors are stable states since the evolution eq. is

$$\frac{\partial v(t)}{\partial t} = -iHv(t)$$

Eigenstates of $\lambda_1(N)$ called elementary ***excitations***.

Spectral Gap: $\Delta_N = \lambda_1(N) - \lambda_0(N)$
Energy to pay to jump from ground states to excited states

Quantum mechanics in 2 slides



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Spectral Gap problem:

How does the spectral gap behave as N goes to infinity?

Does the system have gap? i.e is there a c>0 such that $\Delta_N > c$ for all N?

Why is it interesting?

Spectral Gap in **condensed matter physics**:

- It defines the concept of quantum ***phase***, *phase transition, phase diagram, …*

Spectral Gap in **quantum information and computation**:

- It measures the efficiency in both adiabatic and dissipative quantum computation and quantum state engineering

Spectral Gap in **high energy physics:**

- It is connected (via discretization) with particle masses and hence with the Yang-Mills millenium problem.

OUR RESULT

Our result (informal statement)

Problem (Spectral Gap):

Input: nearest-neighbor interaction h

Output: decide if H has a gap or not.

Theorem:

The Spectral Gap problem is undecidable.

⇕

There is no algorithm that on input h decides it

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The Spectral Gap problem is undecidable.

$\Updownarrow$

There is no algorithm that on input h decides it

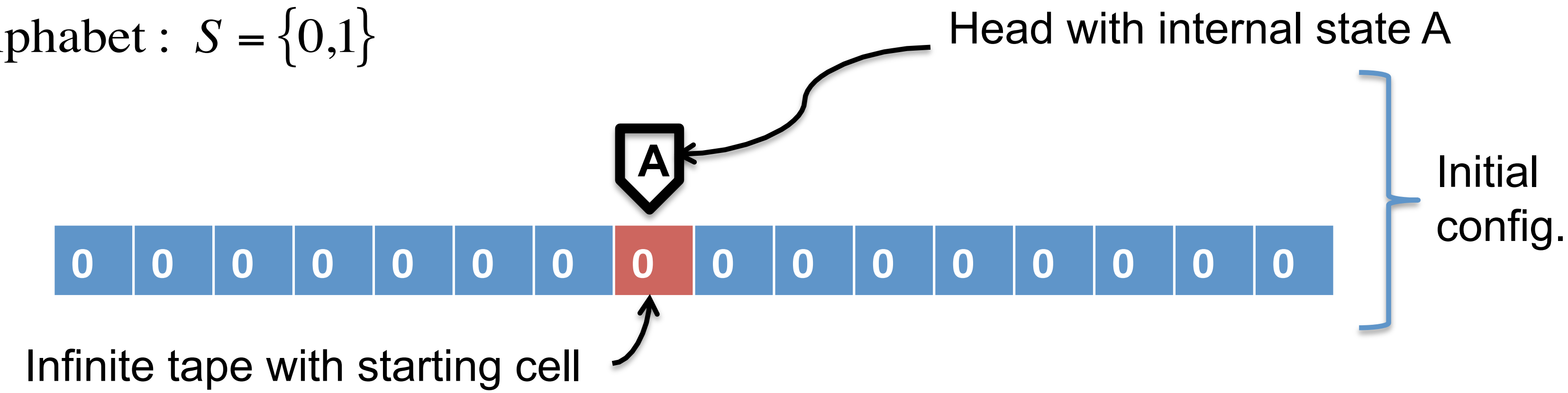
Corollary: There exist nearest neighbor interactions for which the existence or absence of gap cannot be proven within the axioms of mathematics.

Proof by reduction from the halting problem of a Turing Machine.

Turing Machines (TM) in 2 slides

Finite number of internal states : $Q = \{A, B, C, \ldots\} \cup \{\text{halting state H}\}$

Finite alphabet : $S = \{0,1\}$



Instructions : $\delta : Q \times S \to Q \times \{L, R\} \times S$

Turing Machines $\Leftrightarrow$ Natural numbers

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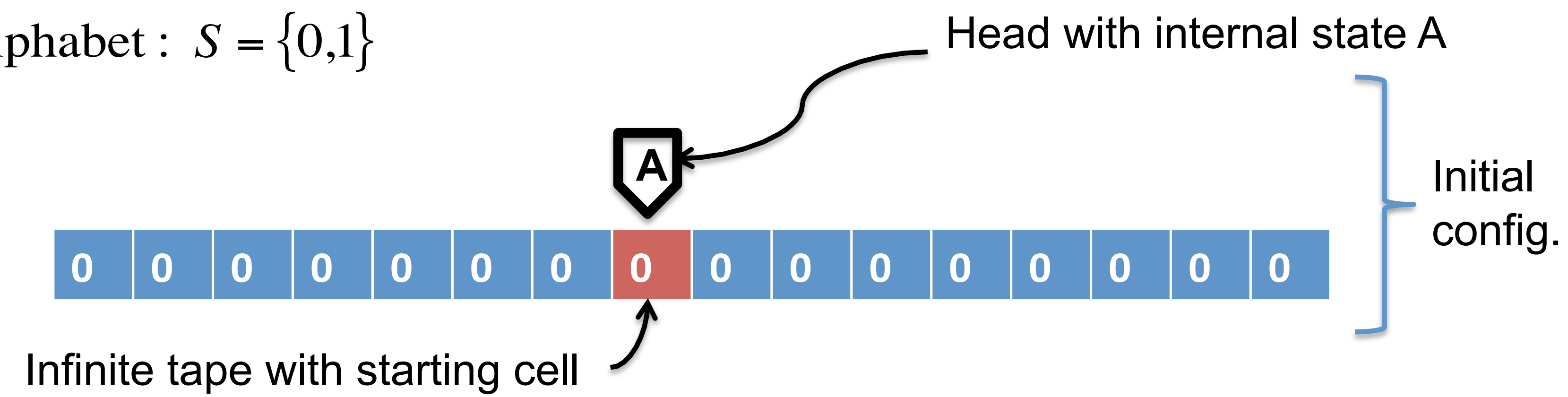
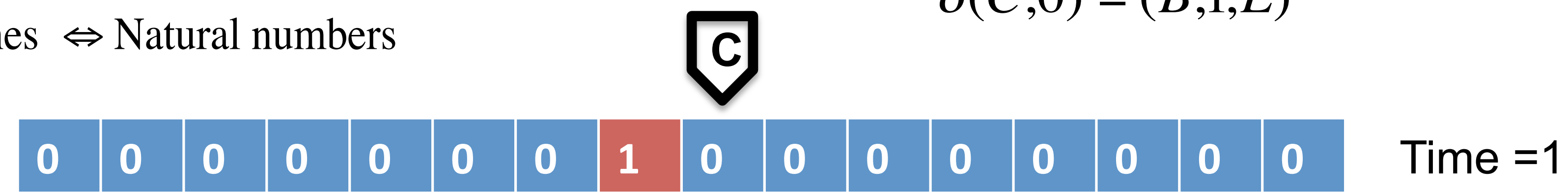


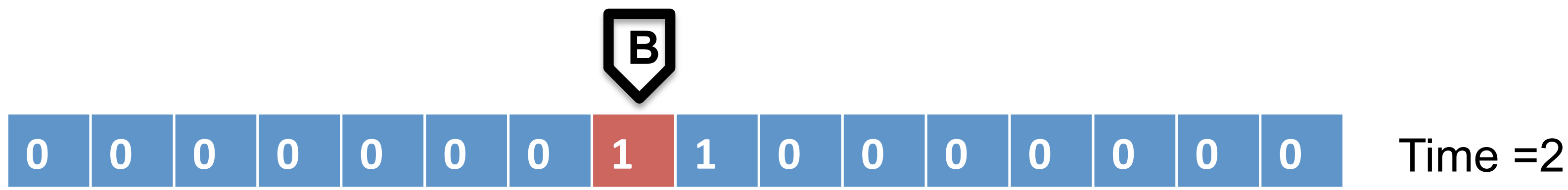
Instructions: $\delta : Q \times S \rightarrow Q \times \{L, R\} \times S$

E.g. $\delta(A,0) = (C,1,R)$

$\delta(C,0) = (B,1,L)$

Turing Machines $\Leftrightarrow$ Natural numbers

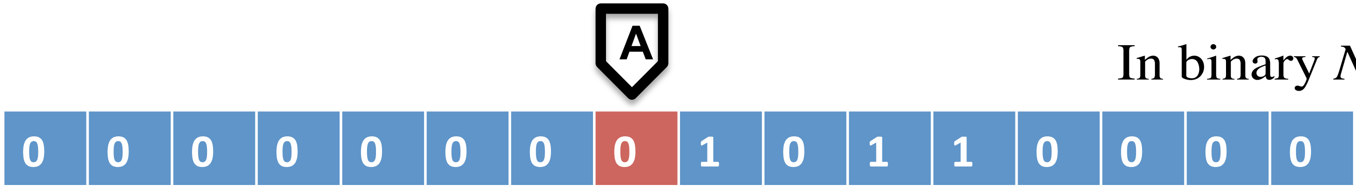




The halting problem of a TM

A TM **halts on input N** if it eventually enters the halting state
when the TM starts with the head in the starting cell and starting internal state, and with the tape initialized in N, written in binary just at the right of the starting cell.

In binary $N = 1101$



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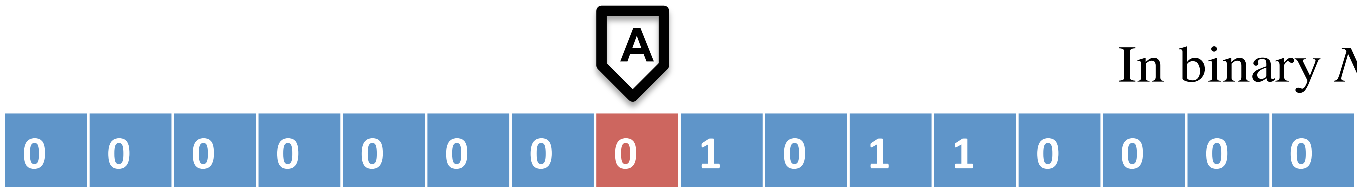

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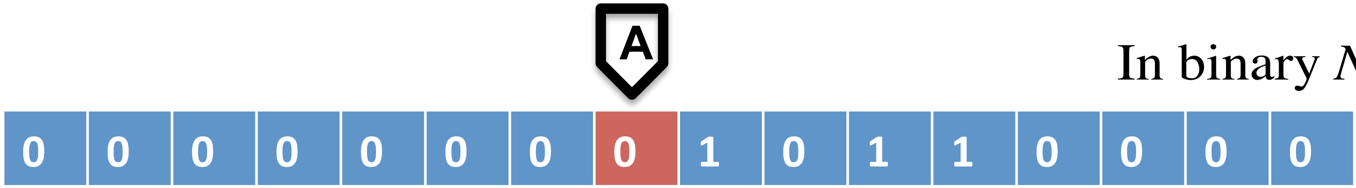

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Theorem (1936, Turing): There exists a TM M, called *universal* (UTM), so that it halts on input N iff the TM=N halts on input 0.

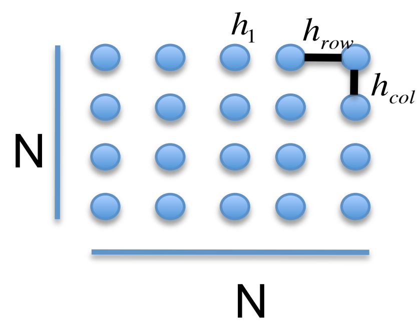
Corollary: There is no algorithm that on input a natural number N, decides whether the UTM halts or not on input N.

Our result (formal statement)

Theorem:

the Hamiltonian given by

$$H(n) = \sum_{rows}\sum_{c} h_{row}^{(c,c+1)} + \sum_{columns}\sum_{r} h_{row}^{(r,r+1)} + \sum_{i} h_1^i$$



Have the following properties

1. All terms $h_{col}(n), h_{row}(n), h_1(n)$ have operator norm bounded by 1.
2. If the UTM halts on input n, then in the thermodynamic limit $0 \in \mathrm{spec}(H) \subset \{0\} \cup [1,\infty)$ (that is, the gap is ≥ 1)
3. If the UTM does not halt on input n, then in the thermodynamic limit $\mathrm{spec}(H) = [0,\infty)$

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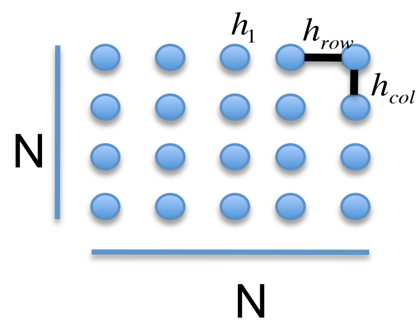
Given the binary rep. of a natural number $n = n_1 n_2 .. n_{|n|}$, we call $\varphi = 0, n_{|n|} .. n_2 n_1 \in Q$

Theorem:

$$\begin{cases} h_{col}(n) = A \\ h_{row}(n) = B + \beta\left(e^{i\pi\varphi} C + e^{i\pi 2^{-|\varphi|}} D + h.c.\right) \\ h_1(n) = \alpha(n)\mathrm{Id} \end{cases}$$

Only dependency on n



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Theorem: We give explicitly a dimension d, $d^2 \times d^2$ matrices A,B,C,D and a rational number β so that

- A, B are hermitian and with coefficients in $Z + \beta Z + \frac{\beta}{\sqrt{2}} Z$
- C, D have integer coefficients

And if we define, for each natural number n,
($\alpha(n)$ an algebraic computable number).

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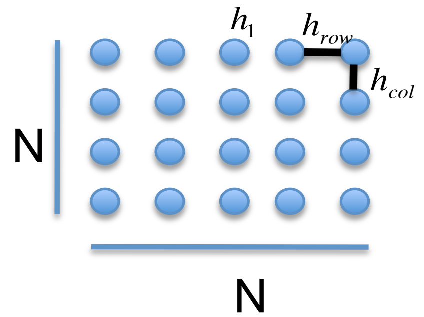

Then the Hamiltonian given by

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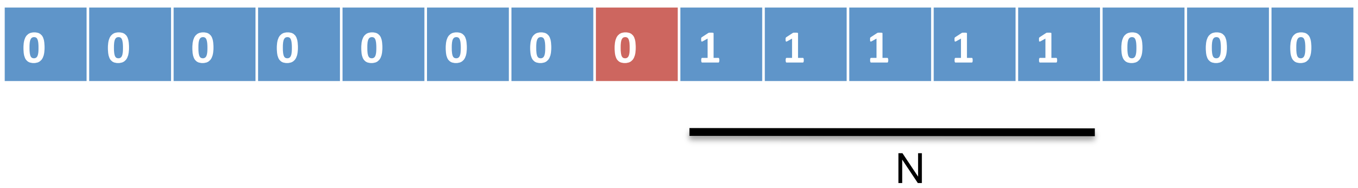
Ingredient 1 (Writing n on the tape). Construction for each n of a QUANTUM Turing Machine *with size independent of n*, so that, for each N>|n|, on input



Outputs always n (in binary)

Ingredients of the proof

Ingredient 2. (a Hamiltonian whose spectrum depends on whether the UTM on input n halts or not)

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Construction, for each classical or quantum Turing Machine, of a 1D nearest neighbor interaction whose ground state encodes the evolution of the TM.

By adding a penalty term to the halting state, one gets that the ground state has energy 0 if the TM does not halt and energy exp(-N) if it halts (N size of the system).

By choosing the QTM of Ingredient 1 followed by the UTM we get a Hamiltonian which has different ground energies depending on the behavior of the UTM on input n.

Ingredients of the proof

Ingredient 3. (Amplifying the spectral difference between halting and not-halting)

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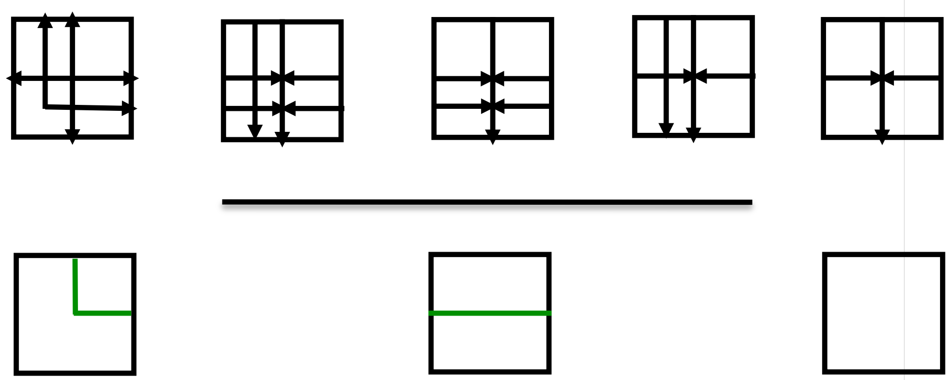
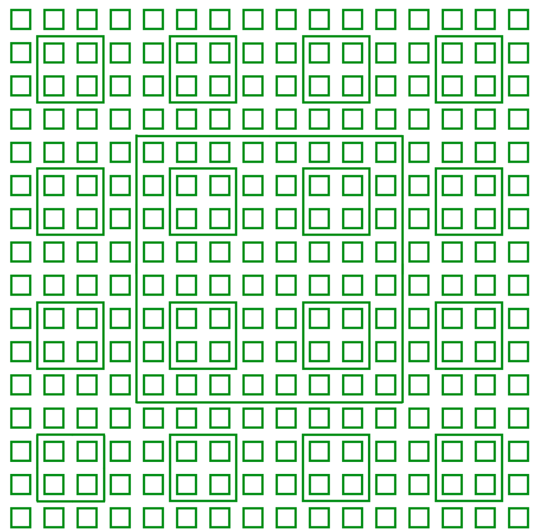
For that we rely on Robinson's aperiodic tiling which allows us to have infinitely many copies of the previous Hamiltonian in all possible system sizes.

We also use the trivial fact that we can encode any valid tiling in the ground state of a nearest neighbor Hamiltonian defined in terms of the tiles.

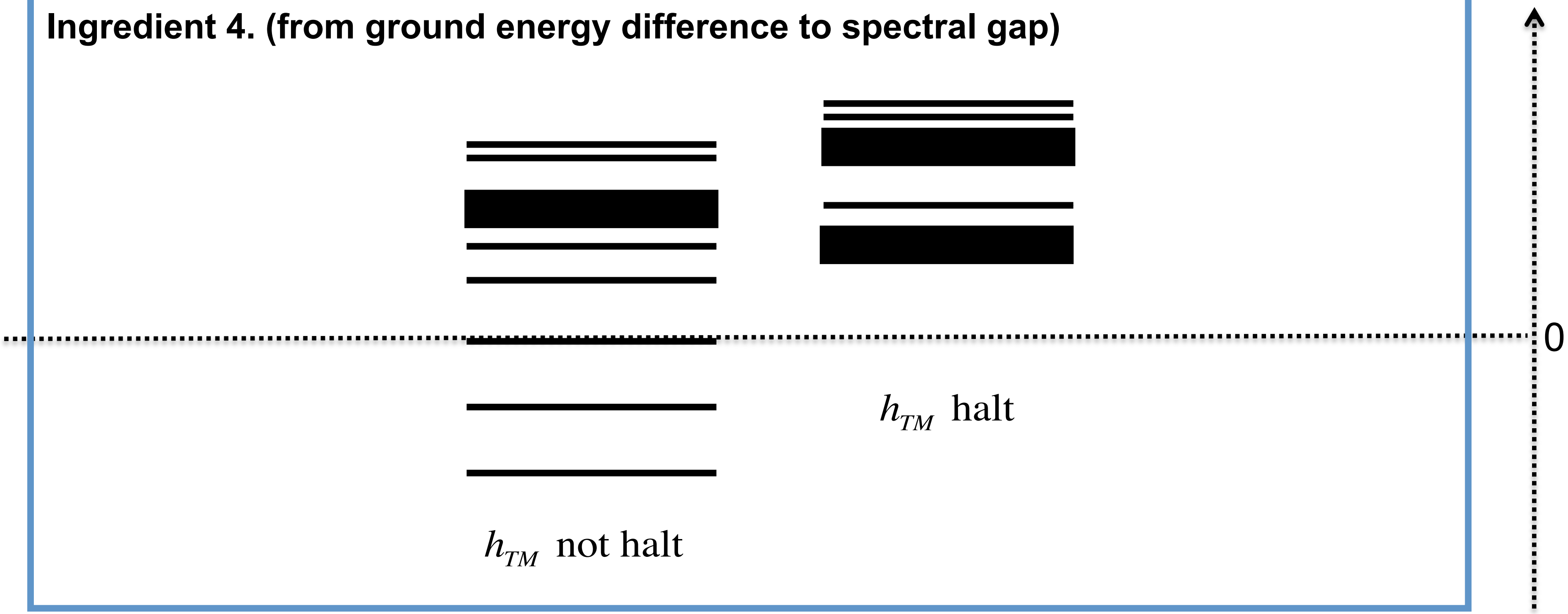




Ingredients of the proof

After Ingredient 3 we have a nearest neighbor interaction so that in a square region of size L we have a difference in energy of order L^2 depending on whether the UTM on input n halts or not. By adding a term $h = \alpha Id$ we can make one energy positive and the other negative.

Ingredient 4. (from ground energy difference to spectral gap)



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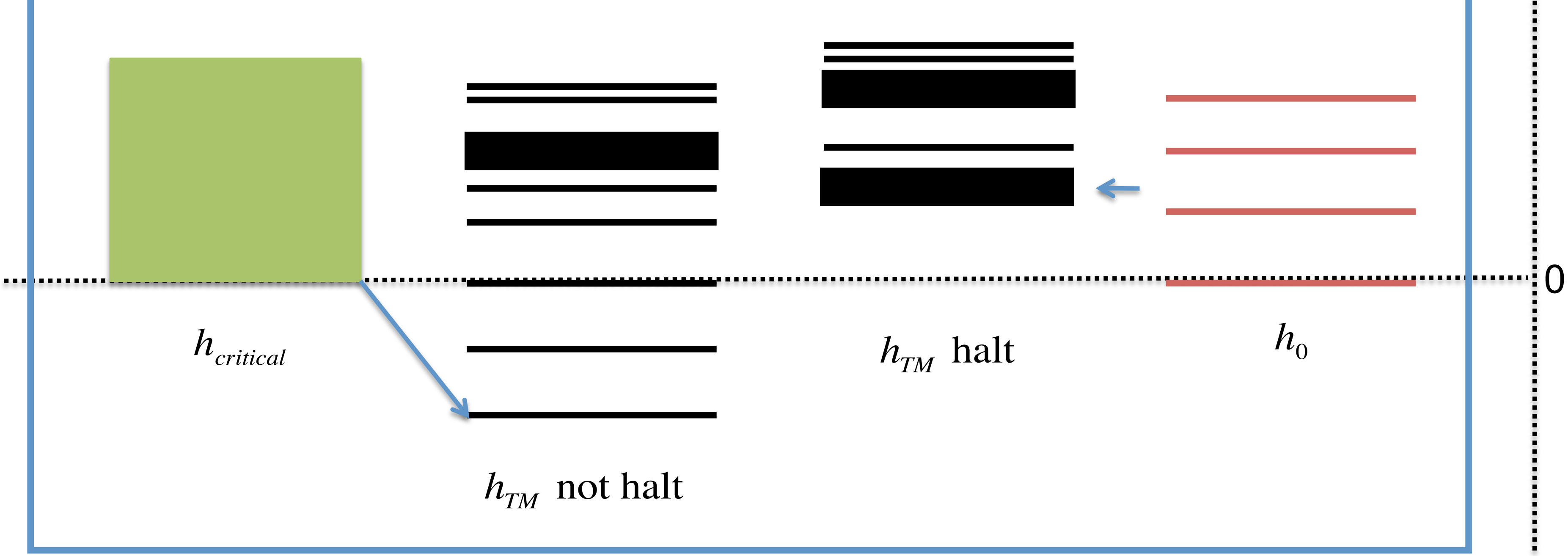

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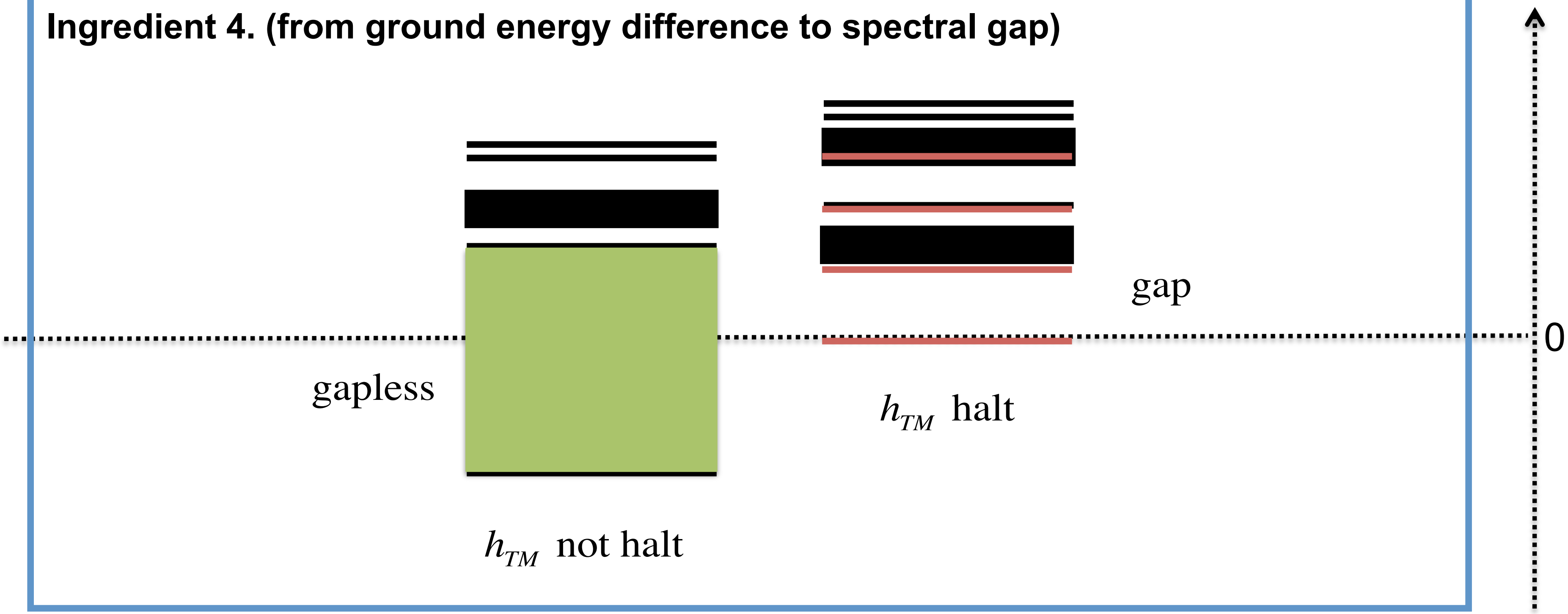
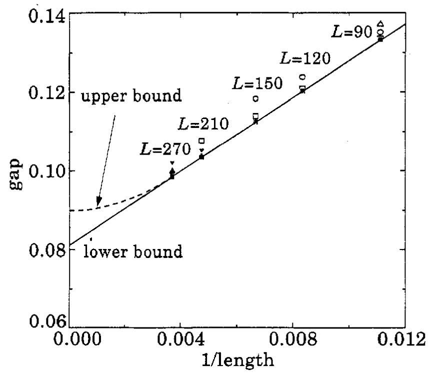
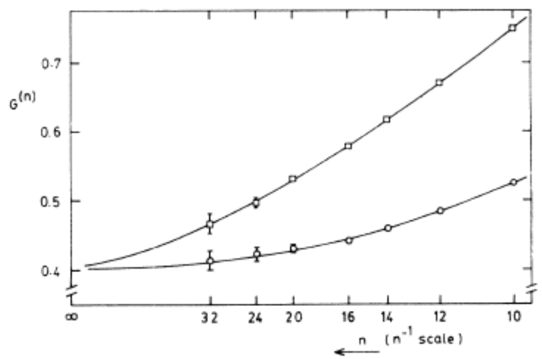


Implications of the result

In condensed matter physics, almost all knowledge is numerical. Got by increasing the system size and extrapolating the result. Example: Haldane's conjecture.



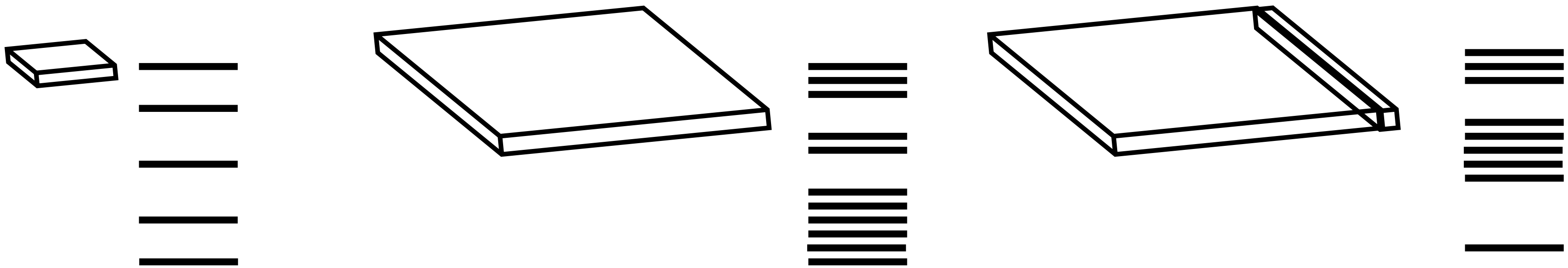
Schollwöck and co. d=5, DMRG (1995)



Nichtingale and co. d=3, MC (1986)

Our result implies that there exist systems that look gapless for all systems sizes $< L_c$

But a gap opens from L_c on. Moreover, this (uncomputable) critical size can be arbitrarily large.



Open questions

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- The number of people that have benefited from Richard's sharp and generous ideas.

- The number of people that have felt at home in Kent thanks to Richard's hospitality.

THANKS RICHARD!

… and HAPPY BIRTHDAY!