Parameter dependence of solutions of differential equations on spaces of distributions

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Statement of the problem

Linear partial differential operator (P.D.O.) with constant coefficients

$$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

$f_\lambda \in \mathscr{D}'(\Omega)$ depends on $\lambda \in U$ holomorphically, smoothly, real analytically, continuously,...

Does there exist $u_\lambda \in \mathscr{D}'(\Omega)$ depending on $\lambda \in U$ in the same way such that

$$P(D)u_\lambda = f_\lambda, \qquad \forall \lambda \in U?$$

Notation

- $P(z) := \sum_{|\alpha| \leq m} a_\alpha z^\alpha, a_\alpha \in \mathbb{C}$
- $P_m(z) := \sum_{|\alpha| = m} a_\alpha z^\alpha$

$$P(D) := \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

where $\partial_j := \dfrac{\partial}{\partial x_j}$, $\quad D_j := \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, $\quad D^\alpha := D_1^{\alpha_1} \ldots D_N^{\alpha_N}$

- $\mathcal{E}(\Omega)$, $\mathcal{D}'(\Omega)$, $C(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{E}_{\{\omega\}}(\Omega)$.

Surjectivity of $P(D)$

Malgrange, Ehrenpreis, Hörmander ($\leq$ 1962)

Theorem 1. $P(D)$ is surjective on $\mathcal{E}(\Omega)$ if and only if Ω is P-convex:

$$\forall K \subset \Omega \quad \exists L \subset \Omega \quad \forall \varphi \in \mathcal{D}(\Omega) \quad \mathrm{supp} P(D)\varphi \subset K \Rightarrow \mathrm{supp}\varphi \subset L$$

- Every convex open set is P-convex.
- If P is elliptic, every open set is P-convex.

Surjectivity of $P(D)$

Malgrange, Ehrenpreis, Hörmander ($\leq$ 1962)

Theorem 2. $P(D)$ is surjective on $\mathcal{D}'(\Omega)$ if and only if Ω is P convex and P-convex for singular supports:

$$\forall K \subset \Omega \;\; \exists L \subset \Omega \; \forall \mu \in \mathcal{E}'(\Omega) \quad \text{sing supp} P(D)\mu \subset K \Rightarrow \text{sing supp}\mu \subset L$$

The singular support is the complement in Ω of the largest open set on which μ is $\mathcal{C}^\infty$.

Historical Remarks

- **Trèves, 1962**: smooth and holomorphic dependence
- **F.E. Browder, 1962**: real analytic dependence
- **F. Mantlik, 1990-92**: parameter dependence of fundamental solutions
- **Grothendieck, late 1950's, Vogt, 1985**: case of $P(D)$ on $\mathcal{E}(\Omega)$
- **Domanski, Bonet**: early results for continuous and real analytic dependence in 1996.

Clarifying the dependence

$G \subset \mathbb{R}^N$, open.
$G \to \mathcal{D}'(\Omega),\ \lambda \to f_\lambda$ is $\mathcal{C}^\infty$ if for each $\varphi \in \mathcal{D}(\Omega)$ the map on G
$\lambda \to \langle f_\lambda, \varphi \rangle$ is $\mathcal{C}^\infty$

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$G \subset \mathbb{C}$, open.
$G \to \mathcal{D}'(\Omega)$, $\lambda \to f_\lambda$ is **holomorphic** if for each $\varphi \in \mathcal{D}(\Omega)$ the map on G
$\lambda \to \langle f_\lambda, \varphi \rangle$ is holomorphic.

Clarifying the real analytic dependence

$G \subset \mathbb{R}^N$, $f : G \to E$ is **real analytic** if $u \circ f : G \to \mathbb{C}$ is real analytic for each $u \in E'$.

- This concept does **not** coincide with the fact that f is locally given by a vector valued Taylor series convergent in E.

Take $\mathbb{R} \to \mathcal{D}'(\mathbb{R})$,

$$f_\lambda(x) := \frac{1}{1+\lambda^2 x^2}, \ x \in \mathbb{R},$$

is real analytic but not strongly real analytic in the sense explained above.

Clarifying the real analytic dependence

The coincidence of the two concepts of real analytic vector valued maps $f : G \to E$, $G \subset \mathbb{R}^N$, was investigated by **Kriegl** and **Michor** in 1991. Their research was continued by **Domanski, Bonet** in 1996.

Theorem (Bonet, Domański, 1996)

Let E be a Fréchet space. Every real analytic map $f : G \to E$ is strongly real analytic if and only if the Fréchet space E satisfies condition (DN) od Vogt.
A nuclear Fréchet space satisfies condition (DN) if and only if it is isomorphic to a closed subspace of the Schwartz space S of rapidly decreasing functions on the real line.

Existence of continuous linear right inverse

- Suppose that there exists a solution operator for $P(D)$. This means that there is a continuous, linear operator

$$R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$$

such that $P(D)Ru = u$ for each $u \in \mathcal{D}'(\Omega)$.

- If we try to solve the parameter dependence problem $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with $f_\lambda \in \mathscr{D}'(\Omega)$ depends on $\lambda \in U$, it is enough to take $u_\lambda := Rf_\lambda$.

- Schwartz asked as a problem to characterize the linear P.D.O. with constant coefficients which admit a solution operator.

Existence of continuous linear right inverse

- Hyperbolic operators : **YES**.
- Elliptic operators: **NO**, Grothendieck, late 1950's.
- Parabolic operators: **NO**, Cohoon, 1970.
- Hypoelliptic operators: **NO**, Vogt, 1983.
- **Complete solution**: Meise, Taylor, Vogt, 1990.
  Characterizations in terms of the existence of shifted fundamental solutions with big lacunas in the support, or, for convex domains, in terms of a Phragmén Lindelöf condition for plurisubharmonic functions on the zero variety of the polynomial $P(z)$ in $\mathbb{C}^N$.
- The Zeilon operator $P(D)$ for $P(z) = z_1^3 + z_2^3 + z_3^3$ admits a solution operator on $\mathcal{D}'(\mathbb{R}^3)$ which is not hyperbolic.

Assume that $P(D)$ is surjective on $\mathcal{D}'(\Omega)$.

(1) (Bonet, Domański, 1996)

Ω arbitrary, $K \subset \mathbb{R}^d$ compact.

$$P(D) : C(K, \mathcal{D}'(\Omega)) \to C(K, \mathcal{D}'(\Omega))$$

is surjective.
It is always possible to solve with continuous dependence.
In fact, for all X Banach,

$$P(D) : \mathcal{D}'(\Omega, X) \to \mathcal{D}'(\Omega, X)$$

is surjective.

(2) (Bonet, Domański, 2006,08)

If Ω is convex then the following operators are surjective:

(a) $P(D) : H(U, \mathscr{D}'(\Omega)) \longrightarrow H(U, \mathscr{D}'(\Omega))$ for every Stein manifold U.

(b) $P(D) : C^\infty(U, \mathscr{D}'(\Omega)) \longrightarrow C^\infty(U, \mathscr{D}'(\Omega))$ for every smooth manifold U.

(c) $P(D) : \mathscr{D}'(\Omega, E) \longrightarrow \mathscr{D}'(\Omega, E)$ for every FN space $E \in (\Omega)$ (for instance, $E \simeq H(U), C^\infty(U), \mathscr{S}, \Lambda_r(\alpha)$).

(d) $P(D) : \mathscr{D}'(\Omega, E') \longrightarrow \mathscr{D}'(\Omega, E')$ for every FN space $E \in (DN)$ (for instance, $E' \simeq \mathscr{S}', H(\{0\}), \Lambda'_\infty(\alpha), \mathscr{D}'(U)$).

For elliptic $P(D)$ the converse to (d) holds (Vogt, 1983).
Examples of spaces with (Ω): $H(U)$, $C^\infty(U)$, $\mathscr{S}$, $\Lambda_r(\alpha)$.

Main Results

(3) (Domański, 2009)

Let Ω be convex and let U be a real analytic manifold.

$$P(D) : \mathcal{A}(U, \mathcal{D}'(\Omega)) \to \mathcal{A}(U, \mathcal{D}'(\Omega)) \quad \text{is surjective}$$

$$\Updownarrow$$

$$P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$$

admits a continuous linear right inverse, **in one of the following cases**:

1. $P(D)$ is homogeneous,
2. $P(D)$ is of order two.
3. $PD)$ is hypoelliptic.

A functional analytic approach

$$H(U, \mathscr{D}'(\Omega)) \leftrightarrow H(U)\hat{\otimes}_\varepsilon \mathscr{D}'(\Omega) = L(H(U)', \mathscr{D}'(\Omega))$$

$$H(U, \mathscr{D}'(\Omega)) \ni u \longleftrightarrow T \in L(H(U)', \mathscr{D}'(\Omega))$$

$$T(\delta_\lambda) := u(\lambda), \quad P(D)(T) := P(D) \circ T$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\, P(D) & \longrightarrow & \mathscr{D}' & \xrightarrow{P(D)} & \mathscr{D}' & \longrightarrow & 0 \\ & & & & & & \uparrow T & & \\ & & & & & & H(U)' & & \end{array}$$

The map

$$P(D) \;:\; L(H(U)', \mathscr{D}'(\Omega)) \longrightarrow L(H(U)', \mathscr{D}'(\Omega))$$

is surjective if and only if every T lifts.

A functional analytic approach

Theorem

Let E be a Fréchet Schwartz space and $R : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be surjective.

$$R \otimes \mathrm{id}_{E} : \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E \simeq \mathscr{D}'(\Omega, E)$$

$$R \otimes \mathrm{id}_{E'} : \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E' \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E' \simeq \mathscr{D}'(\Omega, E')$$

Then

$$\mathrm{Ext}^1(E, \ker R) = 0 \quad \Leftrightarrow \quad R \otimes \mathrm{id}_{E'} \text{ is surjective,}$$

$$\mathrm{Ext}^1(E', \ker R) = 0 \quad \Rightarrow \quad R \otimes \mathrm{id}_{E} \text{ is surjective}$$

If additionally $E \in (\Omega)$ then

$$\mathrm{Ext}^1(E', \ker R) = 0 \quad \Leftrightarrow \quad R \otimes \mathrm{id}_{E} \text{ is surjective.}$$

A functional analytic approach

In the statement $\mathrm{Ext}^1(E, F) = 0$ for (PLS)-spaces E and F means that every short exact sequence of (PLS)-spaces

$$0 \to F \to G \to E \to 0$$

splits. This means that the quotient map has a continuous linear right inverse.

(PLS)-spaces are countable projective limits of (DFS)-spaces, like all the spaces which appear in this lecture.

A functional analytic approach

Theorem

If Ω convex then $\ker P(D) \subseteq \mathscr{D}'(\Omega)$ has (PA) and $(P\Omega)$, i.e., it has dual interpolation estimate for small θ:

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m, \theta < \theta_0\ \exists\ k, C\ \forall\ y \in X_N' :$$

$$\|y\|_{M,m}^{*}\ \leq\ C\left(\|y\|_{K,k}^{*}{}^{(1-\theta)} \cdot \|y\|_{N,n}^{*}{}^{\theta}\right).$$

Invariants like this, and the theory of the splitting of short exact sequences of (PLS)-spaces, permit us to conclude when $\mathrm{Ext}^1(E', \ker P(D)) = 0$

Open problems

1. Show that $\Omega \times \mathbb{R}$ is P-convex for singular support if Ω is P-convex for singular support for arbitrary Ω. In other words, does the surjectivity of $P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ imply that $P(D)$ is surjective on $\mathcal{D}'(\Omega \times \mathbb{R})$?

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2. Does the surjectivity of $P(D) : \mathcal{A}(U, \mathcal{D}'(\Omega)) \to \mathcal{A}(U, \mathcal{D}'(\Omega))$ imply that $P(D)$ has a continuous linear right inverse on $\mathcal{D}'(\Omega)$?

Related open problems

1. Does the space of quasianalytic functions of Roumieu type $\mathcal{E}_{\{\omega\}}(\Omega)$ have a basis?

   Domański and Vogt, Studia Math. 2000, proved that $\mathcal{A}(\Omega)$ does not have a basis.
   Bonet and Domański, J. Funct. Anal. 2006, proved that $\mathcal{E}_{\{\omega\}}(\Omega)$ satisfies certain topological invariant called $(P\overline{\overline{\Omega}})$. But the structure of Fréchet subspaces of $\mathcal{E}_{\{\omega\}}(\Omega)$ is not known. In the case of $\mathcal{A}(\Omega)$, they are isomorphic to subspaces of $H(\mathbb{D}^d)$.

Related open problems

1. Characterize the convolution operators on a space of quasianalytic functions of Beurling or Roumieu type which admit a continuous linear right inverse.

   In the non-quasianalytic case, this was obtained by Meise, Vogt, Math. Ann. 1987, Braun, Meise, Vogt, Math. Nachr. 1996, and Bonet, C. Fernández, Meise, Anal. Acad. Math. Sci Fenn. Math. 2000. For real analytic functions it was solved by Langenbruch, Studia Math. 1994.

   Recent progress in the non-quasianalytic case was obtained by Bonet, Meise in 2008. The full problem remains open. The problem behind it is the structure of the space of quasianalytic functions on compact intervals.

References

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