Algunos resultados del tipo de Farkas

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Outline

- The classical Farkas' lemma

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- Approximate infinite Farkas-type results

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$$(\mathrm{P}) \quad \min f(x) \text{ s.t. } x \in A,$$

where A is the *feasible set*, with $\emptyset \neq A \subset X$ (the *decision space*), and $f : X \to \overline{\mathbb{R}}$ is the *objective function*.

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- By extension: each characterization of the containment of two sets, $A \subset B$, can be seen as an extended Farkas' lemma.
- The expression "Farkas' lemma" appears in the title (abstract) of more than 50 (180) papers reviewed in MathScinet.

The classical Farkas' lemma

- Consider a particle moving within a body $F = \left\{ x \in \mathbb{R}^3 : f_t(x) \leq 0 \ \forall t \in T \right\}$ (T finite, $f_t \in \mathcal{C}^1$ $\forall t \in T$) owing to the action of a conservative field with potential function $f \in \mathcal{C}^1$.

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- Ostrogradski asserted in 1838 that, if $a\in F$ is an equilibrium point (i.e., a *local minimum* of f on F), then

$$\begin{array}{c}\left\{x\in\mathbb{R}^3 : \langle\nabla f_t(a),x\rangle\leq 0\ \forall t\in T(a)\right\}\\ \subset\left\{x\in\mathbb{R}^3 : \langle\nabla f(a),x\rangle\geq 0\right\}\end{array}\tag{1}$$

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(true whenever $\{\nabla f_t(a), t\in T(a)\}$ is linearly independent).
- He also asserted that

$$(1)\Leftrightarrow -\nabla f(a)\in\operatorname{cone}\{\nabla f_t(a), t\in T(a)\}.$$

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- 1902: he gives the 1st correct proof of the linear/linear Farkas' lemma: given $A = \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq 0 \ \forall t \in T\} \neq \emptyset$, T finite,

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- 1911: Minkowski proves the affine/affine Farkas' lemma: given $A = \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq b_t \ \forall t \in T\} \neq \emptyset$, T finite,

$$A \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$$
$$\Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}$$

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Semi-infinite Farkas-type results

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 is semi-infinite when either card T or $\dim X$ is finite (but not both).

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- 1924: Haar considers $X = C(I)$, where $I \subset \mathbb{R}$ is a compact interval, equipped with the scalar product $\langle f, g \rangle = \int_I f(s)\, g(s)\, ds$, $a \in X$, and $A = \{x \in X : \langle a_t, x \rangle \leq 0 \ \forall t \in T\}$ such that $\{a_t, t \in T\}$ is linearly independent ($\Rightarrow$ T finite).

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- 1969: Charnes-Cooper-Kortanek prove a strong duality theorem for (LSIP) under the following CQ:

$$(\textbf{FM}) \quad \mathrm{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\} \text{ is closed.}$$

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- In many LSIP applications T is compact, $t \mapsto a_t$ and $t \mapsto b_t$ are continuous, and $\exists \overline{x}$ such that $\langle a_t, \overline{x}\rangle < b \ \forall t \in T$. Then (**FM**) holds.

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- 1981: GLP take $X = \mathbb{R}^n$, T arbitrary, and $A = \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t \ \forall t \in T\} \neq \emptyset$, "showing" the following affine/affine Farkas' lemma:

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- Moreover, under the (**FM**) CQ, we can eliminate "cl" from (3) (*non-asymptotic optimality theorem*).

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- The stability of the inclusion $F \subset G$ is related with the condition $F \subset \operatorname{int} G$.
- F and $\operatorname{int} G$ are e-convex sets (i.e., intersections of open halfspaces).

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- If $K_F^{<}$ is relatively open, then F is closed.
- If F is compact, then $K_F^{<} = \operatorname{int} K_{\operatorname{cl} F}^{\leq}$ (open).

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Some sequels

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- Denote by $\Gamma(X)$ the set of proper lsc convex functions from X.
- 2006: DGL replace the continuous affine functionals by elements of $\Gamma(X)$, exploiting the fact that these functions are the supremum of continuous affine functionals.

- More precisely, defining the *Fenchel conjugate* of $h \in \Gamma(X)$ as

$$h^*(u) = \sup\{\langle u, x\rangle - h(x) : x \in \mathrm{dom}\ h\},$$

one gets $h^{**} = h$.

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- More precisely, defining the *Fenchel conjugate* of $h \in \Gamma(X)$ as

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- This version does not provide a characterization of global optimality.

Infinite Farkas-type results

- We say that the Farkas-Minkowski CQ holds whenever

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- Recall the closedness condition of Burachik-Jeyakumar (2005):

$$\textbf{(CC)} \quad \operatorname{epi} f^* + \operatorname{cl}\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^*\right\} \text{ is weak}^*\text{-closed}$$

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- Let $f \in \Gamma(X)$, C be a closed convex set in X, and S be a preordering closed convex cone in Z, with *positive dual cone*

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- Assume that $\mathcal{H} : X \to Z$ satisfies $z^* \circ \mathcal{H} \in \Gamma(X)$ $\forall z^* \in S^+$ and $A \cap \operatorname{dom} f \neq \emptyset$, where $A := C \cap \mathcal{H}^{-1}(-S)$.

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- The $\varepsilon-$*normal set* to C at a point $a \in C$ is defined by

$$N_{\varepsilon}(C, a) = \partial_{\varepsilon} i_C(a).$$

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- $A \subset [f \geq 0]$ iff there exists a net

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such that

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \leq \varepsilon_i, \ \forall i,$$

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- This result provides an optimality theorem for convex optimization problems of the form

$$\text{(PC)} \quad \text{minimize } f(x) \text{ s.t. } x \in C \text{ and } \mathcal{H}(x) \in -S.$$

Approximate infinite Farkas-type results

- A point $a \in A \cap (\mathrm{dom}\, f)$ is a minimizer of (PC) iff there exist $(\eta_i)_{i \in I} \to 0_+$ and $\forall i \in I$ there also exist

$$x_{1i}^* \in \partial_{\eta_i} f(a),\ x_{2i}^* \in N_{\eta_i}(C, a),\ x_{3i}^* \in \partial_{\eta_i}(z_i^* \circ \mathcal{H})(a),\ z_i^* \in S^+$$

such that

$$0 \leq \langle z_i^*, -\mathcal{H}(a) \rangle \leq \eta_i, \forall i,$$

$$\lim_i (x_{1i}^* + x_{2i}^* + x_{3i}^*) = 0.$$

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  $\forall x^* \in \operatorname{dom} h^*$, there exists a net

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such that

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \leq h^*(x^*) + \varepsilon_i, \ \forall i,$$

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- The corresponding optimality theorem for DC problems of the form

$$(\mathrm{DC}) \quad \begin{cases} \text{minimize} & f(x) - h(x) \\ \text{s.t.} & x \in C,\ \mathcal{H}(x) \in -S, \end{cases}$$

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  is as follows:

- A point $a \in A \cap (\operatorname{dom} f)$ is a minimizer of (DC) iff $\forall x^* \in \operatorname{dom} h^*$ there exists a net

$$(x_{1i}^*, x_{2i}^*, x_{3i}^*, z_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times S^+ \times \mathbb{R}$$

  satisfying

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \leq h^*(x^*) + h(a) - f(a) + \varepsilon_i, \forall i,$$

$$(x_{1i}^* + x_{2i}^* + x_{3i}^*, \varepsilon_i) \to (x^*, 0_+).$$

- If $a \in A \cap \mathrm{dom}\, f \cap \mathrm{dom}\, h$ is a local minimum of (DC), then

$$\partial h(a) \subset \limsup_{\eta \to 0_+} \bigcup_{z^* \in \partial_\eta i_{-S}(\mathcal{H}(a))} \left\{ \partial_\eta f(a) + \partial_\eta (z^* \circ \mathcal{H})(a) + N_\eta(C, a) \right\}$$

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- or, equivalently, $\forall x^* \in \partial h(a)$, there exists a net

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