Problemas abiertos en dinámica de operadores

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Theorem (G. D. Birkhoff, 1929)

There is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that, for any entire function $g : \mathbb{C} \to \mathbb{C}$ and for every $a \in \mathbb{C} \setminus \{0\}$, there is a sequence $(n_k)_k$ in $\mathbb{N}$ such that

$$\lim_k f(z + an_k) = g(z) \text{ uniformly on compact sets of } \mathbb{C}.$$

Birkhoff's result, in terms of dynamics

- $\mathcal{H}(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \,;\ f \text{ is entire}\}$.
- Endow $\mathcal{H}(\mathbb{C})$ with the compact-open topology τ_0 (topology of uniform convergence on compact sets of $\mathbb{C}$).
- Consider the (continuous and linear!) map

$$T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}), \quad f(z) \mapsto f(z+a).$$

- Then there are $f \in \mathcal{H}(\mathbb{C})$ so that the orbit under T_1:

$$Orb(T_a, f) := \{f, T_a f, T_a^2 f, \dots\}$$

  is dense in $\mathcal{H}(\mathbb{C})$.

In this talk we will focus on some of the open problems arising in linear dynamics that concern

- Existence of orbits with some density properties (frequent hypercyclicity and related).
- Dynamical recurrence of operators.
- Existence of invariant measures with respect to an operator, with certain ergodic properties.
- Entropy in the dynamics of operators.
- Dynamics of C_0-semigroups of operators and applications to linear PDEs and infinite systems of linear ODEs.
- Different chaotic properties in linear dynamics.

- An operator $T : X \to X$ is topologically transitive if, for any $U, V \subset X$ non-empty open sets there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Within our context, this is equivalent to hypercyclicity, that is, the existence of vectors $x \in X$ whose orbit under T is dense in X.

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- T is Devaney chaotic if it is topologically transitive, and the following set is dense in X: $\mathrm{Per}(T) := \{\text{periodic points of } T\} = \{x \in x \ ; \ T^n x = x \text{ for some } n\}$.

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- (Li-Yorke) An uncountable subset $S \subset X$ of a metric space (X, d) is called a scrambled set for a dynamical system $f : X \to X$ if for any $x, y \in S$ with $x \neq y$ we have $\liminf_n d(f^n(x), f^n(y)) = 0$ and $\limsup_n d(f^n(x), f^n(y)) > 0$. f is called Li-Yorke chaotic if it admits an scrambled set.

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- (Bayart, Grivaux) T is frequently hypercyclic if there is $x \in X$ such that, for each nonempty open set $U \subset X$,

$$\underline{\mathrm{dens}} N(x, U) := \liminf_n \frac{\left|\{k \leq n \; ; \; T^k x \in U\}\right|}{n} > 0.$$

Problem 1 [Bayart and Grivaux]

If an operator $T \in L(X)$ is invertible and frequently hypercyclic, is T^{-1} frequently hypercyclic?

Problem 2 [Bayart and Grivaux]

If $T_1, T_2 \in L(X)$ are frequently hypercyclic, is $T_1 \oplus T_2$ frequently hypercyclic?

Problem 3 [Bernardes, Bonilla, Müller, P.]

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(Beauzamy): A vector x is said to be irregular for an operator $T \in L(X)$ on a Banach space X if $\sup_n \|T^n x\| = \infty$ and $\inf_n \|T^n x\| = 0$.
(Bermúdez, Bonilla, Martínez-Giménez, P.): $T \in Ł(X)$ is Li-Yorke chaotic if, and only if, T admits irregular vectors.
It is known (Ansari) that any hypercyclic operator admits a dense manifold consisting of (except 0) hypercyclic vectors.

Problem 4 [Bernardes, Bonilla, Müller, P.]

Does dense Li–Yorke chaos imply the existence of a dense irregular manifold for operators?

Definitions

The support of a Borel probability measure μ, denoted by (μ), is the smallest closed subset F of X such that $\mu(F) = 1$. T is ergodic if $T^{-1}(A) = A$ for $A \in \mathfrak{B}$ implies $\mu(A)(1 - \mu(A)) = 0$. T is strongly mixing with respect to μ if

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Several recent studies (Bayart, Grivaux, Matheron) show that there exist ergodic (strongly mixing) T-invariant measures with full support provided that T admits a “good source” of unimodular eigenvalues and eigenvectors.

Problem 5 [Grivaux, Matheron, Menet]

Do there exist ergodic operators on the Hilbert space without eigenvalues?

Let X be an infinite-dimensional Banach space. A family $\{T_t\}_{t\geq 0}$ of linear and continuous operators on X is said to be a C_0-semigroup if $T_0 = Id$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$, and $\lim_{t\to s} T_t x = T_s x$ for all $x \in X$ and $s \geq 0$.

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$$Ax := \lim_{t\to 0^+} \frac{T_t x - x}{t}, \tag{1}$$

exists on a dense subspace of X; denoted by $D(A)$. Then $(A, D(A))$ is called the (infinitesimal) generator of the C_0-semigroup $\{T_t\}_{t\geq 0}$. If $D(A) = X$, then the C_0-semigroup can be rewritten as $\{e^{tA}\}_{t\geq 0}$. Such a semigroup is the corresponding solution C_0-semigroup of the abstract Cauchy problem

$$\left\{ \begin{array}{ll} u'(t,x) & = Au(t,x) \\ u(0,x) & = \varphi(x), \end{array} \right\}. \tag{2}$$

The solutions to this problem can be expressed as $u(t,x) = e^{tA}\varphi(x)$, where $\varphi(x) \in X$.

A C_0-semigroup $\{T_t\}_{t\geq 0}$ on a Banach space X is hypercyclic if there are $x \in X$ such that the orbit $\{T_t x \ ; \ t \geq 0\}$ is dense in X.
It is well-known that if x is a hypercyclic vector for T, then its orbit $\{x, Tx, T^2x, \dots\}$ is a linearly independent set.

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Problem 6 [Conejero, P.]

Is the orbit of a hypercyclic vector for a C_0-semigroup $\{T_t\}_{t\geq 0}$ always a linearly independent set?

Let $T \in L(X)$ on a Banach space X. Let $K \subset X$ be a compact subset and $n \in \mathbb{N}$. For $x, y \in X$ define

$$d^n(x,y) = \max\{\|T^i x - T^i y\| : i = 0, 1, \ldots, n-1\}.$$

Let $\varepsilon > 0$. A finite subset $F \subset X$ is (n, ε)-separated if $d^n(x,y) \geq \varepsilon$ for all $x, y \in F$, $x \neq y$.
Let $s(n, \varepsilon, K)$ be the maximal cardinality of a (n, ε)-separated subset of K. Let

$$h(T, \varepsilon, K) = \limsup_{n \to \infty} \frac{\log s(n, \varepsilon, K)}{n}$$

and

$$h(T, K) = \lim_{\varepsilon \to 0_+} h(T, \varepsilon, K).$$

The Bowen entropy of T is defined by

$$h(T) = \sup\{h(T, K) : K \subset X \text{ compact}\}.$$

(Bayart, Müller, P.): Let $\dim X < \infty$ and $T \in B(X)$. Then

$$h(T) = \sum_{\substack{\lambda \in \sigma(T) \\ |\lambda| > 1}} 2 \log |\lambda|,$$

where each $\lambda \in \sigma(T)$ is counted according to its algebraic multiplicity.

Problem 7 [Bayart, Müller, P.]

Is it possible that

$$\sum_{\substack{\lambda \in \sigma(T) \\ |\lambda| > 1}} 2 \log |\lambda| < h(T) < \infty?$$

If K is compact, let $D(K)$ be the diameter of K and $D_i(K)$ the diameter of $T^i(K)$. For $n \geq 1$, $x, y \in K$, let

$$d^n(x, y, K) = \max\left\{ \frac{\|T^i x - T^i y\|}{\max(1, D_i(K))};\ i = 0, \ldots, n-1 \right\}.$$

A subset $F \subset K$ is (n, ε)-separated if

$$\forall x \neq y \in F,\ d^n(x, y, K) \geq \varepsilon.$$

$$\begin{aligned} s(n, \varepsilon, K) &= \sup\{\text{card}\, F;\ F \subset K \text{ is } (n, \varepsilon)\text{-separated}\} \\ h(T, \varepsilon, K) &= \limsup_{n \to +\infty} \frac{\log s(n, \varepsilon, K)}{n} \\ h(T) &= \sup_{\varepsilon > 0} \sup_{K} h(T, \varepsilon, K). \end{aligned}$$

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Problem 8 [Bayart, Müller, P.]

Do there exist operators with positive and finite operator entropy?

(Read): There exist operators T on ℓ^1 such that **every** $x \in \ell^1$, $x \neq 0$, is a hypercyclic vector for T. In other words, T admits no invariant closed subset, except the trivial ones ($\{0\}$ and ℓ^1).

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Invariant Subset (Subspace) Problem

Do there exist operators on the Hilbert space without non-trivial invariant closed subsets (subspaces)?