Geometric properties of cones having a large dual

Fernando García Castaño

(Joint work in progress with M. A. Melguizo Padial)

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XIII Encuentro de la Red de Análisis Funcional y Aplicaciones
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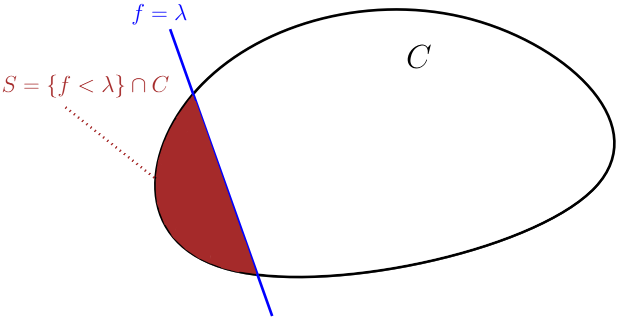
- An slice of a set C is a non empty intersection of C with an open half space of X



Denting points

Definition

Let C be a subset of X, $c \in C$ is said to be a denting point of C if

$$c \notin \overline{\text{conv}}(C \setminus B_\varepsilon(c)), \ \forall \varepsilon > 0.$$

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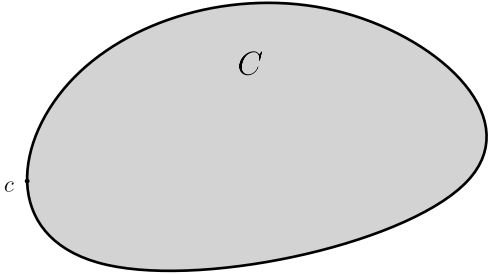

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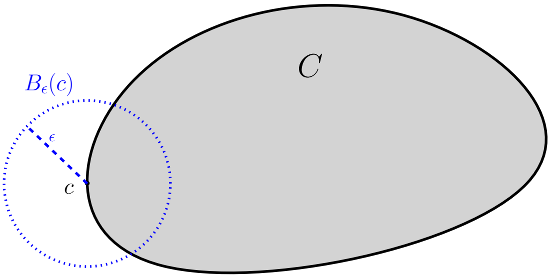

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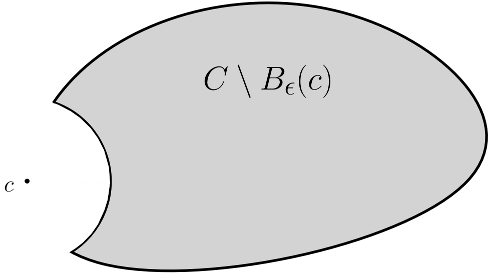

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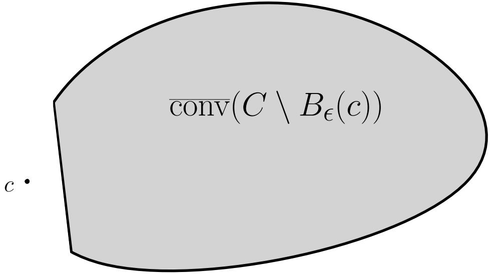

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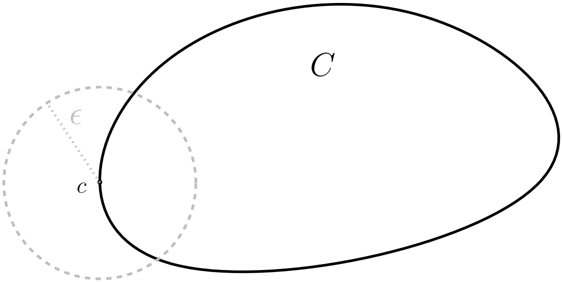

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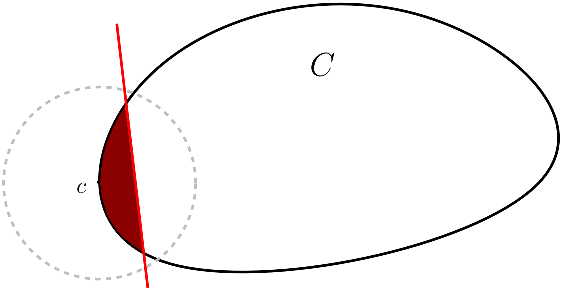

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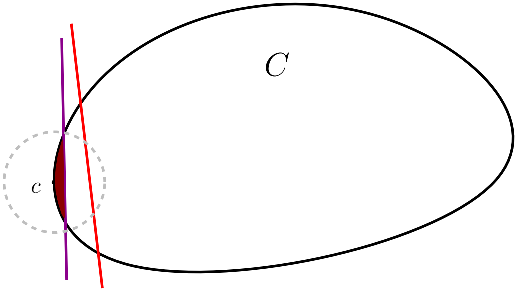

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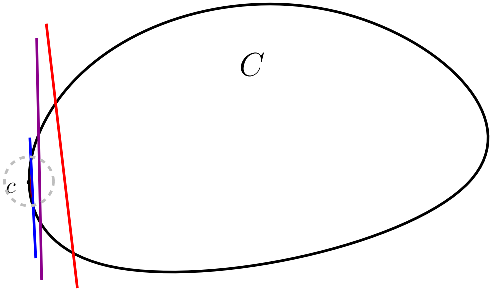

Denting points

Dentability is applied to study

- Radon-Nikodým property
- LUR renorming
- Optimization
- Operators theory

Points of continuity

Definition

Let C be a subset of X, $c \in C$ is said to be a point of continuity for C if the identity map $(C, \text{weak}) \to (C, \| \ \|)$ is continuous at c.

Points of continuity

c is a point of continuity for C if and only if for every open ball $B_\varepsilon(c)$, there exists a weakly open U such that

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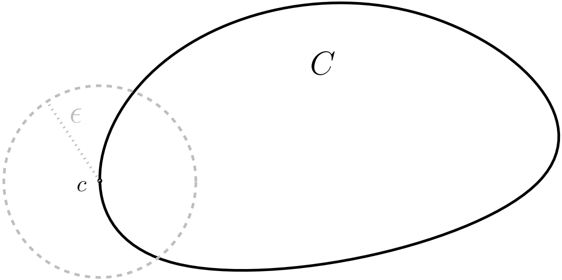

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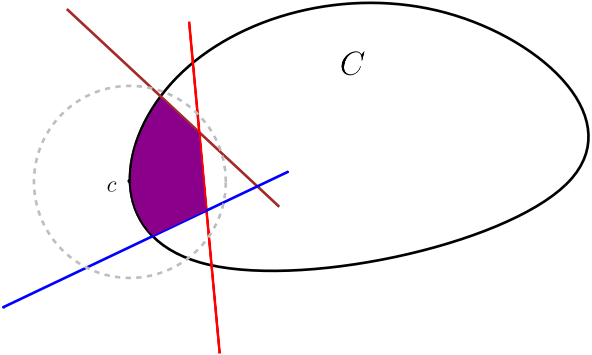

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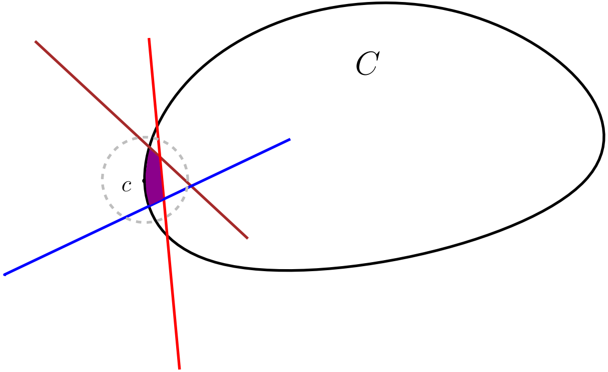

Points of continuity

The notion of point of continuity is applied to

- Provide a geometric proof a fixed point theorem
- Geometric properties related to Radon-Nikodým property
- Optimization

Denting points and points of continuity

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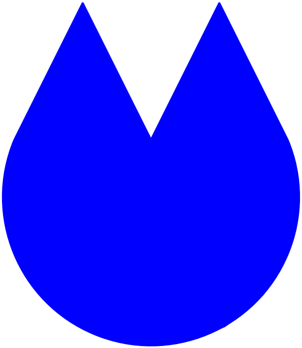

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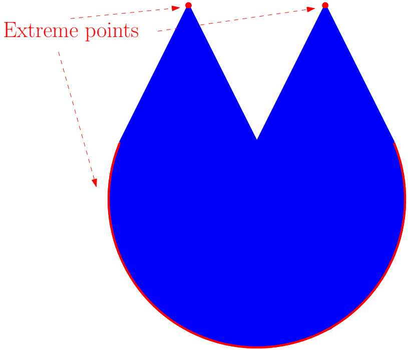

Denting points and points of continuity

Theorem (Lin–Lin–Troyanski, 1985)

Let c be an extreme point of a closed, convex, and bounded subset C of a Banach space. If c is a point of continuity for C, then it is a denting point.

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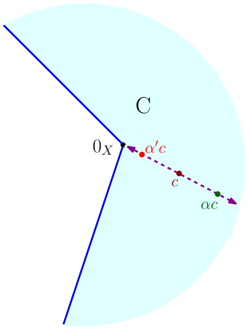
What about cones?

Denting points, points of continuity, and cones

Definition

1. A non empty convex subset C of X is called a cone if

$$\alpha C \subset C, \ \forall \alpha \geq 0$$



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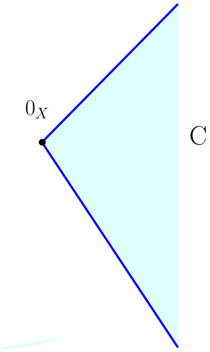
2. A cone C is called pointed if $C \cap (-C) = \{0_X\}$



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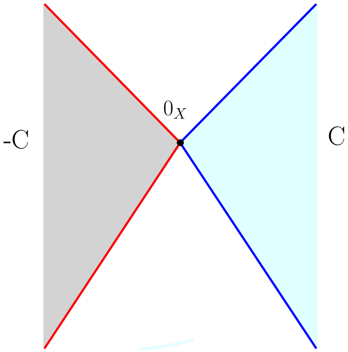

Denting points, points of continuity, and cones

Problem (Gong, 1995)

The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

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A negative answer for Banach spaces

Theorem (Daniilidis, 2000)

Let C be a closed and pointed cone in a Banach space X. Then 0_X is a denting point of C if and only if it is a point of continuity for C.

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The former characterization allowed Daniilidis to prove the equivalence (into the frame of Banach spaces) between two density results of Arrow, Barankin and Blackwell's type. One due to Petschke (1990) and another due to Gong (1995).

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A positive answer for non closed cones

Example (GC–Melguizo–Montesinos, 2015)

Let us define $X := \mathbb{R}^2$ and $C := \mathbb{R} \times (0, +\infty) \cup \{(0,0)\}$ which is a pointed cone. Then 0_X is point of continuity for C but it is not a denting point.

Denting points, points of continuity, and cones

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$$C^* := \{f \in X^* : f(c) \geq 0, \forall c \in C\}$$

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$$[x, y] := \{z \in X \colon x \leq z \leq y\}$$

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Theorem (Kountzakis–Polyrakis, 2006)

Let X be a normed space such that $\exists f \in C^*$ such that $X^* = \overline{\cup_{n\geq 1}[-nf, nf]}$. Then 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C.

Denting points, points of continuity, and cones

Problem (Gong, 1995)

The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

Given $C \subset X \Rightarrow \widetilde{C}$ denotes the closure of C in (X^{**}, weak^*)

Theorem (GC-Melguizo-Montesinos, 2015)

Let X be a normed space, 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C and $\widetilde{C} \subset X^{**}$ is pointed.

Theorem 1 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a pointed cone. The following are equivalent:

(i) 0_X is a denting point of C.

(ii) There exist $n \in \mathbb{N}$, $\{f_i\}_{i=1}^n \subset C^*$, and $\{\lambda_i\}_{i=1}^n \subset (0, +\infty)$ such that the set, $\cap_{i=1}^n \{f_i < \lambda_i\} \cap C$, is bounded.

(iii) 0_X is a point of continuity for C and $\overline{C^* - C^*} = X^*$ (i.e., C^* is quasi-generating).

(iv) $\exists f \in C^*$ such that $X^* = \cup_{n \geq 1}[-nf, nf]$ (i.e., C^* has an order unit).

(v) There exists $\{f_n\}_{n \geq 1} \subset C^*$ such that $X^* = \cup_{n \geq 1}[-nf_n, nf_n]$.

Corollary 1 (GC-Melguizo)

Let X be a normed space with a quasi-generating order cone $C \subset X$. If the origin is denting in C, then the following statements hold true:

- **(i)** Every linear and positive operator $T : X^* \to X^*$ is continuous. In addition, if T is not a multiple of the identity, then it has a nontrivial hyperinvariant subspace.
- **(ii)** If a positive contraction $T : X^* \to X^*$ has 1 as an eigenvalue, then there exits an $0 < f \in X^{**}$ such that $T'f = f$.

Corollary 2 (GC-Melguizo)

Let X be a normed space and C a pointed cone. If 0_X is a point of continuity for C and $C^* \subset X^*$ is quasi-generating, then each weakly compact subset of X has super efficient points.

Proposition 1 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a pointed cone. If C is closed, then $\overline{C^* - C^*}^{\text{weak}^*} = X^*$.

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In general, closed cones do not have quasi-generating dual cones

Definition

Let X be a normed space and $C \subset X$ a cone. It is said that 0_X is a weakly strongly extreme point of C if given two sequences $(c_n)_n$ and $(\tilde{c}_n)_n$ in C such that $\lim_n (c_n + \tilde{c}_n) = 0$, then weak$-\lim_n c_n = 0$.

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Proposition 2 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a pointed cone. If 0_X is a weakly strongly extreme point of C, then $\overline{C^* - C^*} = X^*$.

Definition

A cone C in a normed space X is said to be normal whenever $0 \leq x_n \leq y_n$ in X and $\lim_n y_n = 0$ imply $\lim_n x_n = 0$.

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Proposition 3 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a pointed cone. If C is normal, then 0_X is a weakly strongly extreme point of C.

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Corollary 3 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a normal pointed cone. Then 0_X is a point of continuity for C if and only if it is a denting point of C.

Definition

A norm $\| \ \|$ on a vector space X is called to be strictly convex if given $x, y \in X$ with $\| x \| = \| y \| = 1$ and $\| x + y \| = 2$, we get $x = y$.

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Corollary 4 (GC-Melguizo)

Let X be a normed space a C a pointed cone. If the norm of X^{**} is strictly convex, then $\overline{C^* - C^*} = X^*$. As a consequence, 0_X is a point of continuity for C if and only if it is a denting point of C.

Example 1 (GC-Melguizo)

Let Γ be an abstract nonempty set, consider the vector space

$$c_{00}(\Gamma) := \{(x_\gamma)_{\gamma\in\Gamma} \in l_\infty(\Gamma)\colon \{\gamma \in \Gamma\colon x_\gamma \neq 0\} \text{ is finite } \},$$

the non-complete normed space $(c_{00}(\Gamma), \| \ \|_\infty)$, where

$$\| (x_\gamma)_{\gamma\in\Gamma} \|_\infty := \sup\{|x_\gamma|\colon \gamma \in \Gamma\},$$

and the order cone

$$c_{00}(\Gamma)^+ := \{(x_\gamma)_{\gamma\in\Gamma} \in c_{00}(\Gamma)\colon x_\gamma \geq 0, \forall\gamma \in \Gamma\}.$$

Then the dual cone $(c_{00}(\Gamma)^+)^* \subset (c_{00}(\Gamma), \| \ \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $c_{00}(\Gamma)^+$.

Example 2 (GC-Melguizo)

Let us consider the non-complete normed space $(C_{00}(\mathbb{R}), \| \ \|_\infty)$, where $\| f \|_\infty := \sup\{|f(x)| \colon x \in \mathbb{R}\}$ and the order cone

$$C_{00}(\mathbb{R})^+ := \{f \in C_{00}(\mathbb{R}) \colon f(x) \geq 0, \forall x \in \mathbb{R}\}.$$

Then the dual cone $(C_{00}(\mathbb{R})^+)^* \subset (C_{00}(\mathbb{R}), \| \ \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $C_{00}(\mathbb{R})^+$.

Example 3 (GC-Melguizo)

Let us fix any $k \geq 1$, consider the vector space $C^k[a,b]$ of all functions on $[a,b]$ that have k continuous derivatives, the non-complete normed space $(C^k[a,b], \| \ \|_\infty)$, where $\| f \|_\infty := \sup\{|f(x)| \colon x \in [a,b]\}$, and the order cone

$$C^k[a,b]^+ := \{f \in C^k[a,b] \colon f(x) \geq 0, \ \forall x \in [a,b]\}.$$

Then the dual cone $(C^k[a,b]^+)^* \subset (C^k[a,b], \| \ \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $C^k[a,b]^+$.

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