Extended Eigenvalues for bilateral weighted shifts

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Contents

Definition

If $AX = \lambda XA$ for some $X \neq 0$, λ is called an extended eigenvalue and X an (extended) eigenoperator of A.

Scott Brown (1979) and Kim, Moore and Pearcy (1979), independently.

If an operator A on a Banach space has a non-zero compact eigenoperator, then A has a nontrivial, hyperinvariant subspace.

Lomonosov (1973)

If A commutes with a non-zero compact operator then A has a non-trivial hyperinvariant subspace.

Extended eigenvalue has taken on a life of its own.

- Further results of Lomonosov-type [4, 5].
- Studies of the extended eigenvalues and eigenoperators for interesting classes of naturally occurring operators [1, 2, 3, 1, 2]

We continue this latter thread

Extended eigenvalues for Cesàro operators J. Math. Anal. App, **429**, 2015, 623-657 (with Lacruz, Petrovic and Zabeti)

Extended eigenvalues for bilateral weighted shifts J. Math. Anal. App, **444**, 2016, 1591-1602 (with Lacruz and Muñoz)

Cesàro operators on ℓ^2, $L^2[0,1]$ and $L^2[0,\infty)$

$$(C_0 f)(n) = \frac{1}{n+1}\sum_{k=0}^{n} f(k) \quad (C_1 f)(x) = \frac{1}{x}\int_0^x f(s)\,ds$$

$$(C_\infty f)(x) = \frac{1}{x}\int_0^x f(s)\,ds$$

Question

Extended eigenvalues for Cesàro operators?

Main Theorem

The set of extended eigenvalues for C_∞ it reduces to the singleton $\{1\}$, for C_1 it is the interval $(0,1]$ and for C_0 is the interval $[1,\infty)$.

Extended eigenvalues for C_∞.

Definition

A bounded linear operator U on a complex Hilbert space H is a bilateral shift of multiplicity one provided that there is an orthonormal basis (e_n) of H such that $Ue_n = e_{n+1}$ for all $n \in \mathbb{Z}$.

Brown-Halmos- Shields. C_∞ on $L^2[0,\infty)$

They proved that C_∞ is a bounded linear operator, and they also proved that $I - C_\infty^\star$ is unitarily equivalent to a bilateral shift of multiplicity one.

Theorem

Let U be a bilateral shift of multiplicity one, and let λ be a complex number with $\lambda \neq 1$. Then the equation $(I - U^\star)X = \lambda X(I - U^\star)$ has only the trivial solution $X = 0$.

Lemma

Let X be an operator satisfying $(I - U^\star)X = \lambda X(I - U^\star)$, and let $\cdots, X_{-1}, X_0, X_1, X_2, \cdots$ be the rows of the matrix of X. Then $X_{n+1} = (\lambda U + 1 - \lambda)X_n$, for all $n \in \mathbb{Z}$. Consequently, for any $m, n \in \mathbb{N}$, $X_{m+n} = (\lambda U + 1 - \lambda)^n X_m$. In particular, if $m = 0$, $X_n = (\lambda U + 1 - \lambda)^n X_0$,for all $n \in \mathbb{N}$.

Theorem

The set of extended eigenvalues for the infinite continuous Cesàro operator C_∞ defined on the complex Banach space $L^p[0,\infty)$ reduces to the singleton $\{1\}$.

Theorem

There exists a Schauder basis $\{e_n\}, n \in \mathbb{Z}$ on $L^q[0,\infty)$ such that $(1 - 2/qC_\infty^\star)e_n = e_{n+1}$ for all $n \in \mathbb{Z}$.

Tools

1. To prove that λ is an extended eigenvalue for T, there is no choice but to show the existence of and operator X_λ such that $TX_\lambda = \lambda X_\lambda T$: a) Constructively b) Using Baire Category's theorem.
2. (Rosenblum) If $\sigma(A) \cap \sigma(B) = \emptyset$ then $X = 0$ is the only solution of the equation $AX - XB = 0$. If λ is an extended eigenvalue then $\sigma(T) \cap \sigma(\lambda T) \neq \emptyset$.
3. Semigroup techniques were used by Biswas to discard extended values for the Volterra operator.

Extended eigenvalues for C_1 and C_0.

Operators with rich point spectrum

We say that an operator T on a complex Banach space has rich point spectrum provided that $\mathrm{int}\sigma_p(T) \neq \emptyset$, and that for every open disc $D \subset \sigma_p(T)$, the family of eigenvectors $\cup_{z\in D} ker(T-z)$ is a total set.

Theorem

Let us suppose that an operator T on a complex Banach space has rich point spectrum. If λ is an extended eigenvalue for T then we have $\lambda \cdot \mathrm{int}\, \sigma_p(T) \subset \mathrm{clos}\, \sigma_p(T)$.

$C_0^\star$ and C_1 have rich point spectrum.

Proposition

On ℓ^p spaces the extended eigenvalues for C_0 is contained on $[1, \infty)$.

Proposition

If λ is an extended eigenvalue for C_1 on $L^p[0,1]$ then λ is real and $0 < \lambda \leq 1$.

Extended eigenvalues for C_1 on $L^p[0,1]$

Theorem

If $0 < \lambda \leq 1$ then λ is an extended eigenvalue for the Cesàro operator C_1 on $L^p[0,1]$ and a corresponding extended eigenoperator is the weighted composition operator $X_0 \in \mathcal{B}(L^p[0,1])$ defined by

$$(X_0 f)(x) = x^{(1-\lambda)/\lambda} f(x^{1/\lambda}).$$

Extended eigenvalues for C_0 on ℓ^2

Kriete-Trutt (1974/75)

There exists a positive finite measure defined on the Borel subsets of the complex plane and supported on $\overline{\mathbb{D}}$ and a unitary operator $U : \ell^2 \to H^2(\mu)$ such that $C_0 = U^\star(I - M_z)U$, ($H^2(\mu)$ denotes the closure of the polynomials on $L^2(\mu)$).

Theorem

If $\lambda \geq 1$ then λ is an extended eigenvalue for $I - M_z$ and a corresponding extended eigenoperator is the composition operator X defined by the expression $(Xf)(z) = f\left(\frac{\lambda-1}{\lambda} + \frac{z}{\lambda}\right)$.

Corollary

On ℓ^2 the set of extended eigenvalues for C_0 is the subset $[1, \infty)$

The results applies for bilateral weighted shifts

$$We_n = w_n e_{n+1}, \;\; n \in \mathbb{Z}$$

Theorem

Let us suppose that an operator T on a complex Banach space is similar to αT for some complex number α. If λ is an extended eigenvalue for T then $\lambda\alpha$ is an extended eigenvalue for T.

Corollary

If W is a bilateral weighted shift then every $\lambda \in \mathbb{T}$ is an extended eigenvalue for W .

Theorem

if λ is an extended eigenvalue of a bilateral weighted shift W whose point spectrum has non-empty interior then $|\lambda| = 1$.

Question (Shields'1974)

Let W be a invertible bilateral weighted shift. Is there exist a non-trivial closed subspace invariant for W and W^{-1}? Is there exist a non trivial invariant subspace for $W + W^{-1}$?

Question

Which are the extended eigenvalues for a bilateral weighted shift and their corresponding extended eigenoperators?

Intertwining relations

Definition

A bounded operator A intertwines with a bounded operator B provided there exists a bounded operator $X \neq 0$ such that $AX = XB$.

Question

Let A, B two bilateral weighted shifts. When A intertwines with B?

Shields'74

An operator X intertwines two bilateral weighted shifts A and B with sequences of weights $(\alpha_n) n \in \mathbb{Z}$ and $(\beta_n), n \in \mathbb{Z}$ if and only if

$$\beta_j x_{i+1,j+1} = \alpha_i x_{i,j}$$

where $x_{i,j} = \langle Xe_j, e_i \rangle$ are the coefficients of the matrix of X with respect to the canonical basis on $\ell^2(\mathbb{Z})$.

Theorem

Let A and B be two injective bilateral weighted shifts with sequences of weights $(\alpha_n), n \in \mathbb{Z}$ and $(\beta_n), n \in \mathbb{Z}$. Then, A intertwines with B, if and only if there exist $k \in \mathbb{Z}$ and a constant M such that

$$\left| \frac{\alpha_k \cdots \alpha_{k+n-1}}{\beta_0 \cdots \beta_{n-1}} \right| \leq M \quad \text{and} \quad \left| \frac{\beta_1 \cdots \beta_{n-1}}{\alpha_{k-1} \cdots \alpha_{k-n}} \right| \leq M$$

Theorem

Let W be a injective bilateral weigthed shift. Then, the set of extended eigenvalues for W has only one of the following pictures: $\mathbb{C} \setminus \mathbb{D}$ or $\mathbb{C} \setminus \{0\}$ or $\overline{\mathbb{D}} \setminus \{0\}$, or $\mathbb{T}$.

Theorem

Let A be an injective bilateral weighted shift and let λ extended eigenvalue, with $|\lambda| \neq 1$. Then every extended eigenoperator for A corresponding to λ is strictly lower triangular.

Theorem

Let A be an injective bilateral weighted shift and let $\lambda \in \mathbb{T}$. Then every extended eigenoperator X for A corresponding to λ factors as $X = D_\lambda B$ for some $B \in \{A\}'$. $(D_\lambda e_n = \lambda^{-n} e_n)$.

Questions

1. Show that if X is an extended eigenoperator for C_1 on $L^p[0,1]$ then there exists $R \in \{C_1\}'$ such that $X = X_0 R$, where X_0 is a fixed eigenoperator.
2. Show that if $1 < p < \infty$ and if λ is real and $\lambda \geq 1$ then λ is an extended eigenvalue for C_0 on ℓ^p.
3. How we can weaken the conditions of intertwining in Brown and Kim-Mooore-Pearcy's theorem on the special case of the bilateral weighted shift in order to obtain new results on hyperinvariant subspace for bilateral weighted shifts.

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