ÁLGUNOS PROBLEMAS O CUESTIONES EN RELACIÓN CON ÁLGEBRAS DE BANACH

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ÁLGEBRAS DE BANACH

- A álgebra $\begin{cases} \text{compleja} \\ \text{asociativa} \\ \text{conmutativa} \end{cases}$, espacio Banach $\Rightarrow \|ab\| \leq \|a\|\,\|b\|$ $\forall a,b \in A$

- $(e_j)_{j\in J}$ unidad aproximada: $a e_j \xrightarrow[j]{} a, \forall a$; SEMIGRUPO $\begin{cases} t \in (0,\infty) \mapsto a^t \in A \\ a^{s+t} = a^s a^t \ (s,t>0). \end{cases}$

- Spec $A := \{\varphi \in A' \mid \varphi(ab) = \varphi(a)\varphi(b)\ \forall a,b\} \setminus \{0\}$, $\forall a \in A$,

$\sigma(a) := \{\lambda \in \mathbb{C} \mid \nexists\ (\lambda - a)^{-1} \text{ en } A\} \overset{\text{TEO-}}{\underset{\text{REMA}}{\equiv}} \{\varphi(a) \mid \varphi \in \operatorname{Spec} A\}$

Rad A: $a \in A \ni \sigma(a) = 0$ $[\Leftrightarrow \|a^n\|^{1/n} \xrightarrow[n]{} 0]$

A radical si $A = \operatorname{Rad} A$, A semisimple si $\operatorname{Rad} A = (0)$.

Q1.- Problema del ideal cerrado (PIC)

Dada A, ¿$\exists\ I \underset{\text{cerrado}}{\overset{\text{ideal}}{\subseteq}} A \ni (0) \neq I \neq A$?

(afecta a A radicales sin divisores de cero).

- PIC ⇝ Pba. Sub.$^{\text{o}}$ (hiper-) invariante.

- J. Esterle, LN Maths. 975 (1983)
  - Elements for a classification of commutative radical Banach algebras, pp. 4-65.
  - Quasimultipliers, representations of H^∞, and the PIC for commut. $\mathcal{B}$-algebras, pp. 66-161.

u. aprox. & semigrupos

- $\mathrm{Mul}(A)$: $T \in \mathcal{B}(A) \ni T(ab) = aT(b) \quad \forall a, b \in A$.

- $QM_r(A) := \varinjlim \mathrm{Mul}(B)$, B similar a A.

  $T = \frac{b}{a} \ni \overline{aA} = A \wedge \exists \begin{matrix} \lambda > 0 \\ c \in A \end{matrix} \ni \sup_{n \in \mathbb{N}} \left\| \lambda^n \frac{b^n}{a^n} c \right\|_A < +\infty.$

  $\mathrm{Spec}_Q A := \mathrm{Spec}\, QM_r(A)$ & $\hat{T}(\xi) := \frac{\hat{b}(\xi)}{\hat{a}(\xi)}$

  $(\xi \in \mathrm{Spec}\, A)$

- $A = R$ radical $\Longrightarrow$ $\mathrm{Spec}_Q R \xrightarrow[\text{SOBRE}]{} \mathrm{Spec}\, H^\infty(\mathbb{D})$

  $L^1(\mathbb{R}^+, e^{-t^2})$,

  $L^1_*(0,1)$

  COCIENTES F. ANALÍTICAS

  ¿ $QM_r R$? ¿ $\mathrm{Spec}_Q R$?

- G grupo loc$^{\underline{e}}$ cpto. abeliano, metrizable, no discreto.

TEOREMA (G, 1988).-

(1) $\beta\widehat{G} \underset{\text{HOMEO}}{\hookrightarrow} \mathrm{Spec}_Q\, L^1(G)$.

(2) $\mathrm{Spec}_Q\, L^1(G) \simeq \beta\widehat{G} \iff$ todo $T \in QM_r(L^1(G))$ t.q. $\widehat{T}(\xi) \neq 0$ $(\xi \in \widehat{G})$ es invertible en $QM_r(L^1(G))$.

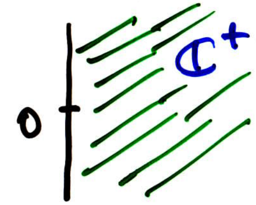
(3) G compacto $\Rightarrow$ $QM_r(L^1(G)) = \left\{ \begin{matrix} \text{seudo-} \\ \text{medidas} \end{matrix} \text{ sobre } G \right\}$ $\wedge$ $\mathrm{Spec}_Q\, L^1(G) \simeq \beta\widehat{G}$.

Q2.- ¿Es $\mathrm{Spec}_Q\, L^1(G) \simeq \beta\widehat{G}$? [tipo Wiener].

• ¿Y para $L^1(\mathbb{R}^+)$?

$f/g \in QM_r(L^1(\mathbb{R}^+)) \xmapsto{\tilde{\mathcal{L}}} \mathcal{L}f/\mathcal{L}g \in \operatorname{Im}\tilde{\mathcal{L}} \subseteq H^\infty(\mathbb{C}^+) \cap C(\overline{\mathbb{C}^+})$

Nevanlinna ¿DENSA?



• $\mathbb{Q}^+ \xrightarrow{\omega} (0,\infty) \ni \omega(s,t) \leq \omega(s)\,\omega(t) \quad \forall s,t \in \mathbb{Q}^+$

$\ell^1(\mathbb{Q}^+,\omega)$: $\sum_{q\in\mathbb{Q}^+} \alpha_q X^q \equiv \sum_{q\in\mathbb{Q}^+} \alpha_q \delta_q \subseteq M(\mathbb{R}^+) \ni$

$\sum_q |\alpha_q|\,\omega(q) < +\infty$; $\ell^1(\mathbb{Q}^+)$ semisimple, $\ell^1(\mathbb{Q}^+, e^{-t^2})$ radical

Q3.- (Esterle 1983)

¿$\exists$ semigrupo continuo no nulo $(a^t)_{t>0}$ en $\ell^1(\mathbb{Q}^+, e^{-t^2})$?

- En $\ell^1(\mathbb{Q}^+)$, NO: $a^t = \sum_{q\in\mathbb{Q}^+} \alpha_q(t)\,\delta_q$. $q_0(t) := \inf\{q \mid \alpha_q(t)\neq 0\}$.

  - $\alpha_{q_0(t)}(t) = 0 \Rightarrow \mathcal{L}(a^t)$ CASI-PERIOD$^{\underline{a}}$ $\xRightarrow[\text{BOHR}]{}$ $\mathcal{L}(a^t)(z_0)=0$ orden k

    $\Rightarrow \mathcal{L}(a^{t/k+1})^{k+1}(z_0) = \mathcal{L}(a^t)(z_0) = 0 \therefore a^s \equiv 0$ (Titschmarsh)

  - $\alpha_{q_0(t)}(t) \neq 0\ \forall t \Longrightarrow t\in(0,\infty) \xrightarrow{q_0} q_0(t) \in \mathbb{Q}^+ \Rightarrow$

    $q_0(s+t) = q_0(s) + q_0(t)\quad \forall s,t$ $\#$ ■

- Para $\ell^1(\mathbb{Q}^+, e^{-t^2})$:

  $\ell^1(\mathbb{Q}^+, e^{-t^2}) \longrightarrow \ell^1(\mathbb{Q}^+, e^{-t^2}) \big/ \{\lambda_q = 0;\ 0<q<1\} = \ell^1(\mathbb{Q}^+\cap(0,1)) \supseteq (a^s)_{s>0}$

  $\rightsquigarrow \left|\mathcal{L}(f \underset{(0,1)}{*} g)(z) - \mathcal{L}f(z)\,.\,\mathcal{L}g(z)\right| \leq \|f\|_1 \|g\|_1 e^{-\operatorname{Re} z}$

& $\left|\mathcal{L}(a^s)(z) - \mathcal{L}(a^{s/k})^k(z)\right| \leq C(s)\, e^{-\operatorname{Re} z} \ldots$ ¿y?

SEMIGRUPOS Y ESTRUCTURA

TEOREMA (Esterle, Sinclair, Ransford).- R radical, $(a^z)_{\mathrm{Re}\, z>0} \subseteq R \ni \overline{a^z R} = R \wedge \int_{-\infty}^{\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2}\, dy < \infty \Longrightarrow a^z \equiv 0.$

Dem.- AHLFORS-HEINS:

$$r^{-1} \log |F(re^{i\theta})| \xrightarrow[r\to\infty]{} c.\cos\theta \longleftrightarrow \lim_{n\to\infty} \|a^{nz}\|^{1/n} = 0$$

∎

COROLARIO (Teor^{a} tauberiano WIENER).- I ideal cerrado en $L^1(\mathbb{R}^n) \ni \hat{f}(\xi) \neq 0$ $(\xi \in \mathbb{R}^n, f \in I) \Longrightarrow I = L^1(\mathbb{R}^n)$

Dem.-

Hipótesis $\equiv Z(I) = \emptyset \longrightarrow L^1(\mathbb{R}^n)/I$ radical $\xrightarrow[\text{REMA}]{\text{TEO-}} I = L^1(\mathbb{R}^n)$

∎

SEMIGRUPOS EN $L^1(\mathbb{R}^n)$

- Gaussiano $G^z = e^{z\Delta} \equiv g^z * (\cdot)$ $\left\{ \begin{array}{l} \Delta := \sum_{j=1}^{n} \partial^2/\partial x_j^2 \\ g^z(x) := (\pi z)^{-n/2} \exp\left[-\frac{\|x\|^2}{4z}\right] \end{array} \right.$

  $\mathrm{Re}\, z > 0$ $\qquad (x \in \mathbb{R}^n)$

- Poisson

  $P^z = e^{-z\sqrt{-\Delta}} \equiv p^z * (\cdot)$, $p^z(x) = c_n (z^2 + \|x\|^2)^{-\frac{n+1}{2}}$

- $\|g^{1+iy}\|_1 = O(|y|^{n/2})$, $\|p^{1+iy}\|_1 = O(|y|^{\frac{n-1}{2}})$, $|y| \to \infty$

  $n \geq 1$ $\qquad n \geq 2$

  PERO $\quad \|p^{1+iy}\|_1 = O(\log |y|)$, $|y| \to +\infty$ en $L^1(\mathbb{R})$

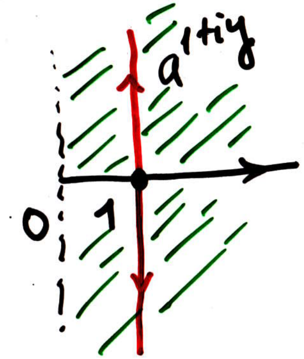
Q4.- (Esterle) ¿ $\exists\, (a^z)_{\mathrm{Re}\, z>0} \subseteq L^1(G) \Rightarrow \sup_y \|a^{1+iy}\|_1 < \infty$

$\Longleftrightarrow$ G compacto ?

TEOREMA (G-White 1997).- G LCA group.

Equivalentes:

(1) $\exists\, (a^z)_{\mathrm{Re}\, z>0}$ holom$^{\underline{o}}$ $\subseteq L^1(G) \ni \sup_{\mathrm{Re}\, z=1} \|a^z\| < \infty$.



(2) $\exists\, E \in L^1(G),\ 0 \neq E = E^2$.

(3) G contiene subgrupo compacto abierto G_0

Dem.- • Si $G = \mathbb{R}$, (1) $\Longrightarrow a^z \equiv 0$. $G = \mathbb{R}^n \times G_0$.

• **Idea**: Use Beurling-Helson: $L^1(\mathbb{R}) \xrightarrow{\theta} L^1(\mathbb{R})$

hom$^{\underline{o}}$ acotado $\Longleftrightarrow (\theta f)^\wedge(\xi) = \hat{f}(\alpha \xi + \beta)\ \forall \xi \in \mathbb{R}$,

con $\alpha, \beta \in \mathbb{R}$.

- Sea $\Phi : L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$
$$f \longmapsto \int_{-\infty}^{\infty} f(y)\, a^{1+iy}\, dy \quad ''$$

$\forall z \in \mathbb{C}^+, \xi \in \mathbb{R},$
$$\widehat{a^z}(\xi) = e^{-z\varphi(\xi)} \therefore (\Phi f)^\wedge(\xi) = e^{-\varphi(\xi)} \hat{f}(\varphi(\xi)),$$
$$\forall f \in L^1(\mathbb{R}), \xi \in \mathbb{R}.$$

$\varphi =$ símbolo

- Adaptación de Katznelson:
  - Indep.^{te} lineal sobre $\mathbb{Q}$ [Kronecker] $\Rightarrow \varphi$ polinomio $(C^{(2)})$
  - Van der Corput $\Rightarrow \varphi$ afín $\rightsquigarrow a^z \equiv 0$ $\blacksquare$

Q5.- TEOREMA (1) $\Longleftrightarrow$ (3), G no abeliano.

Q6.- ¿$\exists\, (a^z) \subseteq L^1(\mathbb{R}) \ni \|a^{1+iy}\|_{L^1} = o(\log|y|),\ |y| \to +\infty$?
$\neq 0$

TEOREMA (G-Ransford 2000).-

(1) $\exists\, \mathfrak{A}$ álg. Banach (BH) $\ni \forall\, \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ $\left\{ \begin{array}{l} \varphi(y) \xrightarrow[|y|\to\infty]{} \infty \\ \inf\limits_{y\in\mathbb{R}} \varphi(y) > 0 \end{array} \right.$

$\exists\, (a^z) \subseteq \mathfrak{A} \ni \|a^{1+iy}\| \xrightarrow[y]{} \infty,\ \|a^{1+iy}\| \leq \varphi(y)$, $\forall y \in \mathbb{R}$.

(2) $\exists\, \mathfrak{B}$ álg. Banach (BH) Separable $\ni \forall\, \mathbb{R} \xrightarrow{\psi} \mathbb{R}$ localm. acotada

$\exists\, (b^z) \subseteq \mathfrak{B} \ni \|b^{1+iy}\| \geq \psi(y)\ \ \forall y \in \mathbb{R}$.

Q7.- ¿Puede obtenerse $\mathfrak{A}$ separable en (1)?

Q8.- (Kahane 1962, Niza) ¿$\exists\, \mathbb{T} \xrightarrow{\phi} \mathbb{R} \ni \|e^{in\phi}\|_{\ell^1(\mathbb{Z})} = o(\log|n|)$, $n \in \mathbb{Z}$, $|n| \to \infty$?

Leblanc, Domar, Lebedev, Olevskii, ... (C.V.)

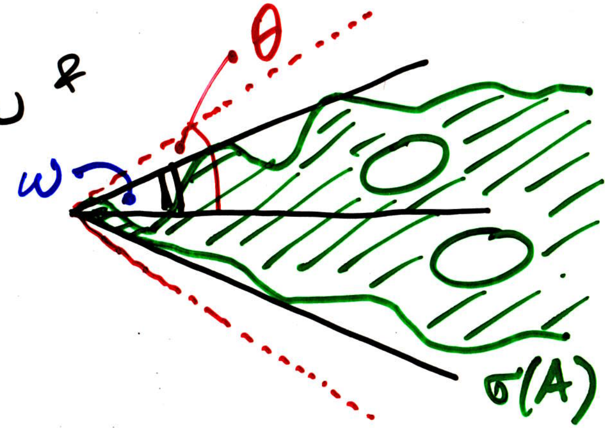
CÁLCULO FUNCIONAL. MODELOS FUNCIONALES

- $\mathcal{H}$ Hilbert ; $0<\omega<\pi$, $A\in\mathcal{C}(\mathcal{H})$

SECTORIAL tipo ω si $\sigma(A)\subseteq S_\omega$ &

$\forall\,\theta \ni \omega<\theta<\pi \quad \exists\, C_\theta \ni$

$$\|(z-A)^{-1}\|\leq C_\theta\,|z|^{-1},\ \forall z\notin S_\theta$$



- A admite H^∞-cálculo funcional si

$$\exists\quad H^\infty(\dot{S}_\theta)\to\mathcal{B}(\mathcal{H})\quad \text{homom}^{\underline{o}}\ \text{acotado}$$
$$f\ \mapsto\ f(A)\qquad [\text{Cauchy}]$$

• Nagy-Foias: $T\in\mathcal{B}(\mathcal{H}),\ \|T\|\leq 1,\ T^{n}x \xrightarrow[n]{} 0\ (x\in\mathcal{H})$

(*):
$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{V^{-1}} & \mathcal{H}^2(\mathbb{D};E)/\delta\mathcal{H}^2(\mathbb{D};F) \\ \underline{T}\downarrow & & \downarrow \tilde{M}_z \\ \mathcal{H} & \xleftarrow{V} & \mathcal{H}^2(\mathbb{D};E)/\delta\mathcal{H}^2(\mathbb{D};F) \end{array}$$

$f(T):=V\circ\tilde{M}_{f(\cdot)}\circ V^{-1}$

• G-Miana-Yakubovich 2011: Nagy-Foias <u>SECTORIALES</u> [Control]

• Cowling-Doust-McIntosh-Yagi 1996: <u>$\omega=0$</u>,

$\exists\, H^{\infty}$-CF para T tq $\|f(T)\|\leq\theta^{-\alpha}\|f\|_{S_{\theta_0}}$ $\Longleftrightarrow$ $\exists\, \Lambda^{\alpha}_{\infty,1}(\mathbb{R}^{+})\to\mathcal{B}(\mathcal{H})$, $f\mapsto f(T)$ $\left(\Longleftarrow\ \text{G-Miana 2008, MIKHLIN}\right)$

BESOV

Q9.- ¿Diagrama (*) tipo Besov cuando $\omega=0$?

¿Y para espacios de <u>Banach</u>?

REPRESENTACIONES

- G grupo de Lie. $\pi : G \to \mathcal{B}(E)$ fuert^{e} cont^{e} $\ni$
  Banach
  $\sup_{t\in G} \|\pi(t)\|_{op} < \infty \rightsquigarrow \pi : L^1(G) \to \mathcal{B}(E)$ homom^{o} acotado

$$\pi(f)x := \int_G f(t)\,\pi(t)x\,dt \quad (x\in E;\ f\in L^1(G)).$$

$E_\infty := \{x\in E : \pi(\cdot)x \in C^{(\infty)}(G;E)\}$ espacio Fréchet

- π (top^{e}) irreducible : $F \leq E$ cerrado $\ni$
  $\pi(t)F \subseteq F$ $(t\in G) \Rightarrow F=(0)$ ó $F=E$.
- $E = \mathcal{H}$ Hilbert $\Longrightarrow$ toda $\pi : \underset{NIL}{G} \to \mathcal{B}(\mathcal{H})$ unitaria irreducible es CCR : $\overline{\pi(L^1(G))}^{\|.\|_{op}} = \mathcal{K}(\mathcal{H})$.

TEOREMA (D. Beltita, I. Beltita - G 2016).-

G nilpotente, E reflexivo, $\pi \begin{cases} \text{fuert}^{\underline{e}}\ \text{cont}^{\underline{e}} \\ \sup_G \|\pi(t)\|_{op} < \infty \end{cases}$

& π irreducible $\Rightarrow$

$$\pi(L^1(G))^{-} = \mathcal{F}(E) := \left\{ E \xrightarrow{T} E \ \text{rango finito} \right\}^{-\|\,\|_{op}} \subseteq \mathcal{B}(E)$$

Dem.-

Paulsen $\Rightarrow \exists\ G \xrightarrow{\rho} \mathcal{B}(\mathcal{H})$ unitaria irreducible &

$E_\infty \underset{A^{-1}}{\overset{A}{\underset{\sim}{\longleftrightarrow}}} \mathcal{H}_\infty$ iso t.q.

$$\begin{array}{ccc} E_\infty & \longrightarrow & E_\infty \\ A \downarrow & \circlearrowleft & \downarrow A \\ \mathcal{H}_\infty & \longrightarrow & \mathcal{H}_\infty \end{array}$$

(enlace)

$E \leftrightsquigarrow \mathcal{H}$

$E_\infty \leftrightarrow \mathcal{H}_\infty$

TRANSFER^{a}

$\blacksquare$

Q10.- ¿ $\pi(L^1(G))^{-} = E \hat{\otimes} E'$?

Q11.- ¿ $E_\infty \xrightarrow{A} \mathcal{H}_\infty \rightsquigarrow$ TEORÍA "C^*" en ESP. BANACH ?